Homework 4

1. Section 9

9.9 Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.
(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.
(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

9.12 Assume all $s_n \neq 0$ and that the limit $L = \lim |\frac{s_{n+1}}{s_n}|$ exists.
(a) Show that if $L < 1$, then $\lim s_n = 0$. *Hint*: Select a so that $L < a < 1$ and obtain N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.
(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$. *Hint:* Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

9.14 Let $p > 0$. Use Exercise 9.12 to show

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a| \leq 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a < -1. \end{cases}$$

Hint: For the $a > 1$ case, use Exercise 9.12(b).

9.16 Use Theorems 9.9 and 9.10 or Exercises 9.9–9.15 to prove the following:
(a) $\lim \frac{n^4+8n}{n^2+9} = +\infty$
(b) $\lim[\frac{2^n}{n^2} + (-1)^n] = +\infty$

2. Section 10

10.1 Which of the following sequences are increasing? decreasing? bounded?
(a) $\frac{1}{n}$ (b) $\frac{(-1)^n}{n^2}$
(c) n^5 (d) $\sin(\frac{n}{7})$
(e) $(-2)^n$ (f) $\frac{n}{3^n}$.

10.2 Prove Theorem 10.2 for bounded decreasing sequences.

10.4 Discuss why Theorems 10.2 and 10.11 would fail if we restricted our world of numbers to the set $\mathbb{Q}$ of rational numbers.

10.5 Prove Theorem 10.4(ii).

10.7 Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$. See also Exercise 1.11.

10.10 Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.
(a) Find s_2, s_3 and s_4.
(b) Use induction to show $s_n > \frac{1}{2}$ for all n.
(c) Show (s_n) is a decreasing sequence.
(d) Show $\lim s_n$ exists and find $\lim s_n$.

3. Supplement Homework

S1. Give an example of each of the following, or argue that such a request is impossible.
(a) A Cauchy sequence that is not monotone.
(b) A Cauchy sequence with an unbounded subsequence.
(c) A divergence monotone sequence with a Cauchy subsequence.
(d) An unbounded sequence containing a subsequence that is Cauchy.