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INVARIANT CUBATURE FORMULAE FOR SPHERES AND BALLS BY COMBINATORIAL METHODS*

SANGWOO HEO† AND YUAN XU‡

Abstract. Invariant cubature formulae for a class of weight functions on the simplex T^d are derived using combinatorial methods, extending the formulae in [Grundmann and Möller, *SIAM J. Numer Anal.*, 15 (1978), pp. 282–290] for the unit weight function on T^d. These formulae are used to derive cubature formulae on the surface of the sphere S^d and on the unit ball B^d using connections between cubature formulae on T^d, B^d and S^d.

Key words. cubature formulae, on the unit sphere, on the simplex, on the ball, combinatorial identities

AMS subject classifications. 65D32, 05A19

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1. Introduction.

The purpose of this paper is to construct invariant cubature formulae on the simplex T^d and on the unit ball B^d of $\mathbb{R}^d$, as well as on the (surface of) sphere S^d of $\mathbb{R}^{d+1}$. For $\mathbf{x} \in \mathbb{R}^d$ we denote by $|\mathbf{x}|$ the usual Euclidean norm and by $|\mathbf{x}|_1 = |x_1| + \cdots + |x_d|$ the ℓ^1 norm of $\mathbf{x}$. Then $S^d = \{\mathbf{y} \in \mathbb{R}^{d+1} : \ |\mathbf{y}| = 1\}$, $B^d = \{\mathbf{x} \in \mathbb{R}^d : \ |\mathbf{x}| \le 1\}$, and

$$T^d = \{\mathbf{x} \in \mathbb{R}^d : x_1 \ge 0, \ldots, x_d \ge 0, 1 - |\mathbf{x}|_1 \ge 0\}.$$

In [3], Grundmann and Möller constructed a family of cubature formulae of arbitrary odd degree for the unit weight function on T^d using combinatorial methods. Recently in [8, 9] we showed that cubature formulae on T^d, B^d, and S^d are related and, in fact, often equivalent. However, formulae for the unit weight function on T^d are related to formulae with respect to $|x_1 \cdots x_{d+1}|d\omega$ on S^d and $|x_1 \cdots x_d|d\mathbf{x}$ on B^d. In order to derive formulae for the unit weight functions on S^d and B^d, we need formulae on T^d for the weight functions $1/\sqrt{x_1 \cdots x_d(1 - |\mathbf{x}|_1)}$ and $1/\sqrt{x_1 \cdots x_d}$, respectively. In this paper we shall construct cubature formulae on T^d for a family of weight functions that include both of them as well as the unit weight function.

The formulae in [3] and some of their extensions in the following section are invariant under the symmetric group on T^d; that is, invariant under the affine transforms of T^d onto itself. The corresponding formulae on S^d are invariant under the octahedral group, which we denote by $\mathcal{B}_{d+1}$ since it is the Weyl group of type B_{d+1}. This is the symmetric group of the unit cube $\{\pm 1, \pm 1, \pm 1\}$ in $\mathbb{R}^3$. It contains permutations of coordinates and the sign changes (the group $\mathcal{B}_{d+1}$ is semiproduct of the symmetric group $\mathcal{S}_{d+1}$ and the abelian group $\mathbb{Z}_2^{d+1}$). We can write

$$(1.1) \qquad \sum_{\sigma \in \mathcal{B}_{d+1}} f(\mathbf{x}\sigma) = \sum f(\pm x_{\sigma_0}, \ldots, \pm x_{\sigma_d}), \qquad \mathbf{x} = (x_0, \ldots, x_d) \in \mathbb{R}^{d+1},$$

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†Department of Mathematics, University of Southern Indiana, Evansville, IN 47712 (heo@usi.edu).
‡Department of Mathematics, University of Oregon, Eugene, OR 97403-1222 (yuan@math.uoregon.edu). The research of this author was supported by National Science Foundation grant DMS-9802265.

where the sum is over all possible sign changes and all permutations $\sigma = (\sigma_0, \dots, \sigma_d)$ of $(0, 1, \dots, d)$. A cubature formula invariant under $\mathcal{B}_{d+1}$ is a linear combination of the above sums at distinct nodes; such a formula is called *fully symmetric* (cf. [7, p. 128]). Recall that a cubature formula is of degree M if it gives the exact values of integrals for polynomials of degree up to M. Throughout this paper, we use the notation

$$\binom{\alpha}{\beta} = \binom{\alpha_0}{\beta_0} \cdots \binom{\alpha_d}{\beta_d}, \qquad \text{where} \quad \binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)},$$

for $\alpha, \beta \in \mathbb{R}^{d+1}$ and we adopt the convention that $\binom{a}{b} = 0$ whenever $b \geq a+1$. Moreover, for $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d)$ we write $\alpha - 1/2 = (\alpha_0 - 1/2, \dots, \alpha_d - 1/2)$. One of the main results of the paper is the following fully symmetric cubature formula for the surface measure on S^d.

THEOREM 1.1. *Let $s \in \mathbb{N}_0$ and $n = 2s+1$. Then the following is a fully symmetric cubature formula of degree $2n+1$ on S^d:*

(1.2)

$$\int_{S^d} g(\mathbf{y}) d\omega = \frac{\pi^{(d+1)/2}}{2^{2s+d}} \left[\sum_{i=0}^{s} (-1)^i \frac{(n + (d-1)/2 - 2i)^n}{i!\, \Gamma(n + (d+1)/2 - i)} \sum_{\substack{|\beta|_1 = s-i, \\ \beta_0 \geq \cdots \geq \beta_d}} \binom{\beta - 1/2}{\beta} \right.$$
$$\left. \times \sum_{\sigma \in \mathcal{B}_{d+1}} g\left(\left(\frac{\sqrt{2\beta_0 + 1/2}}{\sqrt{n + (d-1)/2 - 2i}}, \dots, \frac{\sqrt{2\beta_d + 1/2}}{\sqrt{n + (d-1)/2 - 2i}} \right) \sigma \right) \right].$$

We will also give an analogous formula for the unit weight function on B^d (Theorem 2.5). The cubature formulae in Theorem 1.1 form the first family of formulae that are given explicitly for all d and for higher degrees. Among them all formulae for $s \geq 3$ appear to be new. See section 4 for further discussion. For results on cubature formulae on S^d, T^d, and B^d in general, we refer to [2, 4, 6, 7] and the references therein; more recent references are also collected in [1].

The paper is organized as follows. The preliminary and main results are given in section 2. The proof of the combinatorial identity that yields the formulae is presented in section 3. Examples and further comments are given in section 4.

2. Preliminary and main results. Throughout this section we write $\mathbf{y} = (y_0, y_1, \dots, y_d)$ if $\mathbf{y} \in \mathbb{R}^{d+1}$. Given $\mu = (\mu_0, \dots, \mu_d)$, $\mu_i > -1$, let $W_\mu(\mathbf{y}) = |y_0|^{2\mu_0+1} \cdots |y_d|^{2\mu_d+1}$ be a weight function defined on S^d. We use the notation $\gamma(\mu) = \sum_{i=0}^d \mu_i$. When all μ_i are nonnegative, $\gamma(\mu) = |\mu|_1$. In the following we shall write $|\beta|$ instead of $|\beta|_1$ for $\beta \in \mathbb{N}_0^{d+1}$. Associated with W_μ, we define a weight function W_μ^T on T^d and a weight function W_μ^B on B^d by

$$W_\mu^T(\mathbf{x}) = x_1^{\mu_1} \cdots x_d^{\mu_d} (1 - |\mathbf{x}|_1)^{\mu_0} \quad \text{and} \quad W_\mu^B(\mathbf{x}) = |x_1|^{2\mu_1+1} \cdots |x_d|^{2\mu_d+1} (1 - |\mathbf{x}|^2)^{\mu_0},$$

respectively. We note that W_μ on S^d is invariant under the abelian group $\mathbb{Z}_2^{d+1}$ and W_μ^B is invariant under $\mathbb{Z}_2^d$. The following theorem gives the connection between cubature formulae with respect to these weight functions.

THEOREM 2.1. *If there is a cubature formula of degree n for W_μ^T on T^d given by*

$$\int_{T^d} f(\mathbf{u}) W_\mu^T(\mathbf{u}) d\mathbf{u} = \sum_{i=1}^{N} \lambda_i f(\mathbf{u}_i) \tag{2.1}$$

whose nodes $\mathbf{u}_i = (u_{i,1}, \ldots, u_{i,d})$ *are all in* T^d*, then there is a cubature formula of degree* $2n+1$ *for* W_μ^B *on* B^d *given by*

$$\int_{B^d} f(\mathbf{x}) W_\mu^B(\mathbf{x}) d\mathbf{x} = \sum_{i=1}^N \lambda_i \sum_{\varepsilon \in \{-1,1\}^d} f(\varepsilon_1 \sqrt{u_{i,1}}, \ldots, \varepsilon_d \sqrt{u_{i,d}}) / 2^{k(\mathbf{u}_i)}, \tag{2.2}$$

where $k(\mathbf{u})$ *denote the number of nonzero elements in the node* $\mathbf{u}$*; moreover, there is a cubature formula of degree* $2n+1$ *for* W_μ *on* S^d *given by*

$$\int_{S^d} f(\mathbf{y}) W_\mu(\mathbf{y}) d\omega = 2 \sum_{i=1}^N \lambda_i \sum_{\varepsilon \in \{-1,1\}^{d+1}} f(\varepsilon_0 \sqrt{u_{i,0}}, \ldots, \varepsilon_d \sqrt{u_{i,d}}) / 2^{k(\mathbf{u}_i)}, \tag{2.3}$$

where $u_{i,0} = 1 - |\mathbf{u}_i|_1$.

This theorem is proved in [8, 9] for far more general classes of weight functions. It is also shown in [8, 9] that if a cubature formula of the form (2.2) on B^d or a formula of the form (2.3) on S^d is given, then we can use it to generate cubature formulae on the other two domains. The theorem as stated is precisely what is needed in this paper. We are now ready to state our main results, starting with formulae on T^d.

THEOREM 2.2. *Let* $s \in \mathbb{N}_0$ *and* $n = 2s+1$*. Then the following is a cubature formula of degree* n *with respect to* W_μ^T *on* T^d*:*

(2.4)

$$\int_{T^d} f(\mathbf{x}) W_\mu^T(\mathbf{x}) d\mathbf{x} = \frac{\prod_{j=0}^d \Gamma(\mu_j+1)}{2^{2s}} \left[\sum_{i=0}^s (-1)^i \frac{(n+\gamma(\mu)+d-2i)^n}{i!\,\Gamma(n+\gamma(\mu)+d+1-i)} \times \sum_{|\beta|=s-i} \binom{\beta+\mu}{\beta} f\left(\frac{2\beta_1+\mu_1+1}{n+\gamma(\mu)+d-2i}, \ldots, \frac{2\beta_d+\mu_d+1}{n+\gamma(\mu)+d-2i} \right) \right],$$

where the sum over β *is over* $\beta = (\beta_0, \ldots, \beta_d) \in \mathbb{N}_0^{d+1}$ *such that* $|\beta| = s-i$.

We note that only $\binom{\beta_0+\mu_0}{\beta_0}$ in the summation over β depends on β_0. Hence, we can also write the summation over β as

(2.5)

$$\sum_{|\beta|=s-i} = \sum_{\beta_0=0}^{s-i} \left[\binom{\beta_0+\mu_0}{\beta_0} \times \sum_{|\beta'|=s-i-\beta_0} \binom{\beta'+\mu'}{\beta'} f\left(\frac{2\beta_1+\mu_1+1}{n+\gamma(\mu)+d-2i}, \ldots, \frac{2\beta_d+\mu_d+1}{n+\gamma(\mu)+d-2i} \right) \right],$$

where $\beta' = (\beta_1, \ldots, \beta_d)$ and $\mu' = (\mu_1, \ldots, \mu_d)$. The most interesting case of the above formula is when W_μ^T is symmetric. Let $\mathcal{S}_{d+1}(T)$ denote the symmetric group of T^d, which consists of all permutations among $(x_0, x_1, \ldots, x_d)$, where we write $x_0 = 1 - |\mathbf{x}|_1$. We note that W_μ^T is invariant under $\mathcal{S}_{d+1}(T)$ when $\mu_0 = \mu_1 = \cdots = \mu_d$. In this case, the formula in Theorem 2.2 becomes symmetric under $\mathcal{S}_{d+1}(T)$ as well. In order to state the symmetric cubature formulae, we follow the following notation in [3].

We associate each $\mathbf{x} = (x_1, \ldots, x_d) \in T^d$ with $X = (x_0, \ldots, x_d)$ where $x_0 = 1 - |\mathbf{x}|_1$. Let ϕ be the mapping $\phi : \mathbf{x} \mapsto X$. If $\phi(\mathbf{y})$ is in the orbit of X in $\mathcal{S}_{d+1}(T)$ (that is, $\phi(\mathbf{y}) \in \{Xg | g \in \mathcal{S}_{d+1}\}$), then $\mathbf{y}$ is in T^d. We denote the equivalent class of all points $\mathbf{y}$ whose images $\phi(\mathbf{y})$ are in the orbit of X by $((x_0, x_1, \ldots, x_d))$ or simply $((X))$; that is,

$$((X)) = \{(x_{\sigma_1}, \ldots, x_{\sigma_d}) : \sigma = (\sigma_0, \ldots, \sigma_d) \in \mathcal{S}_{d+1}(T)\}.$$

Then all points in $((X))$ are in T^d. We then introduce an abbreviation for arbitrary $f : \mathbb{R}^d \to \mathbb{R}$, as in [3]:

$$\sum f((X)) = \sum_{\mathbf{y} \in ((X))} f(\mathbf{y}).$$

Symmetric cubature formulae are linear combinations of the above sums for various nodes. Note that for symmetric W_μ^T, we have $\gamma(\mu) = (d+1)\mu$. In the following, if $\mu = (\mu_0, \ldots, \mu_d)$ and all μ_i are equal, we denote their common value also by μ.

THEOREM 2.3. *Let $s \in \mathbb{N}_0$ and $n = 2s+1$. Assume that $\mu_0 = \mu_1 = \cdots = \mu_d$. Then the following is a symmetric cubature formula of degree n for W_μ^T on T^d:*

$$(2.6) \quad \int_{T^d} f(\mathbf{x}) W_\mu^T(\mathbf{x}) d\mathbf{x} = \frac{[\Gamma(\mu+1)]^{d+1}}{2^{2s}} \sum_{i=0}^{s} \left[(-1)^i \frac{(2s + (\mu+1)(d+1) - 2i)^n}{i!\, \Gamma(n + (\mu+1)(d+1) - i)} \right.$$
$$\left. \times \sum_{\substack{|\beta| = s-i, \\ \beta_0 \geq \cdots \geq \beta_d}} \binom{\beta+\mu}{\beta} \sum f\left(\left(\frac{2\beta_0 + \mu + 1}{2s + (\mu+1)(d+1) - 2i}, \ldots, \frac{2\beta_d + \mu + 1}{2s + (\mu+1)(d+1) - 2i}\right)\right)\right].$$

When $\mu = 0$, this theorem was established in [3]. The number of nodes in this formula is exceptionally small; see the discussion in [3] and in section 4.

Using Theorem 2.1, we can then obtain cubature formulae on B^d and on S^d. For S^d, we shall state only formulae that are obtained using the formulae in Theorem 2.3, which consists of the most interesting cases. One should have no difficulty writing down formulae on S^d obtained from Theorem 2.2 if needed. When $\mu_0 = \cdots = \mu_d$, the weight function W_μ is symmetric under the octahedral group $\mathcal{B}_{d+1}$. The corresponding formula on S^d is fully symmetric; see (1.1).

THEOREM 2.4. *Let $s \in \mathbb{N}_0$ and $n = 2s+1$. Assume that $\mu_0 = \mu_1 = \cdots = \mu_d$. Then the following is a fully symmetric cubature formula of degree $2n+1$ for W_μ on S^d:*

$$(2.7) \quad \int_{S^d} g(\mathbf{y}) W_\mu(\mathbf{y}) d\omega = \frac{[\Gamma(\mu+1)]^{d+1}}{2^{2s+d}}$$
$$\times \sum_{i=0}^{s} \left[(-1)^i \frac{(2s + (\mu+1)(d+1) - 2i)^n}{i!\, \Gamma(n + (\mu+1)(d+1) - i)} \sum_{\substack{|\beta| = s-i, \\ \beta_0 \geq \cdots \geq \beta_d}} \binom{\beta+\mu}{\beta} \right.$$
$$\left. \times \sum_{\sigma \in \mathcal{B}_{d+1}} g\left(\left(\frac{\sqrt{2\beta_0 + \mu + 1}}{\sqrt{2s + (\mu+1)(d+1) - 2i}}, \ldots, \frac{\sqrt{2\beta_d + \mu + 1}}{\sqrt{2s + (\mu+1)(d+1) - 2i}}\right)\sigma\right)\right].$$

When $\mu = -1/2$, this is the same as Theorem 1.1. For the cubature formulae on B^d, we shall state the result only for the unit weight function $W(\mathbf{x}) = 1$. In this case, we cannot use Theorem 2.3, since a formula in Theorem 2.3 leads to a cubature formula on B^d for the weight function $W_\mu^B(\mathbf{x}) = |x_1 \ldots x_d|^{2\mu+1}(1 - |\mathbf{x}|^2)^\mu$, which does not include the unit weight function. We need to use the cubature formula for W_μ^T with $\mu = (0, -1/2, \ldots, -1/2)$ in Theorem 2.2. The result is as follows.

THEOREM 2.5. *Let $s \in \mathbb{N}_0$ and $n = 2s+1$. Then the following is a fully symmetric cubature formula of degree $2n+1$ for the unit weight function on B^d:*

(2.8)
$$\int_{B^d} f(\mathbf{x})d\mathbf{x} = \frac{\pi^{d/2}}{2^{2s+d}} \sum_{i=0}^{s} \left[(-1)^i \frac{(n+d/2-2i)^n}{i!\Gamma(n+d/2+1-i)} \sum_{\beta_0=0}^{s-i} \sum_{\substack{|\beta'|=s-i-\beta_0, \\ \beta_1 \ge \cdots \ge \beta_d}} \binom{\beta'-1/2}{\beta'} \right.$$
$$\left. \times \sum_{\sigma \in \mathcal{B}_d} f\left(\left(\frac{\sqrt{2\beta_1+1/2}}{\sqrt{n+d/2-2i}}, \ldots, \frac{\sqrt{2\beta_d+1/2}}{\sqrt{n+d/2-2i}} \right) \sigma \right) \right],$$

where $\beta' = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$.

We note that the formula (2.8) is fully symmetric even though the corresponding formula on T^d is not invariant under $\mathcal{S}_{d+1}(T)$.

The proof of these results is equivalent to the proof of a combinatorial identity. To see how it works we need one more definition. For $\mathbf{x} \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$, we use the standard multi-index notation $\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ to denote the monomials. Let Π_n^d denote the space of polynomials of degree at most n in d variables. Evidently, the monomials $\{\mathbf{x}^\alpha : |\alpha| \le n, \ \alpha \in \mathbb{N}_0^d\}$ form a basis of Π_n^d. We shall need another convenient basis given in terms of

$$X^\beta = x_0^{\beta_0} x_1^{\beta_1} \cdots x_d^{\beta_d}, \qquad x_0 = 1 - x_1 - \cdots - x_d \quad \text{and} \quad \beta = (\beta_0, \ldots, \beta_d),$$

where we follow the notation in [3]. Using the multinormal formula, it follows that $\{X^\beta : |\beta| = n, \ \beta \in \mathbb{N}_0^{d+1}\}$ forms a basis of Π_n^d (see [3, p. 283]). The moments of X^α in terms of W_μ^T are given by

$$\int_{T^d} X^\alpha W_\mu^T(\mathbf{x}) d\mathbf{x} = \frac{\prod_{j=0}^d \Gamma(\alpha_j + \mu_j + 1)}{\Gamma(|\alpha| + \gamma(\mu) + d + 1)}, \qquad \alpha \in \mathbb{N}_0^{d+1}.$$

Since (2.4) is a cubature formula of degree $2s+1$, it is exact for all polynomials X^α for $|\alpha| = 2s+1$, which is equivalent to the following identity:

$$\frac{\prod_{j=0}^d \Gamma(\alpha_j + \mu_j + 1)}{\Gamma(|\alpha| + \gamma(\mu) + d + 1)} = \frac{\prod_{j=0}^d \Gamma(\mu_j + 1)}{2^{2s}} \sum_{i=0}^{s} \left[(-1)^i \frac{(|\alpha| + \gamma(\mu) + d - 2i)^{|\alpha|}}{i!\Gamma(|\alpha| + \gamma(\mu) + d + 1 - i)} \right.$$
$$\left. \times \sum_{|\beta|=s-i} \binom{\beta+\mu}{\beta} \left(\frac{2\beta_0 + \mu_0 + 1}{|\alpha| + \gamma(\mu) + d - 2i} \right)^{\alpha_0} \cdots \left(\frac{2\beta_d + \mu_d + 1}{|\alpha| + \gamma(\mu) + d - 2i} \right)^{\alpha_d} \right].$$

Using the notation of binomial coefficients, the above formula can be written as

(2.9)
$$2^{2s} \alpha! \binom{\alpha+\mu}{\alpha} = \sum_{j=0}^{s} (-1)^j \binom{|\alpha| + \gamma(\mu) + d}{j} \sum_{|\beta|=s-j} \binom{\beta+\mu}{\beta} \prod_{i=0}^{d} (2\beta_i + \mu_i + 1)^{\alpha_i},$$

where $\alpha! = \alpha_0! \cdots \alpha_d!$ and $|\alpha| = 2s+1$. Hence, the existence of the cubature formulae (2.4) is equivalent to the validity of (2.9), which we prove in the next section. Here is how this combinatorial formula was discovered. When $\mu = 0$ the combinatorial identity (2.9) is due to Grundmann and Möller in [3], which suggests the possible existence of the formula in general. However, the proof in [3] does not seem to extend to the general setting. We computed a number of examples using a computer algebra system, which confirmed the existence of a formula in the form of (2.9) up to some coefficients, which were determined by further calculations and judicious guessing and were later justified by the proof.

The formula (2.6) follows from (2.4) or (2.9), as in [3], from the fact that any equivalence class $((x_0,\dots,x_d))$ contains exactly one point $(x_{\tau_0},\dots,x_{\tau_d})$ with $x_{\tau_0}\ge\cdots\ge x_{\tau_d}$. Therefore, if $\mu_0=\cdots=\mu_d$ and their common value is denoted by μ, then

$$\sum_{|\beta|=s-i}\binom{\beta+\mu}{\beta}\left(\frac{2\beta_0+\mu+1}{|\alpha|+\gamma(\mu)+d-2i}\right)^{\alpha_0}\cdots\left(\frac{2\beta_d+\mu+1}{|\alpha|+\gamma(\mu)+d-2i}\right)^{\alpha_d}$$
$$=\sum_{\substack{|\beta|=s-i,\\ \beta_0\ge\cdots\ge\beta_d}}\binom{\beta+\mu}{\beta}X^\alpha\left(\left(\frac{2\beta_0+\mu+1}{n+\gamma(\mu)+d-2i},\dots,\frac{2\beta_d+\mu+1}{n+\gamma(\mu)+d-2i}\right)\right),$$

from which the formula (2.6) follows. Both formula (2.7) in Theorem 2.4 and formula (2.8) in Theorem 2.5 follow from Theorem 2.1 and the formulae on T^d in a straightforward manner. Since all nodes in the formulae (2.4) and (2.6) are nonzero, the corresponding number $k(\mathbf{u}_i)$ in using Theorem 2.1 is equal to d for the formula on the ball and equal to $d+1$ for the formula on the sphere for every node.

3. A combinatorial identity. In this section we prove the combinatorial identity (2.9). We start with finding a generating function that gives the right-hand side of (2.9).

LEMMA 3.1. *Let $\mu_i>-1$ and $\alpha_i\in\mathbb{N}_0$, $0\le i\le d$. For each nonnegative integer s and $0\le r<1$, we have*

$$(1-r)^{2s+d+1+\gamma(\mu)}\prod_{i=0}^d\left(r^{-\mu_i}\frac{d}{dr}r^{1+\mu_i}\right)^{\alpha_i}(1-r^2)^{-\mu_i-1}$$
$$=\sum_{m=0}^\infty\left[\sum_{j=0}^m(-1)^j\binom{2s+d+1+\gamma(\mu)}{j}\sum_{|\beta|=m-j}\binom{\beta+\mu}{\beta}\prod_{i=0}^d(2\beta_i+\mu_i+1)^{\alpha_i}\right]r^{2m}.$$

Proof. Let $a>-1$. We start with the binomial series

$$(1-r^2)^{-a-1}=\sum_{n=0}^\infty\binom{n+a}{a}r^{2n},\qquad 0\le r<1.$$

Let $k\in\mathbb{N}_0$. Applying the operator $r^{-a}(d/dr)r^{1+a}$ to the series k times, we conclude that

$$\left(r^{-a}\frac{d}{dr}r^{1+a}\right)^k(1-r^2)^{-a-1}=\sum_{n=0}^\infty\binom{n+a}{a}(2n+1+a)^kr^{2n},\qquad 0\le r<1.$$

Next we multiply the above identity with a replaced by μ_i, n replaced by β_i, and k replaced by α_i for $0\le i\le d$ to conclude that

$$\prod_{i=0}^d\left(r^{-\mu_i}\frac{d}{dr}r^{1+\mu_i}\right)^{\alpha_i}(1-r^2)^{-\mu_i-1}=\sum_{m=0}^\infty\left[\sum_{|\beta|=m}\binom{\beta+\mu}{\beta}\prod_{i=0}^d(2\beta_i+\mu_i+1)^{\alpha_i}\right]r^{2m}.$$

The desired result follows from multiplying the above equation with the series

$$(1-r^2)^{2s+d+1+\gamma(\mu)} = \sum_{n=0}^{\infty}(-1)^n \binom{2s+d+1+\gamma(\mu)}{n} r^{2n}, \qquad 0 \le r < 1.$$

We note that if $2s+d+1+\gamma(\mu)$ is an integer, then the above series is just a finite binomial sum. The result does not change, however, since we adopt the convention that $\binom{a}{b} = 0$ whenever $b \ge a+1$. $\square$

Let f be a real valued function and a be a real number. The kth forward difference of f based on the points $(a+1)/2, 1+(a+1)/2, \ldots, k+(a+1)/2$, denoted by $\Delta^k f_0$, is defined via the finite difference by

$$\Delta^k f_0 = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f\left(\frac{a+1}{2} + j\right). \tag{3.1}$$

In particular, if $f(x) = x^n$, we write $\Delta^k f_0$ as $\Delta^k (y+(a+1)/2)^n|_{y=0}$.

LEMMA 3.2. *Let $a > -1$ and $n \in \mathbb{N}_0$. Then for $0 \le r < 1$, we have*

$$\left(r^{-a}\frac{d}{dr}r^{1+a}\right)^n (1-r^2)^{-a-1} = 2^n \sum_{k=0}^{n} (-1)^{n-k}\binom{k+a}{k}\Delta^k \left(y+\frac{a+1}{2}\right)^n\Big|_{y=0}(1-r^2)^{-k-a-1}.$$

Proof. Let $\mathcal{D}_a = (1/2)r^{-a}(d/dr)r^{1+a}$ as an operator applying on functions of r. We have for any nonnegative integer k,

$$\mathcal{D}_a(1-r^2)^{-a-k-1} = (1+a+k)(1-r^2)^{-a-k-2} - \left(\frac{1+a}{2}+k\right)(1-r^2)^{-a-k-1}.$$

Let I denote the identity operator. Then we can write the above identity as

$$\left(\mathcal{D}_a + \left(\frac{a+1}{2}+k\right)I\right)(1-r^2)^{-a-k-1} = (a+k+1)(1-r^2)^{-a-k-2}.$$

Iterating the above identity for $k = 0, 1, \ldots, n-1$, we conclude that

$$\begin{aligned}\prod_{i=0}^{n}\left(\mathcal{D}_a + \left(\frac{a+1}{2}+i\right)I\right)(1-r^2)^{-a-1} &= (n+a)\cdots(a+1)(1-r^2)^{-a-n-1} \\ &= \frac{\Gamma(n+a+1)}{\Gamma(a+1)}(1-r^2)^{-a-n-1}.\end{aligned}$$

What we need is the action of $\mathcal{D}_a^n$ which can be derived from the above formula as follows. Using the notation (3.1), the Newton form of the interpolation polynomial based on the points $(a+1)/2, 1+(a+1)/2, \ldots, n+(a+1)/2$ can be written as

$$L_n(f;x) = \sum_{k=1}^{n} \frac{1}{k!}\Delta^k f_0 \prod_{i=0}^{k}\left(x - i - \frac{a+1}{2}\right).$$

Since $L_n(f)$ is the unique interpolation polynomial of degree n on those $n+1$ points, it follows that $L_n f = f$ if f is a polynomial of degree at most n. In particular, applying the formula to $f(t) = t^n$ and then set $t = -x$, we obtain that

$$x^n = \sum_{k=1}^{n}(-1)^{n-k}\frac{1}{k!}\Delta^k\left(y+\frac{a+1}{2}\right)^n\Big|_{y=0}\prod_{i=0}^{k}\left(x+i+\frac{a+1}{2}\right).$$

Using this identity with $\mathcal{D}_a$ in place of x, we can then conclude that

$$\begin{aligned}\mathcal{D}_a^n(1-r)^{-a-1} &= \sum_{k=1}^{n}\frac{(-1)^{n-k}}{k!}\Delta^k\left(y+\frac{a+1}{2}\right)^n\Big|_{y=0}\prod_{i=1}^{k}\left(\mathcal{D}_a-i-\frac{a+1}{2}\right)(1-r^2)^{-a-1}\\ &= \sum_{k=1}^{n}(-1)^{n-k}\Delta^k\left(y+\frac{a+1}{2}\right)^n\Big|_{y=0}\frac{\Gamma(k+a+1)}{k!\Gamma(a+1)}(1-r^2)^{-k-a-1},\end{aligned}$$

which is, upon multiplying 2^n, what we need to prove. ∎

LEMMA 3.3. *Let $\mu_i > -1$ and $\alpha_i \in \mathbb{N}_0$, $0 \le i \le d$. Let s be a nonnegative integer. If $|\alpha| = 2s+1$, then*

(3.2)
$$\begin{aligned}\sum_{j=0}^{s}(-1)^j\binom{2s+d+1+\gamma(\mu)}{j}\sum_{|\beta|=s-j}\binom{\beta+\mu}{\beta}\prod_{i=0}^{d}(2\beta_i+\mu_i+1)^{\alpha_i} &= 2^{2s+1}\alpha!\binom{\alpha+\mu}{\alpha}\\ -\sum_{m=0}^{s}(-1)^{s-m}\binom{2s-m}{s}\sum_{|\mathbf{k}|=m}\binom{\mathbf{k}+\mu}{\mathbf{k}}\prod_{i=0}^{d}2^{\alpha_i}\Delta^{k_i}&\left(y+\frac{\mu_i+1}{2}\right)^{\alpha_i}\Big|_{y=0}.\end{aligned}$$

Proof. Multiplying the identity in Lemma 3.2 with $a = \mu_i$ and $n = \alpha_i$ for $i = 0, 1, \ldots, d$, we get

$$\begin{aligned}&\prod_{i=0}^{d}\left(r^{-\mu_i}\frac{d}{dr}r^{1+\mu_i}\right)^{\alpha_i}(1-r^2)^{-\mu_i-1}\\ &=\prod_{i=0}^{d}\sum_{k_i=0}^{\alpha_i}(-1)^{\alpha_i-k_i}\binom{k_i+\mu_i}{k_i}\Delta^{k_i}\left(y+\frac{\mu_i+1}{2}\right)^{\alpha_i}\Big|_{y=0}(1-r^2)^{-k_i-\mu_i-1}\\ &=\sum_{m=0}^{|\alpha|}(-1)^{|\alpha|-m}\sum_{|\mathbf{k}|=m}\binom{\mathbf{k}+\mu}{\mathbf{k}}\prod_{i=0}^{d}\Delta^{k_i}\left(y+\frac{\mu_i+1}{2}\right)^{\alpha_i}\Big|_{y=0}(1-r^2)^{-m-\gamma(\mu)-d-1}.\end{aligned}$$

Multiplying this equation by $(1-r^2)^{2s+d+1+\gamma(\mu)}$, the terms involving r in the summation become $(1-r^2)^{2s-m}$. Hence, upon setting $|\alpha| = 2s+1$, we see that except the term with $m = |\alpha|$, which contains $(1-r^2)^{-1}$, all other terms become polynomials in r. We expand $(1-r^2)^{-1}$ as a standard power series in r and $(1-r^2)^{2s-m}$, $0 \le m \le 2s$, as a polynomial in r using the binomial formula. After exchanging the order of the summations, we conclude that for $|\alpha| = 2s+1$,

$$\begin{aligned}(1-r)^{2s+d+1+\gamma(\mu)}\prod_{i=0}^{d}\left(r^{-\mu_i}\frac{d}{dr}r^{1+\mu_i}\right)^{\alpha_i}(1-r^2)^{-\mu_i-1} &= 2^{2s+1}\alpha!\binom{\alpha+\mu}{\alpha}\sum_{n=0}^{\infty}r^{2n}\\ +2^{|\alpha|}\sum_{j=0}^{s}(-1)^j\Bigg[\sum_{m=0}^{2s-j}(-1)^{2s+1-m}\binom{2s-m}{j}&\\ \times\sum_{|\mathbf{k}|=m}\binom{\mathbf{k}+\mu}{\mathbf{k}}\prod_{i=0}^{d}\Delta^{k_i}\left(y+\frac{\mu_i+1}{2}\right)^{\alpha_i}\Big|_{y=0}\Bigg]r^{2j}.&\end{aligned}$$

The left-hand side of the above formula is the same as the left-hand side of the formula in Lemma 3.1; both formulae give the power series of this function when $|\alpha| = 2s+1$. The desired formula is derived by comparing the coefficients of r^{2s} in the two expansions. □

Let us denote the two sums in (3.2) by L-sum and R-sum, respectively. Then (3.2) becomes

$$\text{L-sum} = 2^{2s+1}\alpha!\binom{\alpha+\mu}{\alpha} - \text{R-sum}. \tag{3.3}$$

Notice that L-sum is the same as the sum in (2.9); we see that the above equation is equivalent to (2.9) if either

$$\text{L-sum} = 2^{2s}\alpha!\binom{\alpha+\mu}{\alpha} \qquad \text{or} \qquad \text{R-sum} = 2^{2s}\alpha!\binom{\alpha+\mu}{\alpha}$$

holds. In fact, the first identity is exactly (2.9). Since R-sum is as complicated as L-sum, it seems that we do not gain much by switching from L-sum to R-sum. It turns out, however, that these two sums are in fact equal; that is, R-sum = L-sum. The proof of this fact will finish our proof of (2.9).

LEMMA 3.4. *Let $\mu_i > -1$ and $\alpha_i \in \mathbb{N}_0$, $0 \le i \le d$. Let s be a nonnegative integer. If $|\alpha| = 2s+1$, then*

$$\sum_{j=0}^{s}(-1)^j\binom{2s+d+1+\gamma(\mu)}{j}\sum_{|\beta|=s-j}\binom{\beta+\mu}{\beta}\prod_{i=0}^{d}(2\beta_i+\mu_i+1)^{\alpha_i}$$
$$= \sum_{m=0}^{s}(-1)^{s-m}\binom{2s-m}{s}\sum_{|\mathbf{k}|=m}\binom{\mathbf{k}+\mu}{\mathbf{k}}\prod_{i=0}^{d}2^{\alpha_i}\Delta^{k_i}\left(y+\frac{\mu_i+1}{2}\right)^{\alpha_i}\Big|_{y=0}.$$

Proof. Our objective is to prove R-sum = L-sum. We start with (3.1), which implies that

$$\Delta_k\left(y+\frac{a+1}{2}\right)^n\Big|_{y=0} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{a+1}{2}+j\right)^n.$$

Multiplying this formula with $k = k_i$ and $n = \alpha_i$ for $i = 0, 1, \ldots, d$, we get

$$\prod_{i=0}^{d}2^{\alpha_i}\Delta^{k_i}\left(y+\frac{\mu_i+1}{2}\right)^{\alpha_i}\Big|_{y=0} = \sum_{p=0}^{|\mathbf{k}|}(-1)^{|\mathbf{k}|-p}\sum_{|\beta|=p}\binom{\mathbf{k}}{\beta}\prod_{i=0}^{d}(2\beta_i+\mu_i+1)^{\alpha_i}.$$

Substituting the above sum into R-sum and changing the order of summations, we get

$$\text{R-sum} = \sum_{m=0}^{s}\binom{2s-m}{s}\sum_{p=0}^{m}(-1)^{s-p}\sum_{|\beta|=p}\prod_{i=0}^{d}(2\beta_i+\mu_i+1)^{\alpha_i}\sum_{|\mathbf{k}|=m}\binom{\mathbf{k}+\mu}{\mathbf{k}}\binom{\mathbf{k}}{\beta}$$
$$= \sum_{p=0}^{s}(-1)^{s-p}\sum_{m=p}^{s}\binom{2s-m}{s}\sum_{|\beta|=p}\prod_{i=0}^{d}(2\beta_i+\mu_i+1)^{\alpha_i}\sum_{|\mathbf{k}|=m}\binom{\mathbf{k}+\mu}{\mathbf{k}}\binom{\mathbf{k}}{\beta}.$$

Since $\binom{k_i}{\beta_i} = 0$ whenever $\beta_i > k_i$, it follows that $\binom{\mathbf{k}}{\beta} = 0$ whenever $|\beta| > |\mathbf{k}|$ (recall that both k_i and β_i are nonnegative integers). Hence, in the last equation we can write the limits of the sum over m as from $m = 0$ to s. Changing summation indices $m = s - n$ and $p = s - j$, we then have

$$\text{R-sum} = \sum_{j=0}^{s}(-1)^j \sum_{|\beta|=s-j} \prod_{i=0}^{d}(2\beta_i + \mu_i + 1)^{\alpha_i} \sum_{n=0}^{s} \binom{s+n}{s} \sum_{|\mathbf{k}|=s-n} \binom{\mathbf{k}+\mu}{\mathbf{k}}\binom{\mathbf{k}}{\beta}.$$

Using the following identity on the binomial coefficients

$$\binom{\mathbf{k}+\mu}{\mu}\binom{\mathbf{k}}{\beta} = \binom{\beta+\mu}{\beta}\binom{\mathbf{k}+\mu}{\beta+\mu},$$

which can be easily verified, we further obtain

$$\text{R-sum} = \sum_{j=0}^{s}(-1)^j \sum_{|\beta|=s-j} \binom{\beta+\mu}{\beta} \prod_{i=0}^{d}(2\beta_i + \mu_i + 1)^{\alpha_i} \sum_{n=0}^{s} \binom{s+n}{s} \sum_{|\mathbf{k}|=s-j} \binom{\mathbf{k}+\mu}{\beta+\mu}.$$

The proof of R-sum = L-sum then follows from the combinatorial identity proved in the following lemma. $\square$

LEMMA 3.5. *Let s and j be nonnegative integers and let $\beta \in \mathbb{N}_0^{d+1}$ and $|\beta| = s - j > 0$. Then*

$$\sum_{n=0}^{s} \binom{s+n}{s} \sum_{|\mathbf{k}|=s-n} \binom{\mathbf{k}+\mu}{\beta+\mu} = \binom{2s+\gamma(\mu)+d+1}{j}.$$

Proof. We prove this identity by the method of generating function. It is easy to verify that if n is a nonnegative integer and $a > -1$ is a real number, then

$$\sum_{k=0}^{\infty} \binom{k+a}{n+a} r^k = \sum_{k=n}^{\infty} \binom{k+a}{n+a} r^k = r^n (1-r)^{-n-a-1}, \qquad 0 \le r < 1.$$

Therefore, it follows that

$$\begin{aligned} \frac{r^{|\beta|}}{(1-r)^{\gamma(\mu)+|\beta|+d+s+2}} &= (1-r)^{s+1} \prod_{i=0}^{d} r^{\beta_i}(1-r)^{-\mu_i-\beta_i-d-1} \\ &= \sum_{q=0}^{\infty} \binom{s+q}{q} r^q \sum_{p=0}^{\infty} \sum_{|\mathbf{k}|=p} \binom{\mathbf{k}+\mu}{\beta+\mu} r^p \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \binom{s+n}{s} \sum_{|\mathbf{k}|=m-n} \binom{\mathbf{k}+\mu}{\beta+\mu} r^m. \end{aligned}$$

On the other hand, we also have

$$\frac{r^{|\beta|}}{(1-r)^{\gamma(\mu)+|\beta|+d+s+2}} = \sum_{p=0}^{\infty} \binom{p+\gamma(\mu)+|\beta|+d+s+1}{p} r^{|\beta|+p}, \qquad 0 \le r < 1.$$

The desired identity follows from comparing the coefficients of r^s in these two series and using the fact that $|\beta| = s - j$. $\square$

The identity (2.9) is proved by Lemma 3.3, Lemma 3.4, and Lemma 3.5.

4. Examples and remarks. First we give several examples of our formulae on S^d and B^d. We shall only give examples for the unit weight function, that is, $d\omega$ on S^d and $d\mathbf{x}$ on B^d. The corresponding formulae are given in (1.2) and (2.8). In the following we denote by $\omega_d = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$ the surface area of S^d and by $V_d = \pi^{d/2}/\Gamma((d+1)/2)$ the volume of B^d.

4.1. Examples on S^d. For $s = 0$, the formula (1.2) is of degree 3 and it takes the form

$$\int_{S^d} f(\mathbf{y})d\omega = \frac{\omega_d}{2^{d+1}} \sum_{\sigma\in\mathcal{B}_{d+1}} f\left(\left(\frac{1}{\sqrt{d+1}}, \ldots, \frac{1}{\sqrt{d+1}}\right)\sigma\right),$$

which appears as the formula U_n: 3-2 in [7, p. 294]. For $s = 1$, the formula (1.2) is of degree 7 and it takes the form

$$\begin{aligned}\int_{S^d} f(\mathbf{y})d\omega = {} & \frac{\omega_d}{2^{d+4}}\frac{(d+5)^2}{(d+1)(d+3)} \sum_{\sigma\in\mathcal{B}_{d+1}} f\left(\left(\frac{\sqrt{5}}{\sqrt{d+5}}, \frac{1}{\sqrt{d+5}}, \ldots, \frac{1}{\sqrt{d+5}}\right)\sigma\right) \\ & - \frac{\omega_d}{2^{d+4}}\frac{(d+1)^2}{(d+3)} \sum_{\sigma\in\mathcal{B}_{d+1}} f\left(\left(\frac{1}{\sqrt{d+1}}, \ldots, \frac{1}{\sqrt{d+1}}\right)\sigma\right),\end{aligned}$$

which appears as the formula U_n: 7-2 in [7, p. 295]. The next case, $s = 2$, gives a formula of degree 11 in the form

$$\begin{aligned}\int_{S^d} f(\mathbf{y})d\omega = {} & \tfrac{3\omega_d}{2^{d+8}}\tfrac{(d+9)^4}{(d+1)(d+3)(d+5)(d+7)} \sum_{\sigma\in\mathcal{B}_{d+1}} f\left(\left(\tfrac{\sqrt{9}}{\sqrt{d+9}}, \tfrac{1}{\sqrt{d+9}}, \ldots, \tfrac{1}{\sqrt{d+9}}\right)\sigma\right) \\ & + \tfrac{2\omega_d}{2^{d+8}}\tfrac{(d+9)^4}{(d+1)(d+3)(d+5)(d+7)} \sum_{\sigma\in\mathcal{B}_{d+1}} f\left(\left(\tfrac{\sqrt{5}}{\sqrt{d+9}}, \tfrac{\sqrt{5}}{\sqrt{d+9}}, \tfrac{1}{\sqrt{d+9}}, \ldots, \tfrac{1}{\sqrt{d+9}}\right)\sigma\right) \\ & - \tfrac{2\omega_d}{2^{d+8}}\tfrac{(d+5)^4}{(d+1)(d+3)(d+7)} \sum_{\sigma\in\mathcal{B}_{d+1}} f\left(\left(\tfrac{\sqrt{5}}{\sqrt{d+5}}, \tfrac{1}{\sqrt{d+5}}, \ldots, \tfrac{1}{\sqrt{d+5}}\right)\sigma\right) \\ & + \tfrac{\omega_d}{2^{d+8}}\tfrac{(d+1)^4}{(d+3)(d+5)} \sum_{\sigma\in\mathcal{B}_{d+1}} f\left(\left(\tfrac{1}{\sqrt{d+1}}, \ldots, \tfrac{1}{\sqrt{d+1}}\right)\sigma\right).\end{aligned}$$

This formula is different from the fully symmetric formula U_n: 11-1 in [7, p. 296]. As far as we know, this formula and all other formulae of degree $4s+3$ for $s \geq 3$ are new.

4.2. Examples on B^d with respect to the unit weight function. For $s = 0$, the formula (2.8) is of degree 3 and takes the form

$$\int_{B^d} f(\mathbf{x})d\mathbf{x} = \frac{V_d}{2^d} \sum_{\sigma\in\mathcal{B}_d} f\left(\left(\frac{1}{\sqrt{d+2}}, \ldots, \frac{1}{\sqrt{d+2}}\right)\sigma\right).$$

This formula appears as the formula S_n: 3-2 in [7, p. 268]. For $s = 1$, the formula (2.8) is of degree 7 and takes the form

$$\begin{aligned}\int_{B^d} f(\mathbf{x})d\mathbf{x} = \frac{V_d}{2^{d+2}} \frac{(d+6)^2}{(d+2)(d+4)} \sum_{\sigma\in\mathcal{B}_d} f\left(\left(\frac{1}{\sqrt{d+6}},\ldots,\frac{1}{\sqrt{d+6}}\right)\sigma\right)\\ + \frac{V_d}{2^{d+3}} \frac{(d+6)^2}{(d+2)(d+4)} \sum_{\sigma\in\mathcal{B}_d} f\left(\left(\frac{\sqrt{5}}{\sqrt{d+6}},\frac{1}{\sqrt{d+6}},\ldots,\frac{1}{\sqrt{d+6}}\right)\sigma\right)\\ - \frac{V_d}{2^{d+3}} \frac{(d+2)^2}{(d+4)} \sum_{\sigma\in\mathcal{B}_d} f\left(\left(\frac{1}{\sqrt{d+2}},\ldots,\frac{1}{\sqrt{d+2}}\right)\sigma\right).\end{aligned}$$

This formula and all other formulae of degree $4s+3$ for $s \ge 2$ in (2.8) appear to be new.

4.3. Further remarks. We denote the number of nodes of cubature formulae of degree M on T^d, S^d and B^d by $N_M(T^d)$, $N_M(S^d)$ and $N_M(B^d)$, respectively. A lower bounds for the formulae of degree $2n+1$ on these domains are given by (see [5, 6, 7])

$$N_{2n+1}(S^d) \ge 2\binom{n+d}{n} \quad \text{and} \quad N_{2n+1}(T^d) \ge \binom{n+d}{n}. \tag{4.1}$$

The lower bound of $N_{2n+1}(T^d)$ is also the lower bound for $N_{2n+1}(B^d)$. We note, however, that in the case of centrally symmetric weight functions on B^d, there is an improved lower bound; see [5, 6]. As in the case of $\mu = 0$ in [3], the number of nodes of the cubature formula (2.6) is given by

$$N_{2s+1}(T^d) = \binom{d+s+1}{s} \quad \text{and} \quad N_{2s+1}(T^d)\Big/\binom{s+d}{s} = 1 + \frac{s}{d+1}.$$

The second equation shows that the number of nodes of (2.6) is small for sufficiently large d. As shown in [9], in using Theorem 2.1, the number of nodes for cubature formulae on B^d and on S^d are relatively small if the cubature formula on T^d has nodes on the boundary (faces) of the simplex. Since the nodes of (2.4) have no zero components, using Theorem 2.1 we see that the number of nodes of the formula (2.7), in particular, the formula (1.2), and the number of nodes of the formula (2.8) are given by

$$N_{4s+3}(S^d) = 2^{d+1}\binom{d+s+1}{s} \quad \text{and} \quad N_{4s+3}(B^d) = 2^{d}\binom{d+s+1}{s}, \tag{4.2}$$

respectively. Note that these formulae are of degree $4s+3$ instead of $2s+1$. The ratios of these numbers and the corresponding lower bounds in (4.1), however, do not converge to 1 as $d \to \infty$. From this point of view, these formulae are not as spectacular as those on the simplex. The numbers are, nevertheless, quantitatively smaller than the product type formulae, which require $2(2s+2)^d$ and $(2s+2)^d$ nodes, respectively.

The drawback of the formulae (2.4), (2.6), (2.7), and (2.8) is that some of the weights of the formula are negative. The effectiveness of a cubature formula is often measured by the quantity $\sum |w_i| / \sum w_i$, where w_i are the weights of the formula. For positive cubature formulae, this number is 1. For the cubature formula (2.4) with all $\mu_i = 0$, as shown in [3], this quantity grows rather fast as either a function of d or a function of s. The similar undesirable feature holds for other cases in (2.4). Let us look at the case of formula (1.2). In this case, we have $\sum w_i = \omega_d = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$.

Hence, if we denote $\sum |w_i|/\sum w_i$ by $A_{d,s}$, then we have with $n = 2s+1$ that

$$A_{d,s} = \frac{\Gamma((d+1)/2)}{2^{2s+1}} \sum_{i=0}^{s} \frac{(n+(d-1)/2-2i)^n}{i!\,\Gamma(n+(d+1)/2-i)} \binom{d/2+s-i}{d/2},$$

where we have used the fact that $\sum_{|\beta|=s-i} \binom{\beta-1/2}{\beta} = \binom{d/2+s-i}{d/2}$. A straightforward computation shows that

$$A_{d,s} = \frac{1}{2s!}\left(\frac{d}{4}\right)^s + \mathcal{O}(d^{s-1}), \qquad d \to \infty.$$

Moreover, by Sterling's formula, $A_{d,s}$ as a function of s satisfies

$$\begin{aligned} A_{d,s} \geq\ & \frac{\Gamma((d+1)/2)}{\Gamma(d/2+1)} \frac{(2s+(d+1)/2)^{2s+1}}{s!2^{2s+1}} \frac{\Gamma(d/2+s+1)}{\Gamma((d+1)/2+2s+1)} \\ & \sim \frac{1}{2\sqrt{2}\pi} \frac{\Gamma((d+1)/2)}{\Gamma(d/2+1)} \frac{e^{(d+1)/2}}{(2s+(d+1)/2)^{(d+2)/2}} \left(\frac{e}{2}\right)^{2s}, \qquad s \to \infty. \end{aligned}$$

This shows that the $A_{d,s}$ increases rapidly as s grows. A similar conclusion holds for the formula on B^d.

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