Homework 8

1. Section 15

15.1 Determine which of the following series converge. Justify your answers.
(a) $\sum \frac{(-1)^n}{n}$ (b) $\sum \frac{(-1)^n n!}{2^n}$

15.2 Repeat Exercise 15.1 for the following.
(a) $\sum [\sin(\frac{n\pi}{6})]^n$ (b) $\sum \sin[(\frac{n\pi}{7})]^n$

15.3 Show $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

15.4 Determine which of the following series converge. Justify your answers.
(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$ (b) $\sum_{n=2}^{\infty} \frac{\log n}{n}$
(d) $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$

15.6 (a) Give an example of a divergent series $\sum a_n$ for which $\sum a_{2n}$ converges.
(b) Observe that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges. See Exercise 14.7.
(c) Give an example of a convergent series $\sum a_n$ for which $\sum a_{2n}$ diverges.

15.7 (a) Prove if (a_n) is a decreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim na_n = 0$. *Hint*: Consider $|a_{N+1} + a_{N+2} + \cdots + a_n|$ for suitable N.
(b) Use (a) to give another proof that $\sum \frac{1}{n}$ diverges.

15.8 Formulate and prove a general integral test as advised in 15.2.

2. Supplement Homework

S1 Give an example of each or explain why the request is impossible referencing the proper theorem(s).
(a) Two series $\sum x_n$ and $\sum y_n$ that both divergent but where $\sum x_n y_n$ converges.
(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.
(c) Two sequence (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but where $\sum y_n$ diverges.
(d) A sequence (x_n) satisfying $0 \le x_n \le 1/n$ where $\sum(-1)^n x_n$ diverges.

S2 Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.
(a) If $\sum a_n$ and $\sum b_n$ converges, then $\sum a_n b_n$ converges.
(b) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.