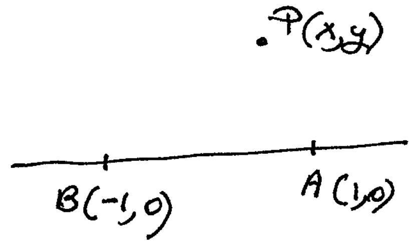
Answers to Questions 1-10 Geometry Exam

1. 360
2. $x = 360 - (\alpha + \beta)$ degrees
3. $\frac{50}{\pi}$ or approximately 15.92
4. (e)
5. $\frac{ab}{a+b}$
6. $\sqrt{40}$ or $2\sqrt{10}$ km.
7. (d)
8. (a)
9. (a)
10. 1

11. Given a segment AB, the set of all points P in a plane such that $\frac{PA}{PB} = 2$ is one of the following: circle, line, elipse or parabola. Which one? Justify your answer.

Answer circle.



$$2 = \frac{PA}{PB} = \frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}},$$

$$4\left((x+1)^2+y^2\right) = (x-1)^2+y^2,$$

$$3x^2+3y^2+10x = -3, \quad \left(x+\frac{10}{6}\right)^2+y^2 = \left(\frac{4}{3}\right)^2. \quad \text{Circle}$$

Remark: It is also possible to prove the above synthetically, but it is much harder.

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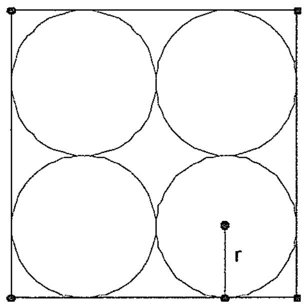
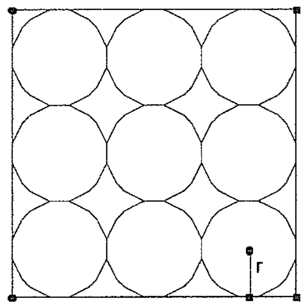


(a)



(b)

a. In the above figure (a), there are 4 congruent circles, each having radius r. The side of the square is $4r$. The percentage of tin wasted would be $\frac{Area(\text{Wasted Tin})}{Area(\text{Square})}$. The wasted tin will remain when the 4 circles are removed. The method to find this is simply: $Area(\text{Wasted Tin}) = Area(\text{Square}) - 4Area(\text{Circle})$. The area formulas for the square and circle are known: $Area(\text{Wasted Tin}) = (4r)^2 - 4\pi \cdot r^2 = 16r^2 - 4\pi \cdot r^2$. Using this to find the percentage of wasted tin shows: $\frac{Area(\text{Wasted Tin})}{Area(\text{Square})} = \frac{16r^2 - 4\pi \cdot r^2}{16r^2} = 1 - \frac{\pi}{4} \approx 21.46\%$

b. In the above figure (b), there are 9 congruent circles, each having radius r. The side of the square is $6r$. The percentage of tin wasted would be $\frac{Area(\text{Wasted Tin})}{Area(\text{Square})}$. Solving as before gives:

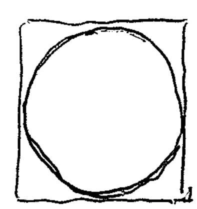
$Area(\text{Wasted Tin}) = Area(\text{Square}) - 4Area(\text{Circle})$

$Area(\text{Wasted Tin}) = (6r)^2 - 9\pi \cdot r^2 = 36r^2 - 9\pi \cdot r^2$. The percentage of wasted tin is:

$\frac{Area(\text{Wasted Tin})}{Area(\text{Square})} = \frac{36r^2 - 9\pi \cdot r^2}{36r^2} = 1 - \frac{\pi}{4} \approx 21.46\%$

c. For the previous examples, the radius of the circle was related to the number of circles included in the interior. When there were 4 circles, the square had side length $4r$. When there were 9 circles, the square had side length $6r$. This will continue as an arithmetic sequence – moving up to 16 circles would create a square with side length of $8r$. In a square with n^2 congruent circles, the square will have side length of $2nr$. In this situation, the percentage of tin wasted will be: $\frac{Area(\text{Wasted Tin})}{Area(\text{Square})} = \frac{4n^2r^2 - n^2\pi \cdot r^2}{4n^2r^2} = 1 - \frac{\pi}{4} \approx 21.46\%$. No matter how many circles are cut out, the percentage of tin will remain the same.

Another approach for all the parts (a), (b) and (c) is to inscribe each circle in its own square. A simple calculation shows that the fraction of the waste is $1 - \frac{\pi}{4}$, no matter what is the size of the square. Consequently for any number of such equal squares the fraction of waste will be the same.



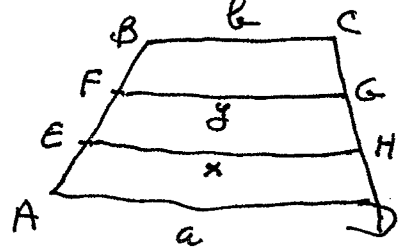
13. Consider the lines $x + y = -1$ and $x - y = 1$, and the lines $x+y+1+k(x-y-1)=0$, where k takes all possible real number values. What do all the lines $x+y+1+k(x-y-1)=0$ have in common? Justify your answer.

They all go through the point of intersection of the two given lines.

Proof if (x_0, y_0) is a common solution of the two given lines then $x_0+y_0+1=0$ and $x_0-y_0-1=0$. Hence (x_0, y_0) satisfies $x+y+1+k(x-y-1)=0$ for all k (since $x_0+y_0+1+k(x_0-y_0-1)=0$)

Remark One could also find the solution of the two given equations and substitute the solution into $x+y+1+k(x-y-1)=0$.

14. $ABCD$ is a trapezoid with bases a and b. If $AE = EF = BF$ and $CG = GH = HD$ find x and y in terms of a and b. Show your work.



$y = \frac{b+x}{2}$ (midsegment theorem),

$x = \frac{y+a}{2}$,

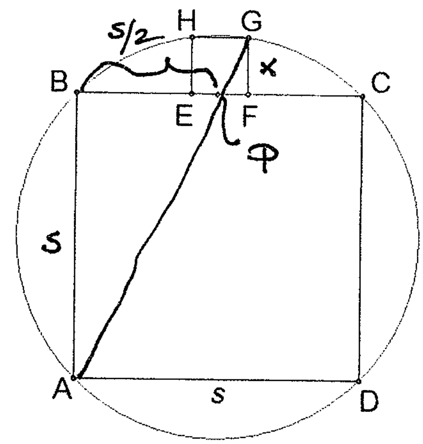
$2y - x = b$

$-y + 2x = a$, $\quad 3y = a + 2b$, $\boxed{y = \frac{a+2b}{3}}$

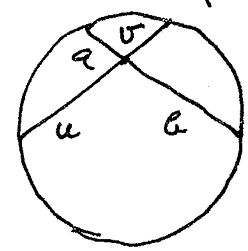
$3x = 2a + b$ $\quad \boxed{x = \frac{2a+b}{3}}$

15. $ABCD$ is a square inscribed in a circle and $EFGH$ is a square with E and H on side BC and F, G points on the circle. Find the side of the smaller square in terms of s. Justify your answer.

$x = \frac{s}{5}$



There are several ways to solve the problem. Use the fact (theorem) th



$uv = ab$ (*)

$BP = \frac{s}{2}$, $PC = \frac{s}{2}$

$(AF)^2 = s^2 + \left(\frac{s}{2}\right)^2 = \frac{5s^2}{2}$,

$AF = \frac{s\sqrt{5}}{2}$

Similarly $PG = \frac{x\sqrt{5}}{2}$

Then By (*) $AF \cdot FG = BP \cdot PC$

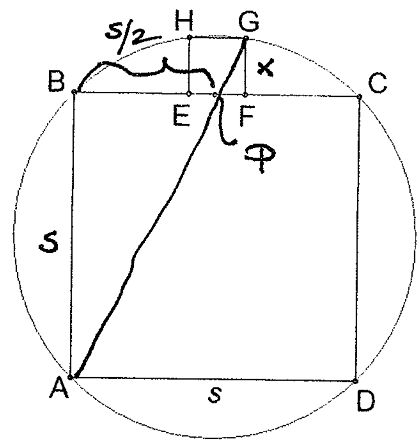
$\frac{s\sqrt{5}}{2} \cdot \frac{x\sqrt{5}}{2} = \left(\frac{s}{2}\right)^2$, $5 \cdot sx = s^2$,

$x = \frac{1}{5}s$

other approaches are possible

15. $ABCD$ is a square inscribed in a circle and $EFGH$ is a square with E and H on side BC and F, G points on the circle. Find the side of the smaller square in terms of s. Justify your answer.

$X = \frac{s}{5}$



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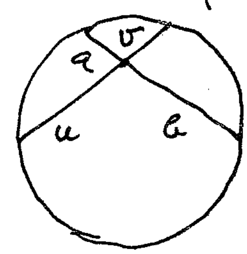

$uv = ab$

$(*)$

$BP = \frac{s}{2}$, $PC = \frac{s}{2}$

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