Strict comparison for crossed products by free minimal actions of $\mathbb{Z}^d$: Supplementary slides

N. Christopher Phillips

University of Oregon
and
Research Institute for Mathematical Sciences, Kyoto

9 September 2011

Conference on C*-Algebras and Related Topics

Research Institute for Mathematical Sciences, Kyoto University

Kyoto, Japan

5–9 September 2011

Appendix 1: Recursive subhomogeneous C*-algebras

Definition

A *recursive subhomogeneous C*-algebra* is a C*-algebra isomorphic to one of the form

$$B = \left[\cdots \left[\left[C_0 \oplus_{C_1^{(0)}} C_1 \right] \oplus_{C_2^{(0)}} C_2 \right] \cdots \right] \oplus_{C_l^{(0)}} C_l,$$

with $C_k = C(X_k, M_{n(k)})$ for compact Hausdorff spaces X_k and positive integers $n(k)$, with $C_k^{(0)} = C\big(X_k^{(0)}, M_{n(k)}\big)$ for compact subsets $X_k^{(0)} \subset X_k$ (possibly empty), and where the maps $C_k \to C_k^{(0)}$ are always the restriction maps and the other maps determining the pullbacks are unital.

An expression like this is a *recursive subhomogeneous decomposition* of B.

The *topological dimension* of the decomposition is $\max\big(\dim(X_0), \dim(X_1), \ldots, \dim(X_l)\big)$.

Appendix 2: Sketch of proof that if B is large in A and B has strict comparison, then so does A.

Theorem

Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be large. Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

We will sketch the proof of the case needed for this talk, which is $\mathrm{rc}(B) = 0$ implies $\mathrm{rc}(A) = 0$. That is, if B has strict comparison of positive elements, then so does A.

Appendix 2: Strict comparison (continued)

This condition says that B is "large" in A:

1. For every $\varepsilon > 0$ and nonzero y in B_+, whenever $a_1, a_2, \ldots, a_n \in A$ satisfy $0 \leq a_j \leq 1$ for $j = 1, 2, \ldots, n$, then there are a continuous function $g \colon X \to [0,1]$ and $b_1, b_2, \ldots, b_n \in A$ such that:
   1. $0 \leq b_j \leq 1$ for $j = 1, 2, \ldots, n$.
   2. $\|b_j - a_j\| < \varepsilon$ for $j = 1, 2, \ldots, n$.
   3. $(1-g)b_j \in B$ for $j = 1, 2, \ldots, n$.
   4. $g \precsim y$.

Recall that for $\tau \in T(A)$, we define $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$ for $a \in M_\infty(A)_+$.

Strict comparison of positive elements means that $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$ implies $a \precsim b$.

We want to show strict comparison for B implies strict comparison for A.

The point is that one can push elements into B by cutting out a piece with small trace, as sketched next.

Appendix 2: Strict comparison (continued)

For a C*-algebra B and $a, b \in B_+$, recall that $a \precsim b$ if there is a sequence $(v_n)_{n\in\mathbb{Z}_{>0}}$ in B such that $\lim_{n\to\infty} v_n b v_n^* = a$.

We describe one technical point. For $\varepsilon > 0$, define $f_\varepsilon \colon [0,\infty) \to [0,\infty)$ by $f_\varepsilon(t) = \max(0,\, t-\varepsilon) = (t-\varepsilon)_+$. For a positive element a of a C*-algebra, define $(a-\varepsilon)_+ = f_\varepsilon(a)$.

Lemma

Let B be a C*-algebra, and let $a, b \in B_+$. Then $a \precsim b$ if and only if $(a-\varepsilon)_+ \precsim b$ for all $\varepsilon > 0$.

This is needed to take care of the approximation in the "largeness" condition on $B \subset A$.

Appendix 2: Strict comparison (continued)

Suppose for simplicity that A has a unique tracial state τ. Then B also has a unique tracial state, namely $\tau|_B$.

Let $a_1, a_2 \in A$ be positive elements such that $d_\tau(a_1) < d_\tau(a_2)$. We want to prove that $a_1 \precsim a_2$.

It is enough to show that $(a_1-\varepsilon)_+ \precsim a_2$ for all $\varepsilon > 0$.

Let $\varepsilon > 0$. Choose $\alpha > 0$ appropriately and nonzero $y \in B_+$ with $d_\tau(y) < \alpha$. Choose $b_1, b_2 \in A$ and $g \in C(X)$ with $0 \leq g \leq 1$ such that:

1. $0 \leq b_1, b_2 \leq 1$.
2. $\|b_1 - a_1\| < \alpha$ and $\|b_2 - a_2\| < \alpha$.
3. $(1-g)b_1, (1-g)b_1 \in B$.
4. $g \precsim y$.

Set

$$c_1 = [(1-g)b_1(1-g) - \alpha]_+ \quad \text{and} \quad c_2 = [(1-g)b_2(1-g) - \alpha]_+.$$

These are in B.

Appendix 2: Strict comparison (continued)

We had $d_\tau(a_1) < d_\tau(a_2)$, and we arranged that

1. $0 \leq b_1, b_2 \leq 1$.
2. $\|b_1 - a_1\| < \alpha$ and $\|b_2 - a_2\| < \alpha$.
3. $(1-g)b_1, (1-g)b_1 \in B$.
4. $g \precsim y$.

We defined

$$c_1 = [(1-g)b_1(1-g)-\alpha]_+ \in B \quad \text{and} \quad c_2 = [(1-g)b_2(1-g)-\alpha]_+ \in B.$$

With a bit of work (and good choice of α), we will get:

$$(a_1-\varepsilon)_+ \precsim c_1 \oplus g, \quad c_2 \precsim a_2, \quad \text{and} \quad d_\tau(c_1) + \alpha < d_\tau(c_2).$$

The condition on g implies $d_\tau(g) \leq d_\tau(y) < \alpha$, so strict comparison for B gives

$$c_1 \oplus g \precsim c_2,$$

whence $(a_1-\varepsilon)_+ \precsim a_2$.

Appendix 2: Strict comparison (continued)

If B has finitely many extreme tracial states, essentially the same method works.

If B has infinitely many extreme tracial states, one has to work a bit harder, using some more machinery, but one gets the same result.

Appendix 3: Rokhlin towers and partition valued functions

To get a partition valued function from a system of Rokhlin towers, let $x \in X$. Every time the orbit of x runs through one of the Rokhlin towers, collect the corresponding values of γ in a set in $\mathcal{P}(x)$. More precisely, the sets in $\mathcal{P}(x)$ are in one to one correspondence with elements $\eta \in \mathbb{Z}^d$ such that $h^{\eta}(x) \in Y_j$ for some j, and the set in $\mathcal{P}(x)$ corresponding to such an element η is $\eta + F_j$.

It is easily seen that $\mathcal{P}$ is bounded and invariant.

To get a system of Rokhlin towers from a bounded invariant partition valued function $\mathcal{P}$, choose finite sets $F_1, F_2, \ldots, F_m \subset \mathbb{Z}^d$ such that every set in every $\mathcal{P}(x)$ is a translate of exactly one of the sets F_j. Define

$$Y_j = \{x \in X : F_j \in \mathcal{P}(x)\}.$$

For $j = 1, 2, \ldots, m$ and $\gamma \in F_j$, we claim that a point $x \in X$ is in $h^{\gamma}(Y_j)$ if and only if the set in $\mathcal{P}(x)$ which contains $0 \in \mathbb{Z}^d$ is $F_j - \gamma$.
This holds because, by invariance of $\mathcal{P}$, we have $x \in h^{\gamma}(Y_j)$ if and only if $F_j - \gamma \in \mathcal{P}(x)$.

Appendix 4: The Følner condition

To prove that the subalgebra $A = \overline{\bigcup_{n=0}^{\infty} A_n}$ is "large", we will need the finite subsets $F_j \subset \mathbb{Z}^d$ that occur in the systems of Rokhlin towers

$$(Y_1, F_1), \, (Y_2, F_2), \, \ldots, \, (Y_m, F_m)$$

to be $(\Sigma_n, \varepsilon_n)$-Følner sets for $\varepsilon_n > 0$ with $\varepsilon_n \to 0$, and for finite sets $\Sigma_n \subset \mathbb{Z}^d$ with $\Sigma_n \nearrow \mathbb{Z}^d$.

Recall that a finite set $F \subset \mathbb{Z}^d$ is a (Σ, ε)-Følner set if

$$\mathrm{card}(F \triangle (\gamma + F)) \leq \varepsilon \cdot \mathrm{card}(F)$$

for all $\gamma \in \Sigma$.

Let $\mathcal{P}$ be the partition valued function corresponding to a system

$$(Y_1, F_1), \, (Y_2, F_2), \, \ldots, \, (Y_m, F_m)$$

of Rokhlin towers. The $F_j \subset \mathbb{Z}^d$ are all (Σ, ε)-Følner if and only if every set in every partition $\mathcal{P}(x)$ is a (Σ, ε)-Følner set.

Appendix 5: $C^*(\mathbb{Z}, X, h)_Y$ is large

Set $A = C^*(\mathbb{Z}, X, h)$ and

$$A_Y = C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X), \, uC_0(X \setminus Y)\big) \subset A.$$

If $Y = \{x_0\}$, we want to show that A_Y is large in A.

We sketch the proof of the condition involving cutdowns. To simplify notation, consider just one element $a \in A$. We want $g \in B$ and c close to a such that $(1 - g)c$, $c(1 - g) \in A_Y$, and such that g is Cuntz subequivalent to some given nonzero positive $z \in A_Y$.

Take c of the form $c = \sum_{n=-N}^{N} f_n u^n$. Let U be a small enough neighborhood of x_0 that any function supported in $\bigcup_{n=-N}^{N} h^n(U)$ is Cuntz subequivalent to z. (This needs some work.) We also want the sets $h^n(U)$ to be disjoint.

Now take g_0 supported in U with $g_0(x_0) = 1$ and $g = \sum_{n=-N}^{N} g_0 \circ h^n$.

One has to check that $(1 - g)c$, $c(1 - g) \in A_Y$. It is at least easy to see that when one writes $(1 - g)c$ or $c(1 - g)$ as $\sum_{n=-N}^{N} k_n u^n$, then $k_1(x_0) = 0$.

Appendix 6: Choosing partition valued functions for actions of $\mathbb{Z}^d$: the topological small boundary property

In general, we choose Y so that, in addition, ∂Y is topologically small. That is, there is $m \in \mathbb{Z}_{\geq 0}$ such that whenever $\gamma_0, \gamma_1, \ldots, \gamma_m$ are $m+1$ distinct elements of $\mathbb{Z}^d$, then

$$h^{\gamma_0}(\partial Y) \cap h^{\gamma_1}(\partial Y) \cap \cdots \cap h^{\gamma_m}(\partial Y) = \varnothing.$$

Let r_0 be the maximum diameter of any set in any $\mathcal{P}(x)$. For r large enough compared to r_0 (the choice $6r_0 + 7$ will do), use point set topology to choose an open set U containing ∂Y which is so small that whenever $\gamma_0, \gamma_1, \ldots, \gamma_m$ are $m+1$ distinct elements of $\mathbb{Z}^d$, all with $\|\gamma_j\|_2 < mr + 1$ (this is the new part), then

$$h^{\gamma_0}(U) \cap h^{\gamma_1}(U) \cap \cdots \cap h^{\gamma_m}(U) = \varnothing.$$

Partition the elements $\gamma \in S_U(x)$ (that is, $\gamma \in \mathbb{Z}^d$ such that $h^{\gamma}(x) \in U$) into "r-clusters" C, that is, maximal sets such that any two points in C can be connected by a chain of elements of $S_U(x)$ such that each element is at distance less than r from the next one.

Equivalently, the clusters are minimal sets such that the distance from one to any other is at least r.

The point of the choice of U above is that it ensures that no r-cluster has more than m elements. (Details omitted.) In particular, r-clusters are finite.

For each r-cluster C, we now group together in a set in $\mathcal{Q}(x)$ all the sets in $\mathcal{P}(x)$ at distance less than $2r_0 + 1$ from $\mathbb{C}$. All leftover sets in $\mathcal{P}(x)$ become sets in $\mathcal{Q}(x)$ without being changed. One can now prove that $\mathcal{Q}$ is semicontinuous.

When $\mathcal{P}$ is (Σ, ε)-Følner, so is $\mathcal{Q}$.

There is still trouble with iteration: at the next step, we will need to know that ∂U was also topologically small.