# Lecture 6: Applications and Problems

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The Second Summer School on Operator Algebras and Noncommutative Geometry 2016

East China Normal University, Shanghai

11–29 July 2016

- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
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# A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
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- The tracial Rokhlin property for actions of finite groups.
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# Recall: The tracial Rokhlin property

## Definition

Let $A$ be an infinite dimensional simple separable unital C*-algebra, and let $\alpha \colon G \to \operatorname{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that, with $e = \sum_{g \in G} e_g$:

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1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
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Recall the simplifications:

1. If $A$ is finite, the last condition can be omitted.
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**Theorem**

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# "Small" in $C^*(G, A, \alpha)$ vs. "small" in $A$

The following result implies that one can make the error projection "small" relative to $C^*(G, A, \alpha)$ by requiring that it be "small" relative to $A$.

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## Theorem

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Apply the tracial Rokhlin property with $F = S \cup \{x\}$. Then $exe \approx \sum_{h \in G} e_h x e_h$ (exercise: check this!) so, provided $\delta$ is small enough, orthogonality of the sum implies that there is some $h \in G$ such that $\|e_h x e_h\| > 1 - \delta$. We will take $a = e_h$. Now $g \neq 1$ implies

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This proves Kishimoto's condition. Exercise: Write out the details.

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Given $x \in A_+$ with $\|x\| = 1$, and a finite set $S \subset A$, we want:

1. $\|axa\| > 1 - \varepsilon$.
2. $\|ab\alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$ and $b \in S$.

For $F \subset A$ finite and $\delta > 0$, the tracial Rokhlin property gives mutually orthogonal projections $e_g$ such that (omitting a condition we won't need):

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Thus $ss^*$ is a projection in $D$ equivalent to the nonzero projection $p \in A$.

# Using Kishimoto's condition (continued)

We want to show that a nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$ contains a nonzero projection equivalent to a projection in $A$.

We have $c \in D_+$, $a \in A_+$, and a nonzero projection $p \in A$, all satisfying

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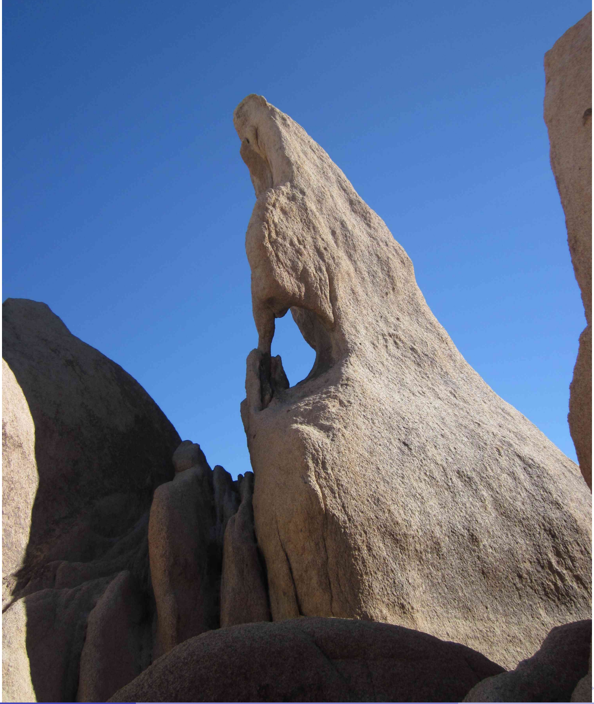

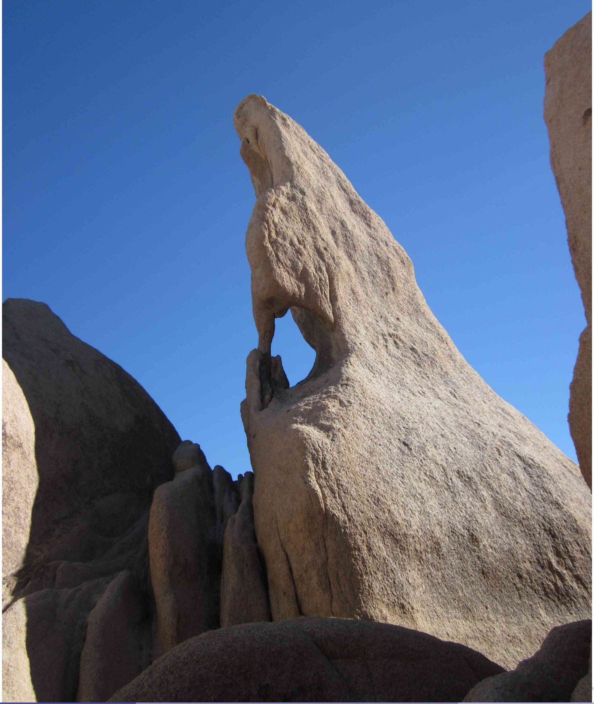

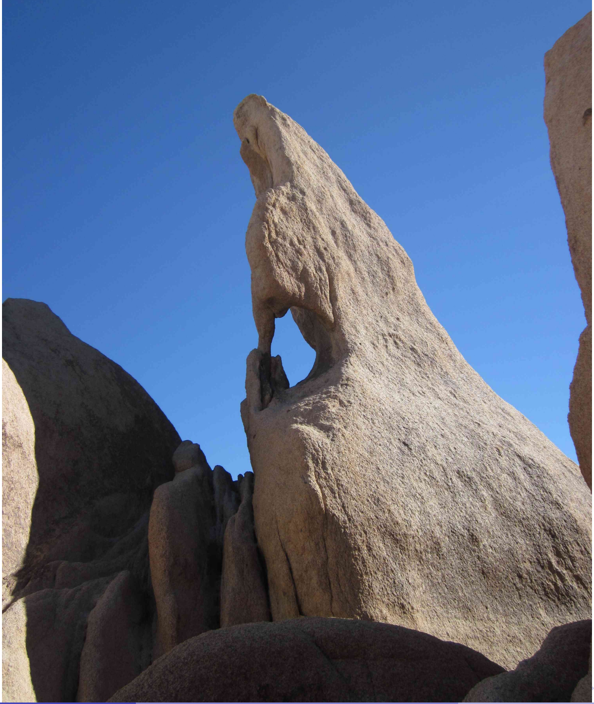

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If $G$ is finite, then strong pointwise outerness is sufficient. If $G$ is exact and discrete, a condition we call "spectral freeness" seems appropriate and works. If $G$ is finite, they are equivalent. Exactness of the group (or at least of the action) is necessary in any case.

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$\alpha\colon G \to \mathrm{Aut}(A)$ is strongly pointwise outer if, for every $g \in G \setminus \{1\}$ and any two $\alpha_g$-invariant ideals $I \subset J \subset A$ with $I \neq J$, the automorphism of $J/I$ induced by $\alpha_g$ is outer.

In order to prove that $C^*_{\mathrm{r}}(G, A, \alpha)$ has only crossed product ideals, one needs at least to know that for every subgroup $H \subset G$ and every $H$-invariant subquotient $J/I$ of $A$, the induced action of $H$ on $J/I$ is pointwise outer. (There is a finite dimensional counterexample if one doesn't consider subgroups.)

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**Question**

If $G$ is exact and discrete, and $\alpha\colon G \to \mathrm{Aut}(A)$ is strongly pointwise outer, does it follow that $C^*_{\mathrm{r}}(G, A, \alpha)$ has only crossed product ideals?

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