Hyperkähler Analogues of Kähler Quotients

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Nicholas James Proudfoot

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Committee in charge:

Professor Allen Knutson, Chair
Professor Robion Kirby
Professor Robert Littlejohn

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Chair Date

Date

Date

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Abstract

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Let $\mathfrak{X}$ be a Kähler manifold that is presented as a Kähler quotient of $\mathbb{C}^n$ by the linear action of a compact group G. We define the *hyperkähler analogue* $\mathfrak{M}$ of $\mathfrak{X}$ as a hyperkähler quotient of the cotangent bundle $T^*\mathbb{C}^n$ by the induced G-action. Special instances of this construction include hypertoric varieties [BD, K1, HS, HP1] and quiver varieties [N1, N2, N3]. One of our aims is to provide a unified treatment of these two previously studied examples.

The hyperkähler analogue $\mathfrak{M}$ is noncompact, but this noncompactness is often "controlled" by an action of $\mathbb{C}^\times$ descending from the scalar action on the fibers of $T^*\mathbb{C}^n$. Specifically, we are interested in the case where the moment map for the action of the circle $S^1 \subseteq \mathbb{C}^\times$ is proper. In such cases, we define the *core* of $\mathfrak{M}$, a reducible, compact subvariety onto which $\mathfrak{M}$ admits a circle-equivariant deformation retraction. One of the components of the core is isomorphic to the original Kähler manifold $\mathfrak{X}$. When $\mathfrak{X}$ is a moduli space of polygons in $\mathbb{R}^3$, we interpret each of the other core components of $\mathfrak{M}$ as related polygonal moduli spaces.

Using the circle action with proper moment map, we define an integration theory on the circle-equivariant cohomology of $\mathfrak{M}$, motivated by the well-known localization theorem of [AB] and [BV]. This allows us to prove a hyperkähler analogue of Martin's theorem [Ma], which describes the cohomology ring of an arbitrary Kähler quotient in terms of the cohomology of the quotient by a maximal torus. This theorem, along with a direct analysis of the equivariant cohomology ring of a hypertoric variety, gives us a method for computing the equivariant cohomology ring of many hyperkähler analogues, including all

quiver varieties.

Professor Allen Knutson
Dissertation Committee Chair

Contents

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Chapter 1

Introduction

We begin with a quick overview of some of the structures that we will consider in this thesis, and the types of questions that we will be asking. Detailed definitions will be deferred until the next chapter.

Let G be a compact Lie group acting linearly on $\mathbb{C}^n$, with moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^*$, and suppose we are given a central (i.e. G-fixed) regular value $\alpha \in \mathfrak{g}^*$ of μ. From this data, we may define the Kähler quotient

$$\mathfrak{X} := \mathbb{C}^n /\!\!/ G = \mu^{-1}(\alpha)/G,$$

which itself inherits the structure of a Kähler manifold. (We may also think of $\mathfrak{X}$ as the geometric invariant theory quotient of $\mathbb{C}^n$ by $G^{\mathbb{C}}$ in the sense of Mumford [MFK, §8]; see Proposition 2.3.) A *hyperkähler* manifold is a riemannian manifold (M, g) equipped with three orthogonal complex structures J_1, J_2, J_3 and three compatible symplectic forms $\omega_1, \omega_2, \omega_3$ such that $J_1 J_2 = -J_2 J_1 = J_3$ for $i = 1, 2$, and 3. The cotangent bundle $T^*\mathbb{C}^n$ has a natural hyperkähler structure, and this structure is preserved by the induced action of G. Furthermore, there exist maps $\mu_i : T^*\mathbb{C}^n \to \mathfrak{g}^*$ for $i = 1, 2, 3$ such that μ_i is a moment map for the action of G with respect to the symplectic form ω_i. We define the *hyperkähler analogue* of $\mathfrak{X}$ to be the hyperkähler quotient

$$\mathfrak{M} := T^*\mathbb{C}^n /\!\!/\!\!/\!\!/ G = \left(\mu_1^{-1}(\alpha) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)\right) \Big/ G.$$

(The set $\mu_2^{-1}(0) \cap \mu_3^{-1}(0) \subseteq T^*\mathbb{C}^n$ is a complex subvariety with respect to J_1, and $\mathfrak{M}$ may be thought of as the geometric invariant theory quotient of this variety by $G^{\mathbb{C}}$.) The quotient $\mathfrak{M}$ is a complete hyperkähler manifold [HKLR], containing $T^*\mathfrak{X}$ as a dense open subset (see Proposition 2.4). The following is a description of some well-known classes of Kähler quotients, along with their hyperkähler analogues.

Toric and hypertoric varieties. These examples comprise the case where G is abelian. The geometry of toric varieties is deeply related to the combinatorics of polytopes; for example, Stanley [St] used the hard Lefschetz theorem for toric varieties to prove certain inequalities for the h-numbers of a simplicial polytope. Hypertoric varieties, introduced by Bielawski and Dancer [BD], interact in a similar way with the combinatorics of real hyperplane arrangements. Following Stanley's work, Hausel and Sturmfels [HS] used the hard Lefschetz theorem on a hypertoric variety to give a geometric interpretation of some previously-known inequalities for the h-numbers of a rationally representable matroid. In Chapter 3 we will explore further combinatorial properties of the various equivariant cohomology rings of a hypertoric variety.

Quiver varieties. For any directed graph, Nakajima [N1, N2, N3] defined a *quiver variety* to be the hyperkähler analogue of the moduli space of framed representations of that graph (see Section 5.1). Examples include the Hilbert scheme of n points in $\mathbb{C}^2$ [N4], the moduli space of instantons on an ALE space [N1], and Konno's hyperpolygon spaces [K2, HP2], which are the hyperkähler analogues of the moduli spaces of n-sided polygons in $\mathbb{R}^3$ with fixed edge lengths. Quiver varieties have received much attention from representation theorists due to the actions of various infinite-dimensional Lie algebras on the cohomology and K-theory of a quiver variety (see, for example, [N3, N5]).

Moduli spaces of bundles and connections. Narasimhan and Seshadri [NS] defined a notion of stability for a vector bundle on a Riemann surface Σ, and proved that the moduli space of stable holomorphic bundles on Σ may be identified with the moduli space of irreducible, flat, unitary connections. Atiyah and Bott presented this space as a Kähler quotient of the affine space of all connections on a fixed bundle E by the gauge group of automorphisms of E. This picture can be complexified by replacing holomorphic bundles with Higgs bundles, and unitary connections with arbitrary ones [Hi]. The correspondence between Higgs bundles, flat connections, and representations of the fundamental group is known as *nonabelian Hodge theory*, and has been studied and generalized extensively by Simpson [Si] in addition to many other authors. These constructions involve taking a quotient of an infinite dimensional affine space by an infinite dimensional group, and therefore lie it is beyond the scope of this work. Many of our techniques, however, can be applied in this context. See for example [H1, HT1, HT2].

Consider the action of the multiplicative group $\mathbb{C}^\times$ on $T^*\mathbb{C}^n$ given by scalar multiplication of the fibers. The action of the compact subgroup $S^1 \subseteq \mathbb{C}^\times$ is hamiltonian with respect to the first symplectic form ω_1, and descends to a circle action on $\mathfrak{M}$ with moment map $\Phi : \mathfrak{M} \to \mathbb{R}$, which is a Morse-Bott function. The geometry and topology associated with this action will be our main object of study. In Chapter 2 we give a detailed discussion of the construction of $\mathfrak{M}$, along with the action of $\mathbb{C}^\times$. In the case where Φ is proper, we describe a reducible subvariety $\mathfrak{L} \subseteq \mathfrak{M}$ called the *core* of $\mathfrak{M}$, onto which $\mathfrak{M}$ retracts S^1-equivariantly. The core $\mathfrak{L}$ has $\mathfrak{X}$ as one of its components, and if $\mathfrak{M}$ is smooth, then $\mathfrak{L}$ is equidimensional of dimension $\dim \mathfrak{X} = \frac{1}{2} \dim \mathfrak{M}$. In particular, the fundamental classes of the components of $\mathfrak{L}$ provide a natural basis for the top degree cohomology of $\mathfrak{M}$. This fact is exploited for hypertoric varieties in [HS], and for quiver varieties in various papers of Nakajima. Building on [HP1], this thesis is the first unified treatment of hyperkähler analogues and their cores, encompassing both hypertoric varieties and quiver varieties.

The geometry of the core of $\mathfrak{M}$ will be one of two major themes that we consider. In the case where $\mathfrak{M}$ is a hypertoric variety, each of the components of the core $\mathfrak{L}$ is itself a toric variety (Lemma 3.8), as first shown in [BD]. In section 3.2, we give an explicit description of the action of $\mathbb{C}^\times$ and the gradient flow of Φ on each piece. The case of hyperpolygon spaces is more interesting. The ordinary polygon space $\mathfrak{X}$ is the moduli space of n-sided polygons in $\mathbb{R}^3$ with fixed edge lengths. In Section 5.2, we show that the other core components are smooth, and may themselves be interpreted as moduli spaces of polygons in $\mathbb{R}^3$ satisfying certain conditions (Theorem 5.11). Thus, for the special case of hyperpolygon spaces, we have solved the following general problem:

Problem 1.1 Given any moduli space $\mathfrak{X}$ that can be constructed as a Kähler reduction (or GIT quotient) of complex affine space, is it possible to interpret the core of the hyperkähler analogue $\mathfrak{X}$ as a union of moduli spaces corresponding to other, related moduli problems?

Our second major theme will be the circle-equivariant cohomology ring of $\mathfrak{M}$. In Chapter 3 we compute the circle-equivariant cohomology ring of a hypertoric variety, and as an application compute the $\mathbb{Z}_2 = Gal(\mathbb{C}/\mathbb{R})$ equivariant cohomology ring of the complement of a complex hyperplane arrangement defined over $\mathbb{R}$. The purpose of Chapter 4 is to extend to the hyperkähler setting a theorem of Martin [Ma], which describes how to compute the cohomology ring of a Kähler quotient $X/\!\!/G$ in terms of the cohomology ring of the abelian quotient $X/\!\!/T$, where $T \subseteq G$ is a maximal torus. The main technical difficulty arises from the fact that Martin's theorem relies heavily on computing integrals, which is not possible on the noncompact hyperkähler analogues that we have defined. Our approach is to make use of the localization theorem of [AB, BV], which allows us to define an integration theory in the circle-equivariant cohomology of S^1-manifolds with compact, oriented fixed point set. This is perhaps the single most important reason for considering the circle action on a hyperkähler analogue. In Section 5.17 we combine the results of Chapters 3 and 4 to compute the equivariant cohomology ring of a hyperpolygon space, and of each of its core components.

Most of Chapter 3 (with the exception of Section 3.5) appeared first in [HP1], and Chapter 4 is a reproduction of [HP]. Chapter 5 is taken primarily from [HP2], with the exception of Section 5.3, which comes from [HP].

Chapter 2

Hyperkähler analogues

Our plan for this chapter is to provide a unified approach to the constructions of hypertoric varieties and quiver varieties, which are the two major classes of examples of hyperkähler analogues of familiar Kähler varieties that appear in the literature. In Section 2.1 we give the basic construction of the hyperkähler analogue $\mathfrak{M}$ of a Kähler quotient $\mathfrak{X} = \mathbb{C}^n /\!\!/ G$, and show that $\mathfrak{M}$ may be understood as a partial compactification of the cotangent bundle to $\mathfrak{X}$ (Proposition 2.4). In Section 2.2, we define a natural action of the group $\mathbb{C}^\times$ on $\mathfrak{M}$, which is holomorphic with respect to one of the complex structures. This action will be our main tool for studying the geometry of $\mathfrak{M}$ in future chapters. Some of this material appeared first in [HP1, §1].

2.1 Hyperkähler and holomorphic symplectic reduction

A *hyperkähler manifold* is a Riemannian manifold (M, g) along with three orthogonal, parallel complex structures, J_1, J_2, J_3, satisfying the usual quaternionic relations. These three complex structures allow us to define three symplectic forms

$$\omega_1(v, w) = g(J_1 v, w),\ \omega_2(v, w) = g(J_2 v, w),\ \omega_3(v, w) = g(J_3 v, w),$$

so that (g, J_i, ω_i) is a Kähler structure on M for $i = 1, 2, 3$. The complex-valued two-form $\omega_2 + i\omega_3$ is closed, nondegenerate, and holomorphic with respect to the complex structure J_1. Any hyperkähler manifold can therefore be considered as a *holomorphic symplectic* manifold with complex structure J_1, real symplectic form $\omega_{\mathbb{R}} := \omega_1$, and holomorphic symplectic form $\omega_{\mathbb{C}} := \omega_2 + i\omega_3$. This is the point of view that we will adopt.

We will refer to an action of G on a hyperkähler manifold M as *hyperhamiltonian* if it is hamiltonian with respect to $\omega_{\mathbb{R}}$ and holomorphic hamiltonian with respect to $\omega_{\mathbb{C}}$, with G-equivariant moment map

$$\mu_{\mathrm{HK}} := \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : M \to \mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}}.$$

The following theorem describes the *hyperkähler quotient* construction, a quaternionic analogue of the Kähler quotient.

Theorem 2.1 [HKLR] *Let M be a hyperkähler manifold equipped with a hyperhamiltonian action of a compact Lie group G, with moment maps μ_1, μ_2, μ_3. Suppose $\xi = \xi_{\mathbb{R}} \oplus \xi_{\mathbb{C}}$ is a central regular value of μ_{HK}, with G acting freely on $\mu_{HK}^{-1}(\xi)$. Then there is a unique hyperkähler structure on the hyperkähler quotient $\mathfrak{M} = M/\!\!/\!\!/\!\!/_{\xi} G := \mu_{HK}^{-1}(\xi)/G$, with associated symplectic and holomorphic symplectic forms $\omega^{\xi}_{\mathbb{R}}$ and $\omega^{\xi}_{\mathbb{C}}$, such that $\omega^{\xi}_{\mathbb{R}}$ and $\omega^{\xi}_{\mathbb{C}}$ pull back to the restrictions of $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$ to $\mu_{HK}^{-1}(\xi)$.*

For a general regular value ξ, the action of G on $\mu_{\mathrm{HK}}^{-1}(\xi)$ will not be free, but only locally free. To deal with this situation we must introduce the notion of a hyperkähler orbifold.

An *orbifold* is a topological space M locally modeled on finite quotients of euclidean space. More precisely, M is a Hausdorff topological space, equipped with an *atlas $\mathcal{U}$ of uniformizing charts*. This consists of a collection of quadruples $(\tilde{U}, \Gamma, U, \phi)$, where $\tilde{U}$ is an open subset of euclidean space, Γ is a finite group acting on $\tilde{U}$ and fixing a set of codimension at least 2, U is an open subset of M, and ϕ is a homeomorphism from $\tilde{U}/\Gamma$ to U. The sets U are required to cover M, and the quadruples must satisfy a list of compatibility conditions, as in [LT].

Given a point $p \in M$, the *orbifold group at p* is the isotropy group $\Gamma_p \subseteq \Gamma$ of a point $\tilde{p} \in \phi^{-1}(p) \subseteq \tilde{U}$ for any quadruple $(\tilde{U}, \Gamma, U, \phi)$ such that U contains p. The orbifold tangent space $T_pM = T_{\tilde{p}}\tilde{U}_{\tilde{p}}$ should be thought of not as a vector space, but rather as a representation of Γ_p (see Proposition 2.8). A differential form on an orbifold may be thought of as a collection of Γ-invariant differential forms on the open sets $\tilde{U}$, subject to certain compatibility conditions. We may define riemannian metrics, complex structures, vector bundles, Kähler structures, and hyperkähler structures on orbifolds in a similar manner.

Example 2.2 Let Z be a smooth manifold, and let G be a compact Lie group acting locally freely on Z. Then Z/G inherits the structure of an orbifold. The orbifold group of an orbit of a point $z \in Z$ is simply the stabilizer $G_z \subseteq G$. Any G-invariant tensor on Z descends to an orbifold tensor on Z/G. Any G-equivariant vector bundle on Z descends to a vector bundle on Z/G. *All of the orbifolds that we consider will be of this form* (and it is not known whether any other examples exist).

These definitions allow for a straightforward extension of Theorem 2.1 to the case where ξ is an arbitrary regular value of μ_{HK}. This implies, by the moment map condition, that G acts locally freely on $\mu_{\text{HK}}^{-1}(0)$, and that the quotient $\mu_{\text{HK}}^{-1}(0)/G$ inherits the structure of a hyperkähler orbifold.

Orbifolds are in many ways just as nice as manifolds; for example, it is possible to adapt Morse theory to the orbifold case, as in [LT], which we will use in the next section. When we say that a certain Kähler or hyperkähler quotient is an orbifold, we wish to express the opinion that it is relatively well behaved, rather than the opinion that it is nasty and singular. For this reason, we will use the positively connoted adjective $\mathbb{Q}$-*smooth* to refer to orbifolds.

We now specialize to the case where $M = T^*\mathbb{C}^n$, and the action of G on $T^*\mathbb{C}^n$ is induced from a linear action of G on $\mathbb{C}^n$ with moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^*$. Choose an identification of $\mathbb{H}^n$ with $T^*\mathbb{C}^n$ such that the complex structure J_1 on $\mathbb{H}^n$ given by right multiplication by i corresponds to the natural complex structure on $T^*\mathbb{C}^n$. Then $T^*\mathbb{C}^n$ inherits a hyperkähler structure. The real symplectic form $\omega_{\mathbb{R}}$ is given by adding the pullbacks of the standard forms on $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$, and the holomorphic symplectic form $\omega_{\mathbb{C}} = d\eta$, where η is the canonical holomorphic 1-form on $T^*\mathbb{C}^n$.

We note that G acts $\mathbb{H}$-linearly on $T^*\mathbb{C}^n \cong \mathbb{H}^n$ (where $n \times n$ matrices act on the left on the space of column vectors $\mathbb{H}^n$, and scalar multiplication by $\mathbb{H}$ is on the right). This action is hyperhamiltonian with moment map $\mu_{\text{HK}} = \mu_{\mathbb{C}} \oplus \mu_{\mathbb{R}}$, where

$$\mu_{\mathbb{R}}(z,w) = \mu(z) - \mu(w) \quad \text{and} \quad \mu_{\mathbb{C}}(z,w)(v) = w(\hat{v}_z)$$

for $w \in T_z^*\mathbb{C}^n$, $v \in \mathfrak{g}_{\mathbb{C}}$, and $\hat{v}_z$ the element of $T_z\mathbb{C}^n$ induced by v. Given a central element $\alpha \in \mathfrak{g}^*$, we refer to the hyperkähler quotient

$$\mathfrak{M} = T^*\mathbb{C}^n /\!\!/\!\!/\!\!/_{(\alpha,0)} G$$

as the *hyperkähler analogue* of the corresponding Kähler quotient

$$\mathfrak{X} = \mathbb{C}^n /\!\!/_{\alpha} G := \mu^{-1}(\alpha)/G.$$

In future sections we will often fix a parameter α and drop it from the notation.

At times it will be convenient to think of Kähler quotients in terms of geometric invariant theory, as follows. Let $G_{\mathbb{C}}$ be an algebraic group acting on an affine variety V, and let $\chi : G_{\mathbb{C}} \to \mathbb{C}^{\times}$ be a character of $G_{\mathbb{C}}$. This defines a lift of the action of $G_{\mathbb{C}}$ to the trivial line bundle $V \times \mathbb{C}$ by the formula

$$g \cdot (v, z) = (g \cdot v, \chi(g)^{-1} z).$$

The *semistable locus* V^s with respect to χ is defined to be the set of points $v \in V$ such that for $z \neq 0$, the closure of the orbit $G_{\mathbb{C}}(v, z) \subseteq V \times \mathbb{C}$ is disjoint from the zero section $V \times \{0\}$ (see [MFK] or [N4]). The *geometric invariant theory* (GIT) quotient $V /\!\!/_{\chi} G_{\mathbb{C}}$ of V by $G_{\mathbb{C}}$ at χ is an algebraic variety with underlying space $V^{ss}/\!\sim$, where $v \sim w$ if and only if the closures $\overline{G_{\mathbb{C}} v}$ and $\overline{G_{\mathbb{C}} w}$ intersect in V^{ss}. The *stable locus* V^{st} with respect to χ is the set of points $v \in V^{ss}$ such that the $G_{\mathbb{C}}$ orbit through v is closed in V^{ss}. Clearly the geometric quotient $V^{st}/G_{\mathbb{C}}$ is an open set inside of the categorical quotient $V /\!\!/_{\chi} G_{\mathbb{C}}$. The following theorem is due to Kirwan in the projective case [Ki]; our formulation of it is taken from [N4, §3] and [MFK, §8].

Theorem 2.3 *Let G be a compact Lie group acting linearly on a complex vector space V with moment map $\mu : V \to \mathfrak{g}^*$. Let $G_{\mathbb{C}}$ be the complexification of G, with its induced action on V. Let χ be a character of G, and let $d\chi$ be the associated element of 5 center of $\mathfrak{g}^*$. Then $v \in V^{ss}$ if and only if $G_{\mathbb{C}} v \cap \mu^{-1}(d\chi) \neq \emptyset$, and the inclusion $\mu^{-1}(d\chi) \subseteq V^{ss}$ induces a homeomorphism from $V /\!\!/_{d\chi} G$ to $V /\!\!/_{\chi} G_{\mathbb{C}}$. Furthermore, $d\chi$ is a regular value of μ if and only if $V^{ss} = V^{st}$.*

Given a regular value $\alpha \in \mathfrak{g}^*$, Theorem 2.3 tells us that we may interpret $V /\!\!/_{\alpha} G$ as a GIT quotient only in the case where α comes from a character of G. We note, however, that the stability and semistability conditions are unchanged when χ is replaced by a high power of itself, hence we may apply Theorem 2.3 whenever some multiple of α comes from a character. Furthermore, the GIT stability condition is locally constant as a function of χ. Hence for any central regular value $\alpha \in \mathfrak{g}^*$, we may perturb α to a "rational" point, thereby interpret $V /\!\!/_{\alpha} G$ as a GIT quotient. Accordingly, we will call an element of V stable with respect to $\alpha \in \mathfrak{g}^*$ if and only if it is stable with respect to χ for all χ such that $d\chi$ is close to a multiple of α, and we will write $V /\!\!/_{\alpha} G_{\mathbb{C}} = V^{ss}/\!\sim$.

We may also use this theorem to formulate the hyperkähler quotient construction purely in terms of algebraic geometry. Theorem 2.3 says that, for α a regular value of $\mu_{\mathbb{R}}$,

$$T^*\mathbb{C}^n /\!\!/_{\alpha} G \;\cong\; T^*\mathbb{C}^n /\!\!/_{\alpha} G_{\mathbb{C}} \;\cong\; (T^*\mathbb{C}^n)^{st}/G_{\mathbb{C}}.$$

Since $\mu_{\mathbb{C}} : T^*\mathbb{C}^n \to \mathfrak{g}^*_{\mathbb{C}}$ is equivariant, we may take its vanishing locus on both sides of the above equation, and we obtain the identity

$$T^*\mathbb{C}^n /\!\!/\!\!/\!\!/_{(\alpha,0)} G \;\cong\; \mu_{\mathbb{C}}^{-1}(0) /\!\!/_{\alpha} G_{\mathbb{C}} \;\cong\; \mu_{\mathbb{C}}^{-1}(0)^{st}/G_{\mathbb{C}}.$$

The following proposition is proven for the case where G is a torus in [BD, 7.1].

Proposition 2.4 *Suppose that α and $(\alpha, 0)$ are regular values for μ and μ_{HK}, respectively. The cotangent bundle $T^*\mathfrak{X}$ is isomorphic to an open subset of $\mathfrak{M}$, and is dense if it is nonempty.*

Proof: Let $Y = \{(z,w) \in \mu_{\mathbb{C}}^{-1}(0)^{st} \mid z \in (\mathbb{C}^n)^{st}\}$, where we ask z to be semistable with respect to α for the action of $G_{\mathbb{C}}$ on $\mathbb{C}^n$, so that $\mathfrak{X} \cong (\mathbb{C}^n)^{st}/G_{\mathbb{C}}$. Let $[z]$ denote the element of $\mathfrak{X}$ represented by z. The tangent space $T_{[z]}\mathfrak{X}$ is equal to the quotient of $T_z\mathbb{C}^n$ by the tangent space to the $G_{\mathbb{C}}$ orbit through z, hence

$$T^*_{[z]}\mathfrak{X} \cong \{w \in T^*_z\mathbb{C}^n \mid w(\hat{v}_z) = 0 \text{ for all } v \in \mathfrak{g}_{\mathbb{C}}\} = \{w \in (\mathbb{C}^n)^* \mid \mu_{\mathbb{C}}(z,w) = 0\}.$$

Then

$$T^*\mathfrak{X} \cong \{(z,w) \mid z \in (\mathbb{C}^n)^{st} \text{ and } \mu_{\mathbb{C}}(z,w) = 0\}/G_{\mathbb{C}} = Y/G_{\mathbb{C}}.$$

By the definition of semistability, Y is an open subset of $\mu_{\mathbb{C}}^{-1}(0)$, and is dense if nonempty. This completes the proof. □

Remark 2.5 We may significantly generalize the construction of hyperkähler analogues as follows. Replace $\mathbb{C}^n$ by a smooth complex variety X, equipped with an action of an algebraic group $G_{\mathbb{C}}$, an ample line bundle L, and a lift of the action to L. Then the cotangent bundle T^*X is holomorphic symplectic, and carries a natural holomorphic hamiltonian action of $G_{\mathbb{C}}$, along with a lift of this action to the pullback of L. We may then define the holomorphic symplectic analogue of the GIT quotient $\mathfrak{X} = X /\!\!/ G_{\mathbb{C}}$ to be the GIT quotient of the zero level of the holomorphic moment map in T^*X, where the semistable sets are defined by the action of $G_{\mathbb{C}}$ on L. Theorem 2.3 tells us that this agrees with our construction if $X = \mathbb{C}^n$, and Proposition 2.4 generalizes to say that the holomorphic symplectic analogue of $\mathfrak{X}$ is a partial compactification of its cotangent bundle.

The reason for relegating this definition to a remark is that when X is not equal to $\mathbb{C}^n$, its cotangent bundle T^*X may not be the best holomorphic symplectic manifold with which to replace it. For example, if X is itself a Kähler quotient, then the holomorphic symplectic analogue of X modulo the trivial group, in the sense of the previous paragraph, would simply be the cotangent bundle to X. But this would (usually) not agree with the hyperkähler analogue of X.

2.2 The $\mathbb{C}^\times$ action and the core

Consider the action of $\mathbb{C}^\times$ on $T^*\mathbb{C}^n$ given by scalar multiplication on the fibers, that is $\tau \cdot (z, w) = (z, \tau w)$. The holomorphic moment map $\mu_{\mathbb{C}} : T^*\mathbb{C}^n \to \mathfrak{g}^*_{\mathbb{C}}$ is $\mathbb{C}^\times$-equivariant with respect to the scalar action on $\mathfrak{g}^*_{\mathbb{C}}$, hence $\mathbb{C}^\times$ acts on $\mu_{\mathbb{C}}^{-1}(0)$. Linearity of the action of G on $\mathbb{C}^n$ implies that the actions of $G_{\mathbb{C}}$ and $\mathbb{C}^\times$ on $T^*\mathbb{C}^n$ commute, therefore we obtain a J_1-holomorphic action of $\mathbb{C}^\times$ on $\mathfrak{M} = \mu_{\mathbb{C}}^{-1}(0) /\!\!/ G_{\mathbb{C}}$. Note that the $\mathbb{C}^\times$ action does *not* preserve the holomorphic symplectic form or the hyperkähler structure on $\mathfrak{M}$; rather we have $\tau^*\omega_{\mathbb{C}} = \tau\omega_{\mathbb{C}}$ for $\tau \in \mathbb{C}^\times$.

If $\mathfrak{M}$ is $\mathbb{Q}$-smooth, then the action of the compact subgroup $S^1 \subseteq \mathbb{C}^\times$ is hamiltonian with respect to the real symplectic structure $\omega_{\mathbb{R}}$, with moment map $\Phi[z, w]_{\mathbb{R}} = \frac{1}{2}|w|^2$. This map is an orbifold Morse-Bott function[1] with image contained in the non-negative real numbers, and $\Phi^{-1}(0) = \mathfrak{X} \subseteq \mathfrak{M}$.

Proposition 2.6 *If the original moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^*$ is proper, then so is $\Phi : \mathfrak{M} \to \mathbb{R}$.*

Proof: We must show that $\Phi^{-1}[0, R]$ is compact for any $R \in \mathbb{R}$. We have

$$\Phi^{-1}[0, R] = \{(z, w) \mid \mu_{\mathbb{R}}(z, w) = \alpha,\ \mu_{\mathbb{C}}(z, w) = 0,\ \Phi(z, w) \leq R\}\big/ G$$

and G is compact, hence it is sufficient to show that $\{(z, w) \mid \mu_{\mathbb{R}}(z, w) = \alpha,\ \Phi(z, w) \leq R\}$

[1] For a detailed discussion of hamiltonian circle actions and Morse theory on orbifolds, see [LT].

is compact. Since $\mu_{\mathbb{R}}(z,w) = \mu(z) - \mu(w)$, this set is a closed subset of

$$\mu^{-1}\left\{\alpha + \mu(w) \,\middle|\, \frac{1}{2}|w|^2 \leq R\right\} \times \left\{w \,\middle|\, \frac{1}{2}|w|^2 \leq R\right\},$$

which is compact by the properness of μ. □

Remark 2.7 In the case where G is abelian and $\mathfrak{X}$ is a nonempty toric variety, properness of μ (and therefore of Φ) is equivalent to compactness of $\mathfrak{X}$.

Suppose that $\mathfrak{M}$ is $\mathbb{Q}$-smooth and Φ is proper. We define the *core* $\mathfrak{L} \subseteq \mathfrak{M}$ to be the union of those $\mathbb{C}^\times$ orbits whose closures are compact. Properness of Φ implies that $\lim\limits_{\tau\to 0} \tau \cdot p$ exists for all $p \in \mathfrak{M}$, hence we may write

$$\mathfrak{L} = \{p \in \mathfrak{M} \mid \lim_{\tau\to\infty} \tau \cdot p \text{ exists}\}.$$

For F a connected component of $\mathfrak{M}^{S^1} = \mathfrak{M}^{\mathbb{C}^\times}$, let U_F be the closure of the set of points $p \in \mathfrak{M}$ such that $\lim\limits_{\tau\to\infty} \tau \cdot p \in F$. In Morse-theoretic language, $U(F)$ is the closure of the unstable orbifold of the critical set F. We may then write $\mathfrak{L}$ as a finite union of irreducible, compact varieties as follows:

$$\mathfrak{L} = \bigcup_{F\subseteq \mathfrak{M}^{\mathbb{C}^\times}} U_F.$$

Proposition 2.8 *The core of $\mathfrak{M}$ has the following properties:*

1. *$\mathfrak{L}$ is an S^1-equivariant deformation retract of $\mathfrak{M}$*

2. *U_F is isotropic with respect to $\omega_{\mathbb{C}}$*

3. *If $\mathfrak{M}$ is smooth at F, then* $\dim U_F = \frac{1}{2}\dim \mathfrak{M}$.

Proof: Let $f : \mathfrak{M} \to [0,1]$ be a smooth, S^1-invariant function with $f^{-1}(0) = \mathfrak{L}$. For all $p \in \mathfrak{M}$ and $t \geq 0$, let $\rho_t(p) = e^{f(p)t} \cdot p$. This defines a smooth family of S^1-equivariant maps $\rho_t : \mathfrak{M} \to \mathfrak{M}$, fixing $\mathfrak{L}$, with $\rho_0 = \mathrm{id}$. The limit $\rho_\infty = \lim\limits_{t\to\infty} \rho_t$ is a well-defined smooth map from $\mathfrak{M}$ to $\mathfrak{L}$, hence (1) is proved.

Suppose that $\mathfrak{M}$ is smooth at F and consider a point $p \in F$. Since p is a fixed point, S^1 acts on $T_p\mathfrak{M}$, and we may write

$$T_p\mathfrak{M} = \bigoplus_{s\in\mathbb{Z}} H_s,$$

where H_s is the s weight space for the circle action. Since $\tau^*\omega_{\mathbb{C}} = \tau\omega_{\mathbb{C}}$ and $\omega_{\mathbb{C}}$ is a nondegenerate bilinear form on $T_p\mathfrak{M}$, $\omega_{\mathbb{C}}$ restricts to a perfect pairing $H_s \times H_{1-s} \to \mathbb{C}$ for all $s \in \mathbb{Z}$. In particular,

$$T_pU_F = \bigoplus_{s\leq 0} H_s$$

is a maximal isotropic subspace of $T_p\mathfrak{M}$, thus proving (2) and (3).

Now suppose that $\mathfrak{M}$ is only $\mathbb{Q}$-smooth at F, and let Γ_p be the orbifold group at p. A circle action on the orbifold tangent space $T_p\mathfrak{M}$ is an action of a group $\widehat{\Gamma}_p$, where $\widehat{\Gamma}_p$ is an extension of S^1 by Γ_p. Let $\widehat{\Gamma}_p^\circ$ be the connected component of the identity in $\widehat{\Gamma}_p$. Then $\widehat{\Gamma}_p^\circ$ is itself isomorphic to a circle, and maps to the original circle S^1 with some degree $d \geq 1$. We now decompose $T_p\mathfrak{M} = \bigoplus H_s$ into $\widehat{\Gamma}_p^\circ$ weight spaces. Again $\omega_{\mathbb{C}}$ is nondegenerate on $T_p\mathfrak{M}$, but now $\hat{\tau}^*\omega_{\mathbb{C}} = \hat{\tau}^d\omega_{\mathbb{C}}$ for $\hat{\tau} \in \widehat{\Gamma}_p^\circ \cong S^1$, hence $\omega_{\mathbb{C}}$ restricts to a perfect pairing $H_s \times H_{d-s} \to \mathbb{C}$ for all $s \in \mathbb{Z}$. It follows that $T_pU_F = \bigoplus_{s\leq 0} H_s$ is isotropic (though not necessarily maximally isotropic[2]) with respect to $\omega_{\mathbb{C}}$. This completes the proof of (2) in the orbifold case. □

Remark 2.9 Proposition 2.8 provides a new way to understand Proposition 2.4 in the case where $\mathfrak{M}$ is $\mathbb{Q}$-smooth and Φ is proper. The Kähler quotient $\mathfrak{X}$ is an $\omega_{\mathbb{C}}$-lagrangian suborbifold of $\mathfrak{M}$, hence $\omega_{\mathbb{C}}$ identifies the normal bundle to $\mathfrak{X}$ in $\mathfrak{M}$ with the cotangent bundle of $\mathfrak{X}$. The Proposition 2.4 follows from the fact that the normal bundle to $\mathfrak{X}$ in $\mathfrak{M}$ can be identified with the dense open set of points in $\mathfrak{M}$ that flow down to $\mathfrak{X} = \Phi^{-1}(0)$ along the gradient of Φ. This also demonstrates that the $\mathbb{C}^\times$ action on $\mathfrak{M}$ restricts to scalar multiplication on the fibers of $T^*\mathfrak{X}$.

Given a space M equipped with the action of a group G, we say that M is *equivariantly formal* if the equivariant cohomology ring $H^*_G(M)$ is a free module over $H^*_G(pt)$. We end the section with the statement of a fundamental theorem which we will use repeatedly throughout the paper. Theorem 2.10 is proven in the compact case in [Ki], and the proof goes through in the noncompact case as long as Φ is proper (see, for example, [H1, §2.2] or [TW, 4.2]).

Proposition 2.10 *Let M be a symplectic orbifold with a hamiltonian circle action such that the moment map $\Phi : M \to \mathbb{R}$ is proper and bounded below, and has finitely many critical values. Then $H^*_{S^1}(M)$ is a free module over $H^*_{S^1}(pt)$. Moreover, if the action of*

[2] See Example 3.14.

S^1 commutes with the action of another torus T, then $H^*_{T\times S^1}(M)$ is a free module over $H^*_{S^1}(pt)$.

Chapter 3

Hypertoric varieties

In this chapter we give a detailed analysis of the construction described in Chapter 2 in the special case where the group G is abelian. In this case the Kähler quotient $\mathfrak{X}$ is called a *toric variety*, and its hyperkähler analogue $\mathfrak{M}$ is called a *hypertoric variety*.[1] These latter spaces were first studied systematically in [BD]; other references include [K1], [HS], and [HP1].

Just as there is a strong relationship between the geometry of toric varieties and the combinatorics of polytopes (see, for example, [St]), the geometry of hypertoric varieties interacts with the combinatorics of real hyperplane arrangements. Hausel and Sturmfels [HS] gave an interpretation of the cohomology ring of a hypertoric variety as the Stanley-Reisner ring of the matroid associated to the corresponding arrangement of hyperplanes (Theorem 3.16). In Section 3.3 we interpret the S^1-equivariant cohomology ring as an invariant of the oriented matroid, a richer combinatorial structure that can be associated to a hyperplane arrangement that is defined over the real numbers (Theorem 3.18 and Remark 3.19). This result is applied in Section 3.4 to obtain a combinatorial presentation of the $\mathbb{Z}_2$-equivariant cohomology ring of the complement of an arrangment of complex hyperplanes defined over $\mathbb{R}$, thus enhancing the classical result of Orlik and Solomon [OS]. In Section 3.5, we use the cogenerator approach of [HS] to explore an aspect of the relationship between the cohomology rings of toric and hypertoric varieties. Most of the material presented here, with the exception of the entirety of Section 3.5, has been taken from [HP1] in a modified form.

[1] Also known as a *toric hyperkähler variety*.

3.1 Hypertoric varieties and hyperplane arrangements

Let $\mathfrak{t}^n$ and $\mathfrak{t}^d$ be real vector spaces of dimensions n and d, respectively, with integer lattices $\mathfrak{t}^n_{\mathbb{Z}} \subseteq \mathfrak{t}^n$ and $\mathfrak{t}^d_{\mathbb{Z}} \subseteq \mathfrak{t}^d$. Let $\{x_1, \ldots, x_n\}$ be an integer basis for $\mathfrak{t}^n_{\mathbb{Z}}$, and let $\{\partial_1, \ldots, \partial_n\}$ be the dual basis for the dual lattice $(\mathfrak{t}^n_{\mathbb{Z}})^* \subseteq (\mathfrak{t}^n)^*$. Suppose given n nonzero integer vectors $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathfrak{t}^d_{\mathbb{Z}}$ that span $\mathfrak{t}^d$ over the real numbers.[2] Define $\pi : \mathfrak{t}^n \to \mathfrak{t}^d$ by $\pi(x_i) = a_i$, and let $\mathfrak{t}^k$ be the kernel of π, so that we have an exact sequence

$$0 \longrightarrow \mathfrak{t}^k \xrightarrow{\iota} \mathfrak{t}^n \xrightarrow{\pi} \mathfrak{t}^d \longrightarrow 0.$$

This sequence exponentiates to an exact sequence of abelian groups

$$0 \longrightarrow T^k \xrightarrow{\iota} T^n \xrightarrow{\pi} T^d \longrightarrow 0,$$

where

$$T^n = \mathfrak{t}^n/\mathfrak{t}^n_{\mathbb{Z}}, \;\; T^d = \mathfrak{t}^d/\mathfrak{t}^d_{\mathbb{Z}}, \;\; \text{and } T^k = \operatorname{Ker}(\pi : T^n \to T^d).$$

Thus T^k is a compact abelian group with Lie algebra $\mathfrak{t}^k$, which is connected if and only if the vectors $\{a_i\}$ span the lattice $\mathfrak{t}^d_{\mathbb{Z}}$ over the integers. It is clear that every closed subgroup of T^n arises in this way.

The restriction to T^k of the standard action of T^n on $T^*\mathbb{C}^n$ is hyperhamiltonian with hyperkähler moment map

$$\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : T^*\mathbb{C}^n \to (\mathfrak{t}^k)^* \oplus (\mathfrak{t}^k_{\mathbb{C}})^*,$$

where

$$\mu_{\mathbb{R}}(z,w) = \iota^* \left(\frac{1}{2} \sum_{i=1}^{n} (|z_i|^2 - |w_i|^2)\partial_i \right) \quad \text{and} \quad \mu_{\mathbb{C}}(z,w) = \iota^*_{\mathbb{C}} \left(\sum_{i=1}^{n} (z_i w_i)\partial_i \right).$$

Given an element $\alpha \in (\mathfrak{t}^k)^*$ with lift $r = (r_1, \ldots, r_n) \in (\mathfrak{t}^n)^*$, the Kähler quotient

$$\mathfrak{X} = \mathbb{C}^n /\!\!/ T^k = \mu^{-1}(\alpha)/T^k$$

is called a *toric variety*, and its hyperkähler analogue

$$\mathfrak{M} = T^*\mathbb{C}^n /\!\!/\!\!/\!\!/ T^k = \left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right)\Big/ T^k$$

[2] In each of [BD, K1, HS, HP1], some additional assumption is placed on the vectors $\{a_i\}$. Sometimes they are assumed to be primitive, and sometimes they are assumed to generate the lattice $\mathfrak{t}^d_{\mathbb{Z}}$ over the integers. Here we make neither assumption.

is called a *hypertoric variety.* Both of these spaces admit an effective residual action of the torus $T^d = T^n/T^k$ which is hamiltonian in the case of $\mathfrak{X}$, and hyperhamiltonian in the case of $\mathfrak{M}$, with hyperkähler moment map

$$\begin{aligned}\bar{\mu}_{\mathbb{R}}[z,w]_{\mathbb{R}} \oplus \bar{\mu}_{\mathbb{C}}[z,w]_{\mathbb{R}} &= \frac{1}{2}\sum_{i=1}^{n}(|z_i|^2 - |w_i|^2 - r_i)\,\partial_i \;\oplus\; \sum_{i=1}^{n}(z_i w_i)\,\partial_i \\ &\in \operatorname{Ker}(\iota^*) \oplus \operatorname{Ker}(\iota_{\mathbb{C}}^*) = (\mathfrak{t}^d)^* \oplus (\mathfrak{t}^d_{\mathbb{C}})^*.\end{aligned}$$

In fact, this property may be used to give intrinsic definitions of toric and hypertoric varieties in certain categories, as demonstrated by the following two theorems.

Theorem 3.1 [De, LT] *Any connected symplectic orbifold of real dimension* $2d$ *which admits an effective, hamiltonian* T^d *action with proper moment map is* T^d*-equivariantly symplectomorphic to a toric variety.*

Theorem 3.2 [Bi] *Any complete, connected, hyperkähler manifold of real dimension* $4d$ *which admits an effective, hyperhamiltonian* T^d *action is* T^d*-equivariantly diffeomorphic, and Taub-NUT deformation equivalent, to a hypertoric variety.*

The data that were used to construct $\mathfrak{X}$ and $\mathfrak{M}$ consist of a collection of nonzero vectors $a_i \in \mathfrak{t}^d_{\mathbb{Z}}$ and an element $\alpha \in (\mathfrak{t}^k)^*$. It is convenient to encode in terms of an arrangement of affine hyperplanes in $(\mathfrak{t}^d)^*$ with some additional structure. A *rational, weighted, cooriented, affine hyperplane* $H \subseteq (\mathfrak{t}^d)^*$ is an affine hyperplane along with a choice of nonzero normal vector $a \in \mathfrak{t}^d_{\mathbb{Z}}$. The word rational refers to integrality of a, and weighted means that a is not required to be primitive. Consider the rational, weighted, cooriented hyperplane

$$H_i = \{v \in (\mathfrak{t}^d)^* \mid v \cdot a_i + r_i = 0\}$$

with normal vector $a_i \in \mathfrak{t}^d_{\mathbb{Z}}$, along with the two half-spaces

$$F_i = \{v \in (\mathfrak{t}^d)^* \mid v \cdot a_i + r_i \geq 0\} \quad \text{and} \quad G_i = \{v \in (\mathfrak{t}^d)^* \mid v \cdot a_i + r_i \leq 0\}. \tag{3.1}$$

Let

$$\Delta = \bigcap_{i=1}^{n} F_i = \{v \mid v \cdot a_i + r_i \geq 0 \text{ for all } i \leq n\}$$

be the (possibly empty) weighted polyhedron in $(\mathfrak{t}^d)^*$ defined by the weighted, cooriented, affine, hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$. Choosing a different lift r' of α corresponds combinatorially to translating $\mathcal{A}$ inside of $(\mathfrak{t}^d)^*$, and geometrically to shifting the

Kähler and hyperkähler moment maps for the residual T^d actions by $r' - r \in \operatorname{Ker} \iota^* = (\mathfrak{t}^d)^*$. Our picture-drawing convention will be to encode the coorientations of the hyperplanes by shading Δ, as in Figure 3.1. In every example that we consider, all hyperplanes will have weight 1; in other words we will choose the primitive integer normal vector inducing the indicated coorientation.

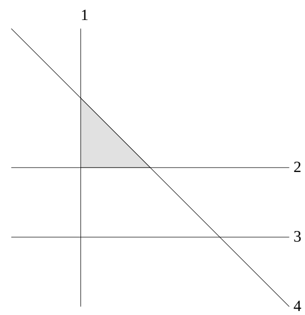

Figure 3.1: A cooriented arrangement representing a toric variety of complex dimension 2, or a hypertoric variety of complex dimension 4, obtained from an action of T^2 on $\mathbb{C}^4$.

We call the arrangement $\mathcal{A}$ *simple* if every subset of m hyperplanes with nonempty intersection intersects in codimension m. We call $\mathcal{A}$ *smooth* if every collection of d linearly independent vectors $\{a_{i_1}, \ldots, a_{i_d}\}$ spans $(\mathfrak{t}^d)^*$. An element $r \in (\mathfrak{t}^n)^*$ or $\alpha \in (\mathfrak{t}^k)^*$ will be called simple if the corresponding arrangement $\mathcal{A}$ is simple.

Theorem 3.3 [BD, 3.2,3.3] *The hypertoric variety $\mathfrak{M}$ is $\mathbb{Q}$-smooth if and only if $\mathcal{A}$ is simple, and smooth if and only if $\mathcal{A}$ is smooth.*

Let us pause to point out the different ways in which $\mathfrak{X}$ and $\mathfrak{M}$ depend on the arrangement $\mathcal{A}$. The toric variety $\mathfrak{X}$ is in fact determined by the weighted polyhedron Δ [LT], and is therefore oblivious to any hyperplane H_i such that Δ is contained in the interior of F_i. Thus the toric variety corresponding to Figure 3.1 is $\mathbb{C}P^2$, the toric variety associated to a triangle. This is not the case for $\mathfrak{M}$; we will see, in fact, that the hypertoric variety of Figure 3.1 is topologically distinct from the one that we would obtain by deleting the third hyperplane. For this reason, it is slightly misleading to call $\mathfrak{M}$ the hyperkähler analogue of $\mathfrak{X}$; more precisely, it is the hyperkähler analogue of *a given presentation* of $\mathfrak{X}$ as a Kähler quotient of a complex vector space.

Just as the toric variety $\mathfrak{X}$ fails to retain all of the data of the arrangement $\mathcal{A}$, there is some data that goes unnoticed by the hypertoric variety $\mathfrak{M}$, as evidenced by the two following results.

Lemma 3.4 *The hypertoric variety $\mathfrak{M}$ is independent, up to T^d-equivariant diffeomorphism,*[3] *of the choice of a simple element $\alpha \in (\mathfrak{t}^k)^*$.*

Lemma 3.5 *The hypertoric variety $\mathfrak{M}$ is independent, up to T^d-equivariant isometry, of the coorientations of the hyperplanes $\{H_i\}$.*

Proof of 3.4: The set of nonregular values for $\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$ has codimension 3 inside of $(\mathfrak{t}^k)^* \oplus (\mathfrak{t}^k_{\mathbb{C}})^*$ [BD], hence we may choose a path connecting the two regular values $(\alpha, 0)$ and $(\alpha', 0)$ for any simple $\alpha, \alpha' \in (\mathfrak{t}^k)^*$, and this path is unique up to homotopy. Since the moment map $\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$ is not proper, we must take some care in showing that two fibers are diffeomorphic. To this end, we note that the norm-square function $\psi(z, w) = \|z\|^2 + \|w\|^2$ is T^n-invariant and proper on $T^*\mathbb{C}^n$. Let $(T^*\mathbb{C}^n)_{reg}$ denote the open submanifold of $T^*\mathbb{C}^n$ consisting of the preimages of the regular values of $\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$. By a direct computation, it is easy to see that the kernels of $d\psi$ and $d\mu_{\mathbb{R}} \oplus d\mu_{\mathbb{C}}$ intersect transversely at any point $p \in (T^*\mathbb{C}^n)_{reg}$. Using the T^n-invariant hyperkähler metric on $T^*\mathbb{C}^n$, we define an Ehresmann connection on $(T^*\mathbb{C}^n)_{reg}$ with respect to $\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$ such that the horizontal subspaces are contained in the kernel of $d\psi$.

This connection allows us to lift a path connecting the two regular values to a horizontal vector field on its preimage in $(T^*\mathbb{C}^n)_{reg}$. Since the horizontal subspaces are tangent to the kernel of $d\psi$, the flow preserves level sets of ψ. Note that the function

$$\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} \oplus \psi : T^*\mathbb{C}^n \to (\mathfrak{t}^k)^* \oplus (\mathfrak{t}^k_{\mathbb{C}})^* \oplus \mathbb{R}$$

is proper. By a theorem of Ehresmann [BJ, 8.12], the properness of this map implies that the flow of this vector field exists for all time, and identifies the inverse image of $(\alpha, 0)$ with that of $(\alpha', 0)$. Since the metric, ψ, and $\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$ are all T^n-invariant, the Ehresmann connection is also T^n-invariant, therefore the diffeomorphism identifying the fibers is T^n-equivariant, and the reduced spaces are T^d-equivariantly diffeomorphic. □

Proof of 3.5: It suffices to consider the case when we change the orientation of a single hyperplane within the arrangement. Changing the coorientation of a hyperplane H_m is equivalent to defining a new map $\pi' : \mathfrak{t}^n \to \mathfrak{t}^d$, with $\pi'(x_i) = a_i$ for $i \neq m$, and $\pi'(x_m) = -a_m$.

[3] Bielawski and Dancer [BD] prove a weaker version of this statement, involving the (nonequivariant) homeomorphism type of $\mathfrak{M}$.

This map exponentiates to a map $\pi' : T^n \to T^d$, with $\mathrm{Ker}(\pi')$ conjugate to $\mathrm{Ker}(\pi)$ inside of $\mathrm{GL}_n(\mathbb{H})$ (the group of quaternion-linear automorphisms of $T^*\mathbb{C}^n \cong \mathbb{H}^n$) by the element $(1,\ldots,1,j,1,\ldots,1) \in \mathrm{GL}_1(\mathbb{H})^n \subseteq \mathrm{GL}_n(\mathbb{H})$, where the j appears in the m^{th} slot. Hence the hyperkähler quotient by $\mathrm{Ker}(\pi')$ is isomorphic to the hyperkähler quotient by $\mathrm{Ker}(\pi)$. $\square$

Example 3.6 The three cooriented arrangements of Figure 2 all specify the same hyperkähler variety $\mathfrak{M}$ up to equivariant diffeomorphism. The first has $\mathfrak{X} \cong \widetilde{\mathbb{C}P^2}$ (the blow-up of $\mathbb{C}P^2$ at a point), and the second and the third have $\mathfrak{X} \cong \mathbb{C}P^2$. Note that if we reversed the coorientation of H_3 in Figure 2(a) or 2(c), then we would get a noncompact $\mathfrak{X} \cong \widetilde{\mathbb{C}^2}$. If we reversed the coorientation of H_3 in Figure 2(b), then $\mathfrak{X}$ would be empty, but the topology of $\mathfrak{M}$ would not change.

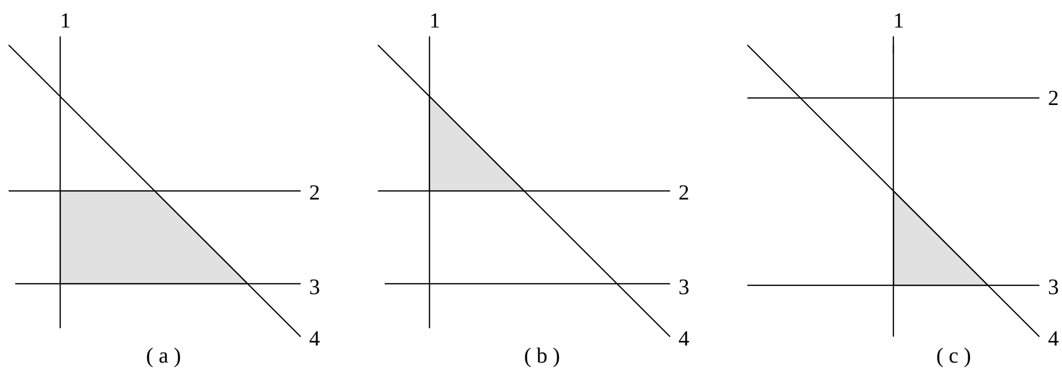

Figure 3.2: Three arrangements related by reversing coorientations and translating hyperplanes.

Our purpose is to study not just the geometry of $\mathfrak{M}$, but the geometry of $\mathfrak{M}$ along with the hamiltonian circle action defined in Section 2.2. In order to define this action, we used the fact that we were reducing at a regular value of the form $(\alpha, 0) \in (\mathfrak{t}^d)^* \oplus (\mathfrak{t}^d_{\mathbb{C}})^*$. Although the set of regular values of $\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$ is simply connected, the set of regular values of the form $(\alpha, 0)$ is disconnected, therefore the diffeomorphism of Lemma 3.4 is not circle-equivariant. Furthermore, left multiplication by the diagonal matrix $(1,\ldots,1,j,1,\ldots,1) \in \mathrm{GL}_n(\mathbb{H})$ is not an S^1-equivariant automorphism of $T^*\mathbb{C}^n \cong \mathbb{H}^n$, therefore the topological type of $\mathfrak{M}$ along with a S^1 action may depend nontrivially on both the locations and the coorientations of the hyperplanes in $\mathcal{A}$. Indeed it must, because we can recover $\mathfrak{X}$ from $\mathfrak{M}$ by taking the minimum $\Phi^{-1}(0)$ of the moment map $\Phi : \mathfrak{M} \to \mathbb{R}$, and we know that $\mathfrak{X}$ depends in an essential way on the combinatorial type of the polyhedron Δ. In this sense, the structure of a hypertoric variety $\mathfrak{M}$ along with a hamiltonian circle action may be regarded as the universal geometric object associated to $\mathcal{A}$ from which both $\mathfrak{M}$ and $\mathfrak{X}$ can be recovered.

3.2 Geometry of the core

In this section we give a combinatorial description of the fixed point set $\mathfrak{M}^{\mathbb{C}^\times} = \mathfrak{M}^{S^1}$ and the core $\mathfrak{L}$ of a $\mathbb{Q}$-smooth hypertoric variety $\mathfrak{M}$. We will assume that Φ is proper. (If Δ is nonempty, this is equivalent to asking that Δ be bounded, or that $\mathfrak{X}$ be compact.) First, we note that the holomorphic moment map $\bar{\mu}_{\mathbb{C}} : \mathfrak{M} \to (\mathfrak{t}^d_{\mathbb{C}})^*$ is $\mathbb{C}^\times$-equivariant with respect to the scalar action on $(\mathfrak{t}^d_{\mathbb{C}})^*$, hence both $\mathfrak{M}^{\mathbb{C}^\times}$ and $\mathfrak{L}$ will be contained in

$$\mathcal{E} = \bar{\mu}_{\mathbb{C}}^{-1}(0) = \Big\{[z,w] \in \mathfrak{M} \,\Big|\, z_i w_i = 0 \text{ for all } i\Big\},$$

which we call the *extended core* of $\mathfrak{M}$. It is clear from the defining equations that the restriction of $\bar{\mu}_{\mathbb{R}}$ from $\mathcal{E}$ to $(\mathfrak{t}^d)^*$ is surjective. The extended core naturally breaks into components

$$\mathcal{E}_A = \Big\{[z,w] \in \mathfrak{M} \,\Big|\, w_i = 0 \text{ for all } i \in A \text{ and } z_i = 0 \text{ for all } i \in A^c\Big\},$$

indexed by subsets $A \subseteq \{1, \ldots, n\}$. When $A = \emptyset$, $\mathcal{E}_A = \mathfrak{X} \subseteq \mathfrak{M}$. More generally, the variety $\mathcal{E}_A \subseteq \mathfrak{M}$ is a d-dimensional Kähler subvariety of $\mathfrak{M}$ with an effective hamiltonian T^d-action, and is therefore itself a toric variety. (It is the Kähler quotient by T^k of an n-dimensional coordinate subspace of $T^*\mathbb{C}^n$, contained in $\mu_{\mathbb{C}}^{-1}(0)$.) The hyperplanes $\{H_i\}$ divide $(\mathfrak{t}^d)^*$ into a union of closed, (possibly empty) convex polyhedra

$$\Delta_A = \bigcap_{i\in A} F_i \;\cap\; \bigcap_{i\in A^c} G_i.$$

Lemma 3.7 *If $w_i = 0$, then $\bar{\mu}_{\mathbb{R}}[z,w]_{\mathbb{R}} \in F_i$. If $z_i = 0$, then $\bar{\mu}_{\mathbb{R}}[z,w]_{\mathbb{R}} \in G_i$.*

Proof: We have

$$\bar{\mu}_{\mathbb{R}}[z,w]_{\mathbb{R}} \cdot a_i + r_i = \mu_{\mathbb{R}}(z,w) \cdot x_i = \frac{1}{2}\Big(|z_i|^2 - |w_i|^2\Big),$$

hence the statement follows from Equation (3.1). □

Lemma 3.8 [BD] *The core component $\mathcal{E}_A$ is isomorphic to the toric variety corresponding to the weighted polytope Δ_A.*

Proof: Lemma 3.7 tells us that $\bar{\mu}_{\mathbb{R}}(\mathcal{E}_A) \subseteq \Delta_A$, and surjectivity of $\bar{\mu}_{\mathbb{R}}|_{\mathcal{E}}$ says that this inclusion is an equality. The lemma then follows from the classification theorems of [De, LT]. $\square$

Although $\mathbb{C}^\times$ does not act on $\mathfrak{M}$ as a subtorus of $T^d_{\mathbb{C}}$, we show below that when restricted to any single component $\mathcal{E}_A$ of the extended core, $\mathbb{C}^\times$ *does* act as a subtorus of $T^d_{\mathbb{C}}$, with the subtorus depending combinatorially on A. This will allow us to give a combinatorial analysis of the gradient flow of Φ on the extended core.

Suppose that $[z,w] \in \mathcal{E}_A$. Then for $\tau \in \mathbb{C}^\times$,

$$\tau[z,w] = [z, \tau w] = [\tau_1 z_1, \ldots \tau_n z_n, \tau_1^{-1} w_1, \ldots \tau_n^{-1} w_n], \text{ where } \tau_i = \begin{cases} \tau^{-1} & \text{if } i \in A, \\ 1 & \text{if } i \notin A. \end{cases}$$

In other words, the $\mathbb{C}^\times$ action on $\mathcal{E}_A$ is given by the one dimensional subtorus $(\tau_1, \ldots, \tau_n)$ of the original torus $T^n_{\mathbb{C}^\times}$, hence the moment map $\Phi|_{\mathcal{E}_A}$ for the action of $S^1 \subseteq \mathbb{C}^\times$ is given by

$$\Phi[z,w] = \left\langle \mu_{\mathbb{R}}[z,w], \sum_{i \in A} a_i \right\rangle.$$

This formula allows us to compute the fixed points of the circle action. For any subset $B \subseteq \{1, \ldots, n\}$, let $\mathcal{E}_A^B$ be the toric subvariety of $\mathcal{E}_A$ defined by the conditions $z_i = w_i = 0$ for all $i \in B$. Geometrically, $\mathcal{E}_A^B$ is defined by the (possibly empty) intersection of the hyperplanes $\{H_i \mid i \in B\}$ with Δ_A.

Proposition 3.9 *The fixed point set of the action of S^1 on $\mathcal{E}_A$ is the union of those toric subvarieties $\mathcal{E}_A^B$ such that $\sum_{i \in A} a_i \in \mathfrak{t}^d_B := \mathrm{Span}_{j \in B}\, a_j$.*

Proof: The moment map $\Phi|_{\mathcal{E}_A^B}$ will be constant if and only if $\sum_{i \in A} a_i$ is perpendicular to the face $\Phi(\mathcal{E}_A^B)$, i.e. if $\sum_{i \in A} a_i$ lies in the kernel of the projection $\mathfrak{t}^d \twoheadrightarrow \mathfrak{t}^d/\mathfrak{t}^d_B$. $\square$

Corollary 3.10 *Every vertex $v \in (\mathfrak{t}^d)^*$ of the polyhedral complex $|\mathcal{A}|$ given by our arrangement is the image of an S^1-fixed point in $\mathfrak{M}$. Every component of $\mathfrak{M}^{S^1}$ maps to a face of $|\mathcal{A}|$.*

Proposition 3.11 *The core $\mathfrak{L}$ of $\mathfrak{M}$ is equal to the union of those subvarieties $\mathcal{E}_A$ such that Δ_A is bounded.*

Proof: Because $\mathbb{C}^\times$ acts on $\mathcal{E}_A$ as a subtorus of the complex torus $T^d_{\mathbb{C}}$, the set

$$\{p \in \mathcal{E}_A \mid \lim_{\tau\to\infty} \tau \cdot p \text{ exists}\}$$

is a (possibly reducible) toric subvariety of $\mathcal{E}_A$, i.e. a union of subvarieties of the form $\mathcal{E}^B_A$. Fix a subset $B \subseteq \{1, \ldots, n\}$. The variety $\mathcal{E}^B_A$ is stable under the $\mathbb{C}^\times$ action, hence if $\mathcal{E}^B_A$ is compact, then $\mathcal{E}^B_A \subseteq \mathfrak{L}$. On the other hand if $\mathcal{E}^B_A$ is noncompact, then properness of Φ precludes it from being part of the core, hence

$$\mathfrak{L} = \{p \in \mathcal{E} \mid \Phi(p) \text{ lies on a bounded face of } |\mathcal{A}|\}.$$

By [HS, 6.7], the bounded complex of $|\mathcal{A}|$ has pure dimension d, and is therefore equal to the union of those polyhedra Δ_A that are bounded. □

Corollary 3.12 *There is a natural injection from the set of bounded regions $\{\Delta_A \mid A \in I\}$ to the set of connected components of $\mathfrak{M}^{\mathbb{C}^\times}$. If $\mathcal{A}$ is smooth, this map is a bijection.*

Proof: To each $A \in I$, we associate the fixed subvariety $\mathcal{E}^B_A$ corresponding to the face of Δ_A on which the linear functional $\sum_{i\in A} a_i$ is minimized, so that $\mathcal{E}_A = U(\mathcal{E}^B_A)$. Proposition 2.8 (3) tells us that if $\mathcal{A}$ is smooth and F is a component of $\mathfrak{M}^{\mathbb{C}^\times}$, then we will have $U(F) = \mathcal{E}_A$ for some $A \subseteq \{1, \ldots, n\}$. □

Example 3.13 In Figure 3, representing a reduction of $T^*\mathbb{C}^5$ by T^3, we choose a metric on $(\mathfrak{t}^2)^*$ in order to draw the linear functional $\sum_{i\in A} a_i$ as a vector in each region Δ_A. We see that $\mathfrak{M}^{S^1}$ has three components, one of them $X \cong \widetilde{\mathbb{C}P}^2$, one of them a projective line with another $\widetilde{\mathbb{C}P}^2$ as its associated core component, and one of them a point with core component $\mathbb{C}P^2$.

Example 3.14 The hypertoric variety represented by Figure 4 has a fixed point set with four connected components (three points and a $\mathbb{C}P^2$), but only three components in its core. This phenomenon can be blamed on the orbifold point p represented by the intersection of H_3 and H_4, which has only a one-dimensional unstable orbifold (to its northwest). In other words, this example illustrates the necessity of the smoothness assumption to obtain a bijection in Corollary 3.12.

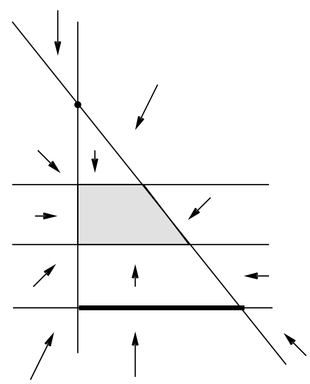

Figure 3.3: The gradient flow of $\Phi : \mathfrak{M} \to \mathbb{R}$.

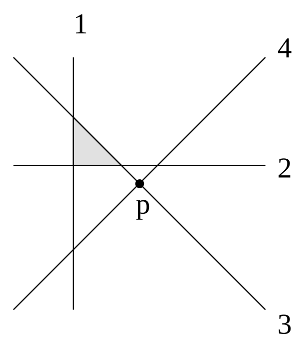

Figure 3.4: A singular example.

3.3 Cohomology rings

In this section we compute the S^1 and $T^d \times S^1$-equivariant cohomology rings of of a $\mathbb{Q}$-smooth hypertoric variety $\mathfrak{M}$, extending the computations of the ordinary and T^d-equivariant rings given in [K1] and [HS]. For the sake of simplicity, and with an eye toward the applications in Chapter 4, we will restrict our attention to the case where Φ is proper (see Remark 2.7). This assumption will be necessary for the application of Proposition 2.10 and the proof of Theorem 3.24.

Remark 3.15 Because we wish to treat the smooth and $\mathbb{Q}$-smooth cases simultaneously, we will work with cohomology over the rational numbers. We note, however, that Konno proves Theorem 3.16 over the integers in the smooth case, and therefore our Theorem 3.18 holds over the integers when $\mathcal{A}$ is smooth. This fact will be significant in Section 3.4, when we will need to reduce our coefficients modulo 2.

Consider the hyperkähler Kirwan maps

$$\kappa_{T^d} : H^*_{T^n}(T^*\mathbb{C}^n) \to H^*_{T^d}(\mathfrak{M}) \quad \text{and} \quad \kappa : H^*_{T^k}(T^*\mathbb{C}^n) \to H^*(\mathfrak{M})$$

induced by the T^n-equivariant inclusion of $\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)$ into $T^*\mathbb{C}^n$. Because the vector space $T^*\mathbb{C}^n$ is T^n-equivariantly contractible, we have

$$H^*_{T^n}(T^*\mathbb{C}^n) = \operatorname{Sym}(\mathfrak{t}^n)^* \cong \mathbb{Q}[\partial_1, \ldots, \partial_n]$$

and

$$H^*_{T^k}(T^*\mathbb{C}^n) = \operatorname{Sym}(\mathfrak{t}^k)^* \cong \mathbb{Q}[\partial_1, \ldots, \partial_n] / \operatorname{Ker}(\iota^*).$$

Theorem 3.16 [K1, HS] *The Kirwan maps κ_{T^d} and κ are surjective, and the kernels of both are generated by the elements*

$$\prod_{i \in S} \partial_i \qquad \textit{for all } S \subseteq \{1, \ldots, n\} \textit{ such that } \bigcap_{i \in S} H_i = \emptyset.$$

Remark 3.17 The kernel of κ_{T^d} is precisely the Stanley-Reisner ideal of the matroid of linear dependencies among the vectors $\{a_i\}$ [HS] (see Remark 3.19).

The inclusion of $\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)$ into $T^*\mathbb{C}^n$ is also S^1-equivariant, hence we may consider the analogous circle-equivariant Kirwan maps

$$\kappa_{T^d \times S^1} : H^*_{T^n \times S^1}(T^*\mathbb{C}^n) \to H^*_{T^d \times S^1}(\mathfrak{M}) \quad \text{and} \quad \kappa : H^*_{T^k \times S^1}(T^*\mathbb{C}^n) \to H^*_{S^1}(\mathfrak{M}),$$

where

$$H^*_{T^n \times S^1}(T^*\mathbb{C}^n) \cong \mathbb{Q}[\partial_1, \ldots, \partial_n, x]$$

and

$$H^*_{T^k}(T^*\mathbb{C}^n) \cong \mathbb{Q}[\partial_1, \ldots, \partial_n, x] / \operatorname{Ker}(\iota^*).$$

The remainder of this section will be devoted to proving the following theorem.

Theorem 3.18 *The circle-equivariant Kirwan maps $\kappa_{T^d \times S^1}$ and κ_{S^1} are surjective, and the kernels of both are generated by the elements*

$$\prod_{i \in S_1} \partial_i \times \prod_{j \in S_2} (x - \partial_j) \qquad \textit{for all disjoint pairs } S_1, S_2 \subseteq \{1, \ldots, n\}$$

$$\textit{such that } \bigcap_{i \in S_1} G_i \;\cap\; \bigcap_{j \in S_2} F_j = \emptyset.$$

Remark 3.19 For all $i \in \{1, \ldots, n\}$, let $b_i = a_i \oplus 0 \in \mathfrak{t}^d \oplus \mathbb{R}$, and let $b_0 = 0 \oplus 1$. The *pointed matroid* associated to $\mathcal{A}$ is a combinatorial object that tells us which subsets of $\{b_0, \ldots, b_n\}$ are linearly dependent. By simplicity of $\mathcal{A}$, this is equivalent to knowing which subsets $S \subseteq \{1, \ldots, n\}$ have the property that

$$\bigcap_{i \in S} H_i = \emptyset,$$

which is in turn equivalent to knowing the dependence relations among the vectors $\{a_i\}$. In particular, it does not depend on the relative positions of the hyperplanes, encoded by the parameter $\alpha \in (\mathfrak{t}^k)^*$.

The *pointed oriented matroid* associated to $\mathcal{A}$ encodes the data of which subsets of $\{\pm b_0, \ldots, \pm b_n\}$ are linearly dependent over the *positive* real numbers. This is equivalent to knowing which pairs of subsets $S_1, S_2 \subseteq \{1, \ldots, n\}$ have the property that

$$\bigcap_{i \in S_1} G_i \;\cap\; \bigcap_{j \in S_2} F_j = \emptyset,$$

which does indeed depend on α. Hence Theorem 3.16 shows that $H^*_{T^d}(\mathfrak{M})$ is an invariant of the pointed matroid of $\mathcal{A}$, and Theorem 3.18 demonstrates that $H^*_{T^d \times S^1}(\mathfrak{M})$ is an invariant of the pointed oriented matroid of $\mathcal{A}$. For more on this perspective, see [H3] and [Pr].

Consider the following commuting square of maps, where ϕ and ψ are each given by setting the image of $x \in H^*_{S^1}(pt) \cong \mathbb{Q}[x]$ to zero.

$$\begin{array}{ccc}
H^*_{T^n \times S^1}(T^*\mathbb{C}^n) & \xrightarrow{\kappa_{T^d \times S^1}} & H^*_{T^d \times S^1}(\mathfrak{M}) \\
\phi \big\downarrow & & \big\downarrow \psi \\
H^*_{T^n}(T^*\mathbb{C}^n) & \xrightarrow{\kappa_{T^d}} & H^*_{T^d}(\mathfrak{M})
\end{array}$$

Lemma 3.20 *The equivariant Kirwan map $\kappa_{T^d \times S^1}$ is surjective.*

Proof: Suppose that $\gamma \in H^*_{T^d \times S^1}(\mathfrak{M})$ is a homogeneous class of minimal degree that is *not* in the image of $\kappa_{T^d \times S^1}$. By Theorem 3.16 κ_{T^d} is surjective, hence we may choose a class $\eta \in \phi^{-1}\kappa_{T^d}^{-1}\psi(\gamma)$. Theorem 2.10 tells us that the kernel of ψ is generated by x, hence $\kappa_{T^d \times S^1}(\eta) - \gamma = x\delta$ for some $\delta \in H^*_{T^d \times S^1}(\mathfrak{M})$. Then δ is a class of lower degree that is not in the image of $\kappa_{T^d \times S^1}$. □

Lemma 3.21 *If $\mathcal{I} \subseteq \operatorname{Ker} \kappa_{T^d \times S^1}$ and $\phi(\mathcal{I}) = \operatorname{Ker} \kappa_{T^d}$, then $\mathcal{I} = \operatorname{Ker} \kappa_{T^d \times S^1}$.*

Proof: Suppose that $a \in \operatorname{Ker} \kappa_{T^d \times S^1} \smallsetminus \mathcal{I}$ is a homogeneous class of minimal degree, and choose $b \in \mathcal{I}$ such that $\phi(a-b) = 0$. Then $a - b = cx$ for some $c \in H^*_{T^n \times S^1}(T^*\mathbb{C}^n)$. By Proposition 2.10, $cx \in \operatorname{Ker} \kappa_{T^d \times S^1} \Rightarrow c \in \operatorname{Ker} \kappa_{T^d \times S^1}$, hence $c \in \operatorname{Ker} \kappa_{T^d \times S^1} \smallsetminus \mathcal{I}$ is a class of lower degree than a. $\square$

Proof of 3.18: For any element

$$h \in H^2_{T^n \times S^1}(T^*\mathbb{C}^n) \cong \mathbb{Q}\{\partial_1, \ldots, \partial_n, x\},$$

let $\tilde{L}_h = T^*\mathbb{C}^n \times \mathbb{C}_h$ be the $T^n \times S^1$-equivariant line bundle on $T^*\mathbb{C}^n$ with equivariant Euler class h. Let

$$L_h = \tilde{L}_h \Big|_{\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)} \Big/ T^k$$

be the quotient $T^d \times S^1$-equivariant line bundle on $\mathfrak{M}$. We will write $\tilde{L}_i = \tilde{L}_{\partial_i}$ and $\tilde{K} = \tilde{L}_x$, with quotients L_i and K. Since the $T^d \times S^1$-equivariant Euler class $e(L_i)$ is the image of ∂_i under the hyperkähler Kirwan map $H^*_{T^n \times S^1}(T^*\mathbb{C}^n) \to H^*_{T^d \times S^1}(\mathfrak{M})$, we will abuse notation and denote it by ∂_i. Similarly, we will denote $e(K)$ by x. Lemma 3.20 tells us that $H^*_{T^d \times S^1}(\mathfrak{M})$ is generated by $\partial_1, \ldots, \partial_n, x$.

Consider the $T^n \times S^1$-equivariant section $\tilde{s}_i$ of $\tilde{L}_i$ given by the function $\tilde{s}_i(z,w) = z_i$. This descends to a $T^d \times S^1$-equivariant section s_i of L_i with zero-set

$$Z_i := \{[z,w] \in \mathfrak{M} \mid z_i = 0\}.$$

Similarly, the function $\tilde{t}_i(z,w) = w_i$ defines a $T^d \times S^1$-equivariant section of $L_i^* \otimes K$ with zero set

$$W_i := \{[z,w] \in \mathfrak{M} \mid w_i = 0\}.$$

Thus the divisor Z_i represents the cohomology class ∂_i, and W_i represents $x - \partial_i$. Note, that by Lemma 3.7, we have $\mu_{\mathbb{R}}(Z_i) \subseteq G_i$ and $\mu_{\mathbb{R}}(W_i) \subseteq F_i$ for all $i \in \{1, \ldots, n\}$.

Let S_1 and S_2 be a pair of subsets of $\{1, \ldots n\}$ such that $\big(\cap_{i \in S_1} G_i\big) \cap \big(\cap_{j \in S_2} F_j\big) = \emptyset$, and hence

$$\big(\cap_{i \in S_1} Z_i\big) \cap \big(\cap_{j \in S_2} W_j\big) \subseteq \mu_{\mathbb{R}}^{-1}\Big(\big(\cap_{i \in S_1} G_i\big) \cap \big(\cap_{j \in S_2} F_j\big)\Big) = \emptyset.$$

Now consider the vector bundle $E = \bigoplus_{i \in S_1} L_i \ \oplus \ \bigoplus_{j \in S_2} L_j^* \otimes K$ with equivariant Euler class

$$e(E) = \prod_{i \in S_1} \partial_i \times \prod_{j \in S_2} (x - \partial_j).$$

The section $(\oplus_{i\in S_1} s_i) \oplus (\oplus_{i\in S_2} t_i)$ is a nonvanishing equivariant global section of E, hence $e(E)$ is trivial in $H^*_{T^d\times S^1}(\mathfrak{M})$. Theorem 3.16 and Lemma 3.21 tell us that we have found all of the relations. The proofs of the analogous statements for $H^*_{S^1}(\mathfrak{M})$ are identical. □

How sensitive are the invariants $H^*_{T^d\times S^1}(\mathfrak{M})$ and $H^*_{S^1}(\mathfrak{M})$? We can recover $H^*_{T^d}(\mathfrak{M})$ and $H^*(\mathfrak{M})$ by setting x to zero, hence they are at least as fine as the ordinary or T^d-equivariant cohomology rings. The ring $H^*_{T^d\times S^1}(\mathfrak{M})$ does *not* depend on coorientations, for if $\mathfrak{M}'$ is related to $\mathfrak{M}$ by flipping the coorientation of the l^{th} hyperplane H_k, then the map taking ∂_l to $x-\partial_l$ is an isomorphism between $H^*_{T^d\times S^1}(\mathfrak{M})$ and $H^*_{T^d\times S^1}(M')$.[4] The ring *does*, however, depend on α, as we see in Example 3.22.

Example 3.22 We compute the equivariant cohomology ring $H^*_{T^d\times S^1}(\mathfrak{M})$ for the hypertoric varieties $\mathfrak{M}_a$, $\mathfrak{M}_b$, and $\mathfrak{M}_c$ defined by the arrangements in Figure 3.2(a), (b), and (c), respectively. Note that each of these arrangements is smooth, hence Theorem 3.18 holds over the integers, as in Remark 3.15.

$$H^*_{T^d\times S^1}(\mathfrak{M}_a) = \mathbb{Z}[\partial_1,\ldots,\partial_4,x]\big/\left\langle \partial_2\partial_3, \partial_1(x-\partial_2)\partial_4, \partial_1\partial_3\partial_4\right\rangle,$$

$$H^*_{T^d\times S^1}(\mathfrak{M}_b) = \mathbb{Z}[\partial_1,\ldots,\partial_4,x]\big/\left\langle (x-\partial_2)\partial_3, \partial_1\partial_2\partial_4, \partial_1\partial_3\partial_4\right\rangle,$$

$$H^*_{T^d\times S^1}(\mathfrak{M}_c) = \mathbb{Z}[\partial_1,\ldots,\partial_4,x]\big/\left\langle \partial_2\partial_3, (x-\partial_1)\partial_2(x-\partial_4), \partial_1\partial_3\partial_4\right\rangle.$$

As we have already observed, the rings $H^*_{T^d\times S^1}(\mathfrak{M}_a)$ and $H^*_{T^d\times S^1}(\mathfrak{M}_b)$ are isomorphic by interchanging ∂_2 with $x-\partial_2$. One can check that the annihilator of ∂_2 in $H^*_{T^d\times S^1}(\mathfrak{M}_a)$ is the principal ideal generated by ∂_3, while the ring $H^*_{T^d\times S^1}(\mathfrak{M}_c)$ has no degree 2 element whose annihilator is generated by a single element of degree 2. Hence $H^*_{T^d\times S^1}(\mathfrak{M}_c)$ is not isomorphic to the other two rings.

The ring $H^*_{S^1}(\mathfrak{M})$, on the other hand, is sensitive to coorientations as well as the parameter α, as we see in Example 3.23.

Example 3.23 We now compute the ring $H^*_{S^1}(\mathfrak{M})$ for $\mathfrak{M}_a$, $\mathfrak{M}_b$, and $\mathfrak{M}_c$ of Figure 2. Theorem 3.18 tells us that we need only to quotient the ring $H^*_{T^d\times S^1}(\mathfrak{M})$ by $\operatorname{Ker}(\iota^*)$. For $\mathfrak{M}_a$,

[4] The oriented matroid of a collection of nonzero vectors in a real vector space does not change when one of the vectors is negated, hence the independence of $H^*_{T^d\times S^1}(\mathfrak{M})$ on coorientations can be deduced from Remark 3.19.

the kernel of ι_a^* is generated by $\partial_1+\partial_2-\partial_3$ and $\partial_1-\partial_4$, hence we have

$$\begin{aligned} H^*_{S^1}(\mathfrak{M}_a) &= \mathbb{Z}[\partial_2,\partial_3,x]\big/\left\langle \partial_2\partial_3, (\partial_3-\partial_2)^2(x-\partial_2), (\partial_3-\partial_2)^2\partial_3\right\rangle \\ &\cong \mathbb{Z}[\partial_2,\partial_3,x]\big/\left\langle \partial_2\partial_3, (\partial_3-\partial_2)^2(x-\partial_2), \partial_3^3\right\rangle. \end{aligned}$$

Since the hyperplanes of 2(c) have the same coorientations as those of 2(a), we have $\operatorname{Ker}\iota_b^* = \operatorname{Ker}\iota_a^*$, hence

$$\begin{aligned} H^*_{S^1}(\mathfrak{M}_c) &= \mathbb{Z}[\partial_2,\partial_3,x]\big/\left\langle \partial_2\partial_3, (x-\partial_3+\partial_2)^2\partial_2, (\partial_3-\partial_2)^2\partial_3\right\rangle \\ &\cong \mathbb{Z}[\partial_2,\partial_3,x]\big/\left\langle \partial_2\partial_3, (x-\partial_3+\partial_2)^2\partial_2, \partial_3^3\right\rangle. \end{aligned}$$

Finally, since Figure 2(b) is obtained from 2(a) by flipping the coorientation of H_2, we find that $\operatorname{Ker}(\iota_b^*)$ is generated by $\partial_1-\partial_2-\partial_3$ and $\partial_1-\partial_4$, therefore

$$H^*_{S^1}(\mathfrak{M}_b) = \mathbb{Z}[\partial_2,\partial_3,x]\big/\left\langle (x-\partial_2)\partial_3, (\partial_2+\partial_3)^2\partial_2, (\partial_2+\partial_3)^2\partial_3\right\rangle.$$

As in Example 3.22, $H^*_{S^1}(\mathfrak{M}_a)$ and $H^*_{S^1}(\mathfrak{M}_c)$ can be distinguished by the fact that the annihilator of $\partial_2 \in H^*_{S^1}(\mathfrak{M}_a)$ is generated by a single element of degree 2, and no element of $H^*_{S^1}(\mathfrak{M}_c)$ has this property. On the other hand, $H^*_{S^1}(\mathfrak{M}_b)$ is distinguished from $H^*_{S^1}(\mathfrak{M}_a)$ and $H^*_{S^1}(\mathfrak{M}_c)$ by the fact that neither $x-\partial_2$ nor ∂_3 cubes to zero.

3.4 The equivariant Orlik-Solomon algebra

In this section we restrict our attention to smooth arrangements. When $\mathcal{A}$ is smooth, all of the computations of Section 3.3 hold over the integers (see Remark 3.15). Since the rings in question are torsion-free, the presentations are also valid when the coefficients are taken in the field field $\mathbb{Z}_2$.

Let $\mathfrak{M}_{\mathbb{R}} \subseteq \mathfrak{M}$ be the *real locus* $\{[z,w]\in\mathfrak{M} \mid z,w \text{ real}\}$ of $\mathfrak{M}$ with respect to the complex structure J_1. The full group $T^d\times S^1$ does not act on $\mathfrak{M}_{\mathbb{R}}$, but the subgroup $T^d_{\mathbb{R}}\times\mathbb{Z}_2$ does act, where $T^d_{\mathbb{R}} := \mathbb{Z}_2^d \subseteq T^d$ is the fixed point set of the involution of T^d given by complex conjugation.[5] In this section we will study the geometry of the real locus, focusing

[5] It is interesting to note that the real locus with respect to the complex structure J_1 is in fact a complex submanifold with respect to J_3, on which $T^d_{\mathbb{R}}$ acts holomorphically and $\mathbb{Z}_2$ acts anti-holomorphically.

in particular on the properties of the $\mathbb{Z}_2$ action. The following theorem is an extension of the results of [BGH] and [Sc] to the noncompact case of hypertoric varieties.

Theorem 3.24 [HH] *Let $G = T^d \times S^1$ or T^d, and $G_{\mathbb{R}} = T^d_{\mathbb{R}} \times \mathbb{Z}_2$ or $T^d_{\mathbb{R}}$, respectively. Then we have $H^*_G(\mathfrak{M};\mathbb{Z}_2) \cong H^*_{G_{\mathbb{R}}}(\mathfrak{M}_{\mathbb{R}};\mathbb{Z}_2)$ by an isomorphism that halves the grading.*[6] *Furthermore, this isomorphism takes the cohomology classes represented by the G-stable submanifolds Z_i and W_i to those represented by the $G_{\mathbb{R}}$-stable submanifolds $Z_i \cap \mathfrak{M}_{\mathbb{R}}$ and $W_i \cap \mathfrak{M}_{\mathbb{R}}$.*

Consider the restriction Ψ of the hyperkähler moment map $\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$ to $\mathfrak{M}_{\mathbb{R}}$. Since z and w are real for every $[z,w] \in \mathfrak{M}_{\mathbb{R}}$, the map $\mu_{\mathbb{C}}$ takes values in $\mathfrak{t}^d_{\mathbb{R}} \subseteq \mathfrak{t}^d_{\mathbb{C}}$, which we will identify with $i\mathbb{R}^d$, so that Ψ takes values in $\mathbb{R}^d \oplus i\mathbb{R}^d \cong \mathbb{C}^d$. Note that Ψ is $\mathbb{Z}_2$-equivariant, with $\mathbb{Z}_2$ acting on $\mathbb{C}^n$ by complex conjugation.

Lemma 3.25 *The map $\Psi : \mathfrak{M}_{\mathbb{R}} \to \mathbb{C}^d$ is surjective, and the fibers are the orbits of $T^d_{\mathbb{R}}$. The stabilizer of a point $p \in \mathfrak{M}_{\mathbb{R}}$ has order 2^r, where r is the number of hyperplanes in the complexified arrangement $\mathcal{A}_{\mathbb{C}}$ containing the point $\Psi(p)$.*

Proof: For any point $a + bi \in \mathbb{C}^d$, choose a point $[z,w] \in \mathfrak{M}$ such that $\mu_{\mathbb{R}}[z,w] = a$ and $\mu_{\mathbb{C}}[z,w] = b$. After moving $[z,w]$ by an element of T^d we may assume that z and w are real, hence we may assume that $[z,w] \in \mathfrak{M}_{\mathbb{R}}$. Then

$$\Psi^{-1}(a+bi) = \mu_{\mathbb{R}}^{-1}(a) \cap \mu_{\mathbb{C}}^{-1}(b) \cap \mathfrak{M}_{\mathbb{R}} = T^d[z,w] \cap \mathfrak{M}_{\mathbb{R}} = T^d_{\mathbb{R}}[z,w].$$

The second statement follows easily from [BD, 3.1]. $\square$

Let $Y \subseteq \mathfrak{M}_{\mathbb{R}}$ be the locus of points on which $T^d_{\mathbb{R}}$ acts freely, i.e. the preimage under Ψ of the space $\mathcal{M}(\mathcal{A}) := \mathbb{C}^d \setminus \cup_{i=1}^n H_i^{\mathbb{C}}$. The inclusion map $Y \hookrightarrow \mathfrak{M}_{\mathbb{R}}$ induces maps backward on cohomology, which we will denote

$$\phi : H^*_{T^d_{\mathbb{R}}}(\mathfrak{M}_{\mathbb{R}};\mathbb{Z}_2) \to H^*_{T^d_{\mathbb{R}}}(Y;\mathbb{Z}_2) \quad \text{and} \quad \phi_2 : H^*_{T^d_{\mathbb{R}}\times\mathbb{Z}_2}(\mathfrak{M}_{\mathbb{R}};\mathbb{Z}_2) \to H^*_{T^d_{\mathbb{R}}\times\mathbb{Z}_2}(Y;\mathbb{Z}_2).$$

By Theorem 3.24, we have

$$H^*_{T^d_{\mathbb{R}}}(\mathfrak{M}_{\mathbb{R}};\mathbb{Z}_2) \cong H^*_{T^d}(\mathfrak{M};\mathbb{Z}_2) \quad \text{and} \quad H^*_{T^d_{\mathbb{R}}\times\mathbb{Z}_2}(\mathfrak{M}_{\mathbb{R}};\mathbb{Z}_2) \cong H^*_{T^d\times S^1}(\mathfrak{M};\mathbb{Z}_2).$$

[6] In particular, $H^*_G(\mathfrak{M};\mathbb{Z}_2)$ is concentrated in even degree.

Furthermore, since $T^d_{\mathbb{R}}$ acts freely on Y with quotient $\mathcal{M}(\mathcal{A})$, we have

$$H^*_{T^d_{\mathbb{R}}}(Y;\mathbb{Z}_2) \cong H^*(\mathcal{M}(\mathcal{A});\mathbb{Z}_2) \quad \text{and} \quad H^*_{T^d_{\mathbb{R}}\times\mathbb{Z}_2}(Y;\mathbb{Z}_2) \cong H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A});\mathbb{Z}_2),$$

hence we may write

$$\phi : H^*_{T^d}(\mathfrak{M};\mathbb{Z}_2) \to H^*(\mathcal{M}(\mathcal{A});\mathbb{Z}_2) \quad \text{and} \quad \phi_2 : H^*_{T^d\times S^1}(\mathfrak{M};\mathbb{Z}_2) \to H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A});\mathbb{Z}_2).$$

The ring $H^*(\mathcal{M}(\mathcal{A});\mathbb{Z}_2)$ is a classical invariant of the arrangement $\mathcal{A}$ known as the *Orlik-Solomon algebra* (with coefficients in $\mathbb{Z}_2$), and is denoted by $A(\mathcal{A};\mathbb{Z}_2)$ [OT]. The ring $H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A});\mathbb{Z}_2)$ was introduced in [HP1] and further studied in [Pr]. We call it the *equivariant Orlik-Solomon algebra* and denote it by $A_2(\mathcal{M}(\mathcal{A});\mathbb{Z}_2)$.

Proposition 3.26 [Pr, 2.4] *The space $\mathcal{M}(\mathcal{A})$ is $\mathbb{Z}_2$-equivariantly formal, i.e. $A_2(\mathcal{A};\mathbb{Z}_2)$ is a free module over $\mathbb{Z}_2[x] = H^*_{\mathbb{Z}_2}(pt)$.*

Theorem 3.27 *Both ϕ and ϕ_2 are surjective, with kernels*

$$\operatorname{Ker}\phi = \left\langle \partial_i^2 \;\middle|\; i\in\{1,\ldots,n\}\right\rangle \quad \textit{and} \quad \operatorname{Ker}\phi_2 = \left\langle \partial_i\,(x-\partial_i) \;\middle|\; i\in\{1,\ldots,n\}\right\rangle.$$

Proof: Theorem 3.24 tells us that $\phi_2(\partial_i)$ is represented in $H^*_{T^d_{\mathbb{R}}\times\mathbb{Z}_2}(Y;\mathbb{Z}_2)$ by the submanifold $Z_i\cap Y$, and likewise $\phi_2(x-\partial_i)$ by the submanifold $W_i\cap Y$. Since $\mu_{\mathbb{R}}(Z_i\cap W_i)\subseteq H_i$, we have $Z_i\cap W_i\cap Y=\emptyset$, hence $\partial_i(x-\partial_i)$ lies in the kernel of ϕ_2 (and therefore ∂_i^2 lies in the kernel of ϕ).

By Proposition 3.26 and a pair of formal arguments identical to those of Lemmas 3.20 and 3.21, it is sufficient to prove Theorem 3.27 only for ϕ. Quotienting $Z_i\cap Y$ by $T^d_{\mathbb{R}}$, we find that $\phi(\partial_i)$ is represented in $A(\mathcal{A};\mathbb{Z}_2)$ by the submanifold

$$\{v\in\mathcal{M}(\mathcal{A}) \mid v\cdot a_i + r_i \in \mathbb{R}^-\}.$$

The standard presentation of $A(\mathcal{A};\mathbb{Z}_2)$ (see, for example, [OT]) says that these classes generate the ring, and that all relations between them are generated by the monomials of Theorem 3.16 and ∂_i^2 for all i. □

Remark 3.28 Theorems 3.18 and 3.27 combine to give a presentation of the equivariant Orlik-Solomon algebra $A_2(\mathcal{A};\mathbb{Z}_2)$ in the case where $\mathcal{A}$ is rational, simple, and smooth. This presentation first appeared in [HP1]. In [Pr], we generalize this presentation to arbitrary real hyperplane arrangements, and in fact to arbitrary pointed oriented matroids.

Remark 3.29 The ring $A_2(\mathcal{A};\mathbb{Z}_2)$ is a deformation over the affine line $\operatorname{Spec}\mathbb{Z}_2[x]$ from the ordinary Orlik-Solomon algebra $A(\mathcal{A};\mathbb{Z}_2)$ to the Varchenko-Gel′fand ring $VG(\mathcal{A};\mathbb{Z}_2)$ of locally constant $\mathbb{Z}_2$-valued functions on the real points of $\mathcal{M}(\mathcal{A})$ [Pr]. While the rings $A(\mathcal{A};\mathbb{Z}_2)$ and $VG(\mathcal{A};\mathbb{Z}_2)$ depend only on the matroid associated to $\mathcal{A}$, the deformation $A_2(\mathcal{A};\mathbb{Z}_2)$ depends on the richer structure of the pointed oriented matroid (see Remark 3.19).

Example 3.30 Consider the arrangements $\mathcal{A}_a$ and $\mathcal{A}_c$ in Figure 2(a) and 2(c). By Theorem 3.27 and Example 3.22 we have

$$H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}_a);\mathbb{Z}_2) \cong \mathbb{Z}_2[\partial_1,\ldots,\partial_4,x]\Big/\left\langle \begin{array}{c} \partial_1(x-\partial_1),\partial_2(x-\partial_2),\partial_3(x-\partial_3),\partial_4(x-\partial_4),\\ \partial_2\partial_3,\partial_1(x-\partial_2)\partial_4,\partial_1\partial_3\partial_4 \end{array}\right\rangle$$

and

$$H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}_c);\mathbb{Z}_2) \cong \mathbb{Z}_2[\partial_1,\ldots,\partial_4,x]\Big/\left\langle \begin{array}{c} \partial_1(x-\partial_1),\partial_2(x-\partial_2),\partial_3(x-\partial_3),\partial_4(x-\partial_4),\\ \partial_2\partial_3,(x-\partial_1)\partial_2(x-\partial_4),\partial_1\partial_3\partial_4 \end{array}\right\rangle.$$

The map $f: H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}_a);\mathbb{Z}_2) \to H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}_b);\mathbb{Z}_2)$ given by

$$f(\partial_1)=\partial_1+\partial_2, f(\partial_2)=\partial_2+\partial_3+x, f(\partial_3)=\partial_3, f(\partial_4)=\partial_2+\partial_4, \text{ and } f(x)=x$$

is an isomorphism of graded $\mathbb{Z}_2[x]$-algebras, despite the fact that the oriented matroids of the two arrangements differ.[7]

Example 3.31 Now consider the arrangements $\mathcal{A}'_a$ and $\mathcal{A}'_c$ obtained from $\mathcal{A}_a$ and $\mathcal{A}_c$ by adding a vertical line on the far left, as shown in Figure 3.5. Again by Theorem 3.27, we have

$$H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}'_a);\mathbb{Z}_2) \cong \mathbb{Z}_2[\partial_1,\ldots,\partial_4,x]\Big/\left\langle \begin{array}{c} \partial_1(x-\partial_1),\partial_2(x-\partial_2),\partial_3(x-\partial_3),\partial_4(x-\partial_4),\\ \partial_5(x-\partial_5),\partial_2\partial_3,(x-\partial_1)\partial_5,\partial_1(x-\partial_2)\partial_4,\\ \partial_1\partial_3\partial_4,(x-\partial_2)\partial_4\partial_5,\partial_3\partial_4\partial_5 \end{array}\right\rangle$$

and

$$H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}'_c);\mathbb{Z}_2) \cong \mathbb{Z}_2[\partial_1,\ldots,\partial_4,x]\Big/\left\langle \begin{array}{c} \partial_1(x-\partial_1),\partial_2(x-\partial_2),\partial_3(x-\partial_3),\partial_4(x-\partial_4),\\ \partial_5(x-\partial_5),\partial_2\partial_3,(x-\partial_1)\partial_5,(x-\partial_1)\partial_2(x-\partial_4),\\ \partial_1\partial_3\partial_4,(x-\partial_2)\partial_4\partial_5,\partial_3\partial_4\partial_5 \end{array}\right\rangle.$$

[7] We thank Graham Denham for finding this isomorphism.

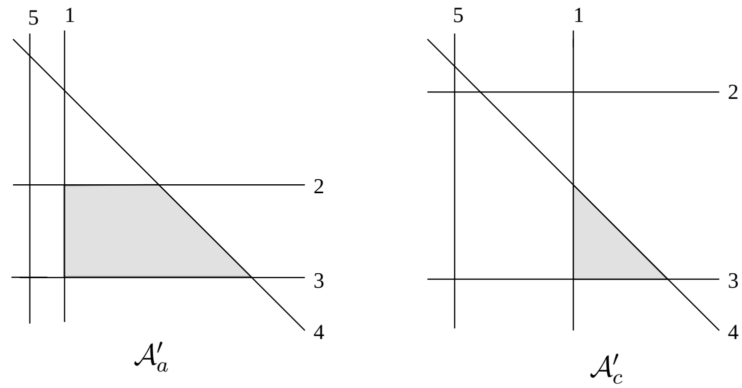

Figure 3.5: Adding a vertical line to $\mathcal{A}_a$ and $\mathcal{A}_c$.

We have used Macaulay 2 [M2] to check that the annihilator of the element $\partial_2 \in H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}'_a);\mathbb{Z}_2)$ is generated by two linear elements (namely ∂_3 and $x - \partial_2$) and nothing else, while there is no element of $H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}'_c);\mathbb{Z}_2)$ with this property. Hence the two rings are not isomorphic.

3.5 Cogenerators

Consider the Kähler Kirwan map

$$K_\alpha : \mathrm{Sym}(\mathfrak{t}^k)^* \cong H^*_{T^k}(\mathbb{C}^n) \to H^*(\mathfrak{X}_\alpha)$$

induced by the T^k-equivariant inclusion of $\mu^{-1}(\alpha)$ into $\mathbb{C}^n$. In this section we would like to consider simultaneously the Kirwan maps K_α for many different values of α, so almost all of the notation that we use will have a subscript or superscript indicating the parameter $\alpha \in (\mathfrak{t}^k)^*$ or a lift $r \in (\mathfrak{t}^n)^*$. An exception to this rule will be the hyperkähler Kirwan map

$$\kappa : \mathrm{Sym}(\mathfrak{t}^k)^* \to H^*(\mathfrak{M}_\alpha),$$

which, by Lemma 3.4 or Theorem 3.16, is independent of our choice of simple $\alpha \in (\mathfrak{t}^k)^*$. The main result of this section is the following.

Theorem 3.32 *The kernel of the hyperkähler Kirwan map κ is equal to the intersection over all simple α of the kernels of the Kähler Kirwan maps K_α.*

Remark 3.33 Konno [K2, 7.6] proves an analogous theorem about the kernels of the Kirwan maps to the cohomology rings of polygon and hyperpolygon spaces. We may therefore conjecture a generalization of Theorem 3.32 in which T^k is replaced by an arbitrary compact group G. Note that our proof of Theorem 3.32 depends strongly on the combinatorics associated to hypertoric varieties.

We approach Theorem 3.32 by describing the kernels of κ and K_α not in terms of generators, but rather in terms of cogenerators. Given an ideal $\mathcal{I} \subseteq \mathrm{Sym}(\mathfrak{t}^k)^*$, a set of *cogenerators* for $\mathcal{I}$ is a collection of polynomials $\{f_i\} \subseteq \mathrm{Sym}\,\mathfrak{t}^k$ such that

$$\mathcal{I} = \{\partial \in \mathrm{Sym}(\mathfrak{t}^k)^* \mid \partial \cdot f_i = 0 \text{ for all } i\}.$$

The volume function $\mathrm{Vol}\,\Delta^r$ is locally polynomial in r. More precisely, for every simple $r \in (\mathfrak{t}^n)^*$, there exists a degree d polynomial $P^r \in \mathrm{Sym}^d\,\mathfrak{t}^n$ such that for every simple $s \in (\mathfrak{t}^n)^*$ lying in the same connected component of the set of simple elements as r, we have

$$\mathrm{Vol}\,\Delta^s = P^r(s).$$

We will refer to P^r as the *volume polynomial* of Δ^r. The fact that the volume of a polytope is translation invariant tells us that P^r lies in the image of the inclusion $\iota : \mathrm{Sym}^d\,\mathfrak{t}^k \hookrightarrow \mathrm{Sym}^d\,\mathfrak{t}^n$.

Theorem 3.34 [GS, KP] *Let $r \in (\mathfrak{t}^n)^*$ be a simple element with $\iota^*(r) = \alpha$. Then*

$$\mathrm{Ker}\,K_\alpha = \mathrm{Ann}\{\iota^{-1}P^r\} = \big\{\partial \in \mathrm{Sym}(\mathfrak{t}^k)^* \mid \partial \cdot (\iota^{-1}P^r) = 0\big\}.$$

A similar description of the cohomology ring of a hypertoric variety is given in [HS]. For any subset $A \subseteq \{1, \ldots, n\}$, consider the polyhedron Δ^r_A introduced in Section 3.2. If Δ^r_A is nonempty, then it is bounded if and only if the vectors $\{\varepsilon_1(A)a_1, \ldots, \varepsilon_n(A)a_n\}$ span $\mathfrak{t}^d$ over the non-negative real numbers, where $\varepsilon_i(A) = (-1)^{|A \cap \{i\}|}$. We call such an A *admissible*. For all admissible A, there exists a degree d polynomial $P^r_A \in \mathrm{Sym}^d\,\mathfrak{t}^n$ such that for every simple $s \in (\mathfrak{t}^n)^*$ lying in the same connected component of the set of simple elements as r, we have

$$\mathrm{Vol}\,\Delta^s_A = P^r_A(s).$$

Once again, the translation invariance of volume implies that P^r_A lies in the image of the inclusion $\iota : \mathrm{Sym}^d\,\mathfrak{t}^k \hookrightarrow \mathrm{Sym}^d\,\mathfrak{t}^n$. Consider the linear span

$$U^r = \mathbb{Q}\big\{P^r_A \mid A \text{ admissible}\big\}.$$

Theorem 3.35 [HS] *Let $r \in (\mathfrak{t}^n)^*$ be a simple element with $\iota^*(r) = \alpha$. Then*

$$\operatorname{Ker} \kappa = \operatorname{Ann} \iota^{-1} U^r = \left\{ \partial \in \operatorname{Sym}(\mathfrak{t}^k)^* \mid \partial \cdot \iota^{-1} P = 0 \textit{ for all } P \in U^r \right\}.$$

Remark 3.36 It is clear from the statement of Theorem 3.35 that $H^*(\mathfrak{M})$ does not depend on the coorientations of the hyperplanes $\{H_1^r, \ldots, H_n^r\}$, as has been observed in Lemma 3.5 and throughout Section 3.3. Indeed, the polynomials P_A^r for A admissible are simply the volume polynomials of the maximal regions of $|\mathcal{A}|$. What is not clear from this presentation is the independence of $H^*(\mathfrak{M})$ on α. In other words, it is a nontrivial fact that the vector space U^r is independent of the parameter $r \in (\mathfrak{t}^n)^*$.

Proof of 3.32: The statement of Theorem 3.32 equates the kernel of κ, which is cogenerated by the vector space $\iota^{-1}U^r$, with the intersection of the kernels of the maps K_α, each of which is cogenerated by the element $\iota^{-1}P^r$. Intersection of ideals corresponds to linear span on the level of cogenerators, hence we have

$$\bigcap_{\alpha \text{ simple}} \operatorname{Ker} K_\alpha = \operatorname{Ann} \iota^{-1} V, \qquad \text{where} \qquad V = \mathbb{Q}\left\{ P^r \mid r \text{ simple} \right\}.$$

Our plan is to show that $V = U^r$ for any simple r. Recall that the assignment of P^r to r is locally constant on the set of simple elements of $(\mathfrak{t}^n)^*$, hence V is finite-dimensional. Since $P^r \in U^r$ and U^r does not depend on r (see Remark 3.36), it is clear that $V \subseteq U^r$. Thus to prove Theorem 3.32, it will suffice to prove the opposite inclusion, as stated below.

Proposition 3.37 *We have $P_A^r \in V$ for every admissible $A \subseteq \{1, \ldots, n\}$.*

Let $\mathcal{F}$ be the infinite dimensional vector space consisting of all real-valued functions on $(\mathfrak{t}^d)^*$, and let $\mathcal{F}^{bd}$ be the subspace consisting of functions with bounded support. For all subsets $A \subseteq \{1, \ldots, n\}$, let

$$W_A = \mathbb{Q}\left\{ \mathbf{1}_{\Delta_A^r} \mid r \text{ simple} \right\}$$

be the subspace of $\mathcal{F}$ consisting of finite linear combinations of characteristic functions of polyhedra Δ_A^r, and let

$$W_A^{bd} = W_A \cap \mathcal{F}^{bd}.$$

Note that $W_A^{bd} = W_A$ if and only if A is admissible.

Lemma 3.38 *For all $A, A' \subseteq \{1, \ldots, n\}$, $W_A^{bd} = W_{A'}^{bd}$.*

Proof: We may immediately reduce to the case where $A' = A \cup \{j\}$. Fix a simple $r \in (\mathfrak{t}^n)^*$. Let $\tilde{r} \in (\mathfrak{t}^n)^*$ be another simple element obtained from r by putting $\tilde{r}_i = r_i$ for all $i \neq j$, and $\tilde{r}_j = N$ for some $N \gg 0$. Then $\Delta_A^r \subseteq \Delta_A^{\tilde{r}}$, and

$$\begin{aligned} \Delta_A^{\tilde{r}} \smallsetminus \Delta_A^r &= \big\{v \in (\mathfrak{t}^d)^* \mid \varepsilon_i(A')(v \cdot a_i + r_i) \geq 0 \text{ for all } i \leq n \text{ and } v \cdot a_j + N \geq 0\big\} \\ &= \Delta_{A'}^r \cap G_j^N. \end{aligned}$$

Suppose that $f \in \mathcal{F}^{bd}$ can be written as a linear combination of functions of the form $\mathbf{1}_{\Delta_{A'}^r}$. Choosing N large enough that the support of f is contained in G_j^N, the above computation shows that f can be written as a linear combination of functions of the form $\mathbf{1}_{\Delta_A^r}$, hence $W_{A'}^{bd} \subseteq W_A^{bd}$. The reverse inclusion is obtained by an identical argument. $\square$

Example 3.39 Suppose that we want to write the characteristic function for the upper triangle $\Delta_{\{1,4\}}$ in Figure 3.2(c) as linear combination of characteristic functions of the shaded regions obtained by translating the hyperplanes in any possible way. Since $\{1,4\}$ has two elements, the procedure described in Lemma 3.38 must be iterated twice, and the result will have a total of $2^2 = 4$ terms, as illustrated in Figure 3.6. The first iteration exhibits $\mathbf{1}_{\Delta_{\{1,4\}}}$ as an element of $W_{\{4\}}^{bd}$ by expressing it as the difference of the characteristic functions of two (unbounded) regions. With the second iteration, we attempt to express each of these two characteristic functions as elements of $W_{\{1,4\}}^{bd} = W_{\{1,4\}}$. This attempt must fail, because each of the two functions that we try to express has unbounded support. But the failures cancel out, and we succeed in expressing the *difference* as an element of $W_{\{1,4\}}$.

Proof of 3.37: By Lemma 3.38, we may write

$$\mathbf{1}_{\Delta_A^r} = \sum_{j=1}^m \eta_j \mathbf{1}_{\Delta^{r(j)}}$$

for any simple r and admissible A, where $\eta_j \in \mathbb{Z}$ and $r(j)$ is a simple element of $(\mathfrak{t}^n)^*$ for all $j \leq m$. Taking volumes of both sides of the equation, we have

$$P_A^r(r) = \sum_{j=1}^m \eta_j P^{r(j)}\big(r(j)\big). \tag{3.2}$$

Furthermore, we observe from the proof of Lemma 3.38 that for all $j \leq m$ and all $i \leq n$, the i^{th} coordinate $r_i(j)$ of $r(j)$ is either equal to r_i, or to some large number number $N \gg 0$. The

Equation (3.2) still holds if we wiggle these large numbers a little bit, hence the polynomial $P^{r(j)}$ must be independent of the variable $r_i(j)$ whenever $r_i(j) \neq r_i$. Thus we may substitute r for each $r(j)$ on the right-hand side, and we obtain the equation

$$P_A^r(r) = \sum_{j=1}^{m} \eta_j P^{r(j)}(r).$$

This equation clearly holds in a neighborhood of r, hence we obtain an equation of polynomials

$$P_A^r = \sum_{j=1}^{m} \eta_j P^{r(j)}.$$

This completes the proof of Proposition 3.37, and therefore also of Theorem 3.32. □

Example 3.40 Let's see what happens when we take volume polynomials in the equation of Figure 3.6. The two polytopes on the top line have different volumes, but the same volume polynomial, hence these two terms cancel. We are left with the equation

$$P^{(0,1,1,0)}_{\{1,4\}} = P^{(0,1,1,0)} - P^{(N,1,1,0)},$$

which translates as

$$\frac{1}{2}\left(-x_1 + x_2 - x_4\right)^2 = \frac{1}{2}\left(x_1 + x_3 + x_4\right)^2 - \left(x_2 + x_3\right)\left(x_1 + x_4 + \frac{1}{2}x_3 - \frac{1}{2}x_2\right).$$

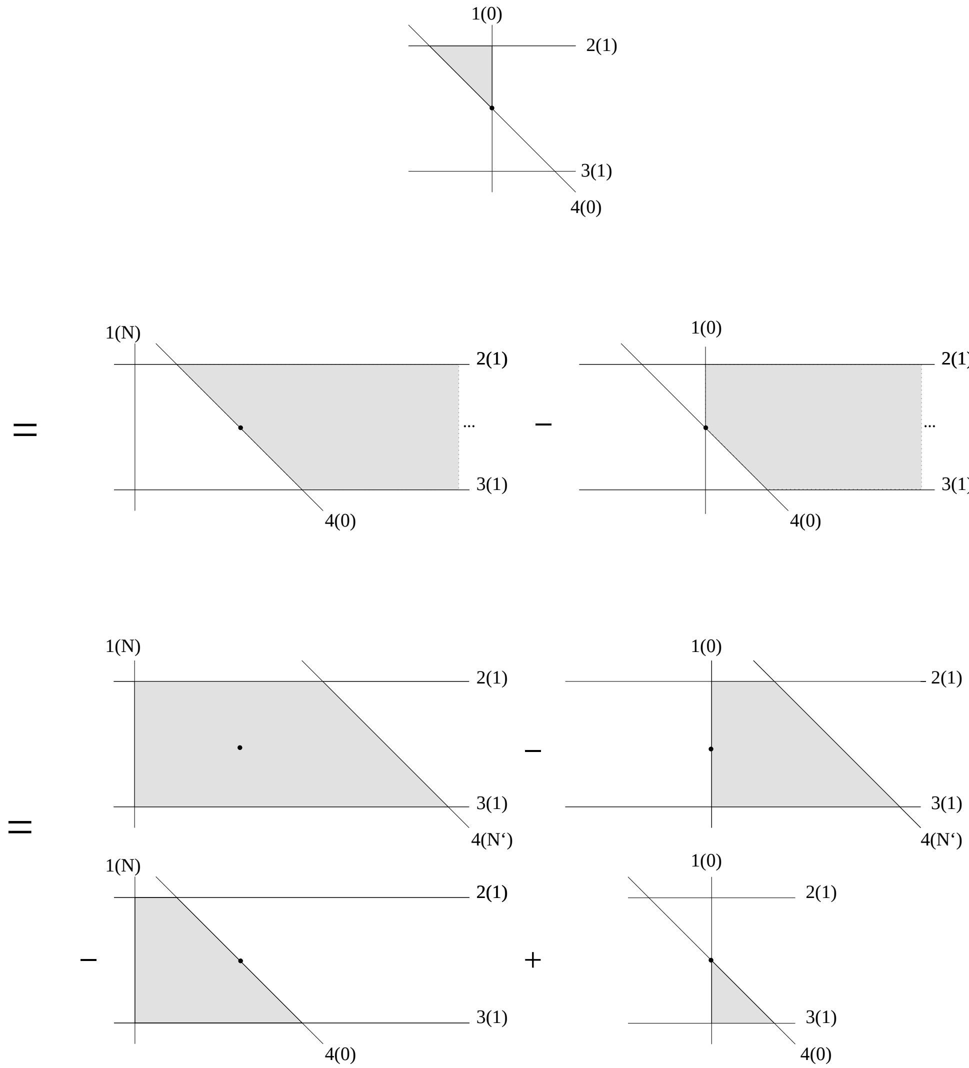

Figure 3.6: An equation of characteristic functions. We write two numbers next to each hyperplane: the first is the index $i \in \{1, \ldots, 4\}$, and the second is the parameter r_i specifying the distance from the origin (denoted by a black dot) to H_i. The two iterations of Lemma 3.38 have produced two undetermined large numbers, which we call N and N'.

Chapter 4

Abelianization

Let X be a symplectic manifold equipped with a hamiltonian action of a compact Lie group G. Let $T \subseteq G$ be a maximal torus, let $\Delta \subset \mathfrak{t}^*$ be the set of roots[1] of G, and let $W = N(T)/T$ be the Weyl group of G. If $\mu : X \to \mathfrak{g}^*$ is a moment map for the action of G, then $pr \circ \mu : X \to \mathfrak{t}^*$ is a moment map for the action of T, where $pr : \mathfrak{g}^* \to \mathfrak{t}^*$ is the standard projection. Suppose that $0 \in \mathfrak{g}^*$ and $0 \in \mathfrak{t}^*$ are regular values for the two moment maps. If the symplectic quotients

$$X/\!\!/G = \mu^{-1}(0)/G \quad \text{and} \quad X/\!\!/T = (pr \circ \mu)^{-1}(0)/T$$

are both compact, then Martin's theorem [Ma, Theorem A] relates the cohomology of $X/\!\!/G$ to the cohomology of $X/\!\!/T$. Specifically, it says that

$$H^*(X/\!\!/G) \cong \frac{H^*(X/\!\!/T)^W}{ann(e_0)},$$

where

$$e_0 = \prod_{\alpha \in \Delta} \alpha \in (\operatorname{Sym} \mathfrak{t}^*)^W \cong H_T^*(pt)^W,$$

which acts naturally on $H^*(X/\!\!/T)^W \cong H_T^*(\mu_T^{-1}(0))^W$. In the case where X is a complex vector space and G acts linearly on X, a similar result was obtained by Ellingsrud and Strømme [ES] using different techniques.

Our goal is to state and prove an analogue of this theorem for hyperkähler quotients. There are two main obstacles to this goal. First, hyperkähler quotients are rarely compact. The assumption of compactness in Martin's theorem is crucial because his proof

[1] Not to be confused with the polyhedron Δ of Chapter 3.

involves integration. Our answer to this problem is to work with equivariant cohomology of *circle compact* manifolds, by which we mean oriented manifolds with an action of S^1 such that the fixed point set is oriented and compact. Using the localization theorem of Atiyah-Bott [AB] and Berline-Vergne [BV], as motivation, we show that integration in rationalized S^1-equivariant cohomology of circle compact manifolds can be defined in terms of integration on their fixed point sets. Section 4.1 is devoted to making this statement precise by defining a well-behaved push forward in the rationalized S^1-equivariant cohomology of circle compact manifolds.

The second obstacle is that Martin's result uses surjectivity of the Kähler Kirwan map from $H^*_G(X)$ to $H^*(X/\!\!/G)$ [Ki]. The analogous map for circle compact hyperkähler quotients is conjecturally surjective, but only a few special cases are known (see Theorems 3.16 and 5.14, and Remarks 5.3 and 5.16). Our approach is to assume that the rationalized Kirwan map is surjective, which is equivalent to saying that the cokernel of the non-rationalized Kirwan map

$$\kappa_G : H^*_{S^1\times G}(M) \to H^*_{S^1}(M/\!\!/\!\!/\!\!/G)$$

is torsion as a module over $H^*_{S^1}(pt)$. This is a weaker assumption than surjectivity of κ_G; in particular, we show in Section 5.1 that this assumption holds for quiver varieties, as a consequence of the work of Nakajima.

Under this assumption, Theorem 4.10 computes the rationalized equivariant cohomology of $M/\!\!/\!\!/\!\!/G$ in terms of that of $M/\!\!/\!\!/\!\!/T$. We show that, at regular values of the hyperkähler moment maps,

$$\widehat{H}^*_{S^1}(M/\!\!/\!\!/\!\!/G) \cong \frac{\widehat{H}^*_{S^1}(M/\!\!/\!\!/\!\!/T)^W}{ann(e)},$$

where $\widehat{H}^*_{S^1}$ denotes rationalized equivariant cohomology (see Definition 4.1), and

$$e = \prod_{\alpha\in\Delta} \alpha(x-\alpha) \in (\operatorname{Sym}\mathfrak{t}^*)^W \otimes \mathbb{Q}[x] \cong H^*_{S^1\times T}(pt)^W \subseteq \widehat{H}^*_{S^1\times T}(pt)^W.$$

Theorem 4.11 describes the image of the non-rationalized Kirwan map in a similar way:

$$H^*_{S^1}(M/\!\!/\!\!/\!\!/G) \supseteq \operatorname{Im}(\kappa_G) \cong \frac{(\operatorname{Im}\kappa_T)^W}{ann(e)},$$

where $\kappa_T : H^*_{S^1\times T}(M) \to H^*_{S^1}(M/\!\!/\!\!/\!\!/T)$ is the Kirwan map for the abelian quotient. In many situations, such as when $M = T^*\mathbb{C}^n$, κ_T is known to be surjective (Theorem 3.16).

This Chapter is a reproduction of [HP, §1-3].

4.1 Integration

The localization theorem of Atiyah-Bott [AB] and Berline-Vergne [BV] says that given a manifold M with a circle action, the restriction map from the circle-equivariant cohomology of M to the circle-equivariant cohomology of the fixed point set F is an isomorphism modulo torsion. In particular, integrals on M can be computed in terms of integrals on F. If F is compact, it is possible to use the Atiyah-Bott-Berline-Vergne formula to *define* integrals on M.

We will work in the category of *circle compact* manifolds, by which we mean oriented S^1-manifolds with compact and oriented fixed point sets. Maps between circle compact manifolds are required to be equivariant.

Definition 4.1 Let $\mathbb{K} = \mathbb{Q}(x)$, the rational function field of $H^*_{S^1}(pt) \cong \mathbb{Q}[x]$. For a circle compact manifold M, let $\widehat{H}^*_{S^1}(M) = H^*_{S^1}(M) \otimes \mathbb{K}$, where the tensor product is taken over the ring $H^*_{S^1}(pt)$. We call $\widehat{H}^*_{S^1}(M)$ the *rationalized* S^1-equivariant cohomology of M. Note that because $\deg(x) = 2$, $\widehat{H}^*_{S^1}(M)$ is supergraded, and supercommutative with respect to this supergrading.

An immediate consequence of [AB] is that restriction gives an isomorphism

$$\widehat{H}^*_{S^1}(M) \cong \widehat{H}^*_{S^1}(F) \cong H^*(F) \otimes_{\mathbb{Q}} \mathbb{K}, \tag{4.1}$$

where $F = M^{S^1}$ denotes the compact fixed point set of M. In particular $\widehat{H}^*_{S^1}(M)$ is a finite dimensional vector space over $\mathbb{K}$, and trivial if and only if F is empty.

Let $i : N \hookrightarrow M$ be a closed embedding. There is a standard notion of proper pushforward

$$i_* : H^*_{S^1}(N) \to H^*_{S^1}(M)$$

given by the formula $i_* = r \circ \Phi$, where $r : H^*_{S^1}(M, M \setminus N) \to H^*_{S^1}(M)$ is the restriction map, and $\Phi : H^*_{S^1}(N) \to H^*_{S^1}(M, M \setminus N)$ is the Thom isomorphism. We will also denote the induced map $\widehat{H}^*_{S^1}(N) \to \widehat{H}^*_{S^1}(M)$ by i_*. Geometrically, i_* can be understood as the inclusion of cycles in Borel-Moore homology.

This map satisfies two important formal properties [AB]:

$$\text{Functoriality: } (i \circ j)_* = i_* \circ j_* \tag{4.2}$$

$$\text{Module homomorphism: } i_*(\gamma \cdot i^*\alpha) = i_\gamma \cdot \alpha \text{ for all } \alpha \in \widehat{H}^*_{S^1}(M), \gamma \in \widehat{H}^*_{S^1}(N). \tag{4.3}$$

We will denote the Euler class $i^*i_*(1) \in \widehat{H}^*_{S^1}(N)$ by $e(N)$. If a class $\gamma \in \widehat{H}^*_{S^1}(N)$ is in the image of i^*, then property (4.3) tells us that $i^*i_*\gamma = e(N)\gamma$. Since the pushforward construction is local in a neighborhood of N in M, we may assume that i^* is surjective, hence this identity holds for all $\gamma \in \widehat{H}^*_{S^1}(N)$.

Let $F = M^{S^1}$ be the fixed point set of M. Since M and F are each oriented, so is the normal bundle to F inside of M. The following result is standard, see e.g. [Ki].

Lemma 4.2 *The Euler class $e(F) \in \widehat{H}^*_{S^1}(F)$ of the normal bundle to F in M is invertible.*

Proof: Let $\{F_1, \ldots, F_d\}$ be the connected components of F. Since $\widehat{H}^*_{S^1}(F) \cong \bigoplus \widehat{H}^*_{S^1}(F_i)$ and $e(F) = \oplus e(F_i)$, our statement is equivalent to showing that $e(F_i)$ is invertible for all i. Since S^1 acts trivially on F_i, $\widehat{H}^*_{S^1}(F_i) \cong H^*(F_i) \otimes_{\mathbb{Q}} \mathbb{K}$. We have $e(F_i) = 1 \otimes ax^k + nil$, where $k = \operatorname{codim}(F_i)$, a is the product of the weights of the S^1 action on any fiber of the normal bundle, and nil consists of terms of positive degree in $H^*(F_i)$. Since F_i is a component of the fixed point set, S^1 acts freely on the complement of the zero section of the normal bundle, therefore $a \neq 0$. Since ax^k is invertible and nil is nilpotent, we are done. □

Definition 4.3 For $\alpha \in \widehat{H}^*_{S^1}(M)$, let

$$\int_M \alpha = \int_F \frac{\alpha|_F}{e(F)} \in \mathbb{K}.$$

Note that this definition does not depend on our choice of orientation of F. Indeed, reversing the orientation of F has the effect of negating $e(F)$, *and* introducing a second factor of -1 coming from the change in fundamental class. These two effects cancel.

For this definition to be satisfactory, we must be able to prove the following lemma, which is standard in the setting of ordinary cohomology of compact manifolds.

Lemma 4.4 *Let $i : N \hookrightarrow M$ be a closed immersion. Then for any $\alpha \in \widehat{H}^*_{S^1}(M), \gamma \in \widehat{H}^*_{S^1}(N)$, we have $\int_M \alpha \cdot i_*\gamma = \int_N i^*\alpha \cdot \gamma$.*

Proof: Let $G = N^{S^1}$, let $j : G \to F$ denote the restriction of i to G, and let $\phi : F \to M$ and $\psi : G \to N$ denote the inclusions of F and G into M and N, respectively.

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ \psi \uparrow & & \uparrow \phi \\ G & \xrightarrow{j} & F \end{array}$$

Then

$$\int_M \alpha \cdot i_*\gamma = \int_F \frac{\phi^*\alpha \cdot \phi^* i_*\gamma}{e(F)},$$

and

$$\int_N i^*\alpha \cdot \gamma = \int_G \frac{\psi^* i^*\alpha \cdot \psi^*\gamma}{e(G)} = \int_G \frac{j^*\phi^*\alpha \cdot \psi^*\gamma}{e(G)} = \int_F \phi^*\alpha \cdot j_*\left(\frac{\psi^*\gamma}{e(G)}\right),$$

where the last equality is simply the integration formula applied to the map $j : G \to F$ of compact manifolds [AB]. Hence it will be sufficient to prove that

$$\phi^* i_*\gamma = e(F) \cdot j_*\left(\frac{\psi^*\gamma}{e(G)}\right) \in \widehat{H}^*_{S^1}(F).$$

To do this, we will show that the difference of the two classes lies in the kernel of ϕ_*, which we know is trivial because the composition $\phi^*\phi_*$ is given by multiplication by the invertible class $e(F) \in \widehat{H}^*_{S^1}(F)$. On the left hand side we get

$$\phi_*\phi^* i_*\gamma = \phi_*(1) \cdot i_*\gamma \quad \text{by (4.3)},$$

and on the right hand side we get

$$\begin{aligned}
\phi_*\left(e(F) \cdot j_*\left(\frac{\psi^*\gamma}{e(G)}\right)\right) &= \phi_*\left(\phi^*\phi_*(1) \cdot j_*\left(\frac{\psi^*\gamma}{e(G)}\right)\right) \\
&= \phi_*(1) \cdot \phi_* j_*\left(\frac{\psi^*\gamma}{e(G)}\right) && \text{by (4.3)} \\
&= \phi_*(1) \cdot i_*\psi_*\left(\frac{\psi^*\gamma}{e(G)}\right) && \text{by (4.2).}
\end{aligned}$$

It thus remains only to show that $\gamma = \psi_*\left(\frac{\psi^*\gamma}{e(G)}\right)$. This is seen by applying ψ^* to both sides, which is an isomorphism (working over the field $\mathbb{K}$) by [AB]. □

For $\alpha_1, \alpha_2 \in \widehat{H}^*_{S^1}(M)$, consider the symmetric, bilinear, $\mathbb{K}$-valued pairing

$$\langle \alpha_1, \alpha_2 \rangle_M = \int_M \alpha_1\alpha_2.$$

Lemma 4.5 (Poincaré Duality) *This pairing is nondegenerate.*

Proof: Suppose that $\alpha \in \widehat{H}^*_{S^1}(M)$ is nonzero, and therefore $\phi^*\alpha \neq 0$. Since F is compact, there must exist a class $\gamma \in \widehat{H}^*_{S^1}(F)$ such that $0 \neq \int_F \phi^*\alpha \cdot \gamma = \int_M \alpha \cdot \phi_*\gamma = \langle \alpha, \phi_*\gamma \rangle_M$. □

Definition 4.6 For an arbitrary equivariant map $f : N \to M$, we may now define the pushforward

$$f_* : \widehat{H}^*_{S^1}(N) \to \widehat{H}^*_{S^1}(M)$$

to be the adjoint of f^* with respect to the pairings $\langle \cdot, \cdot \rangle_N$ and $\langle \cdot, \cdot \rangle_M$. This is well defined because, according to (4.1), $\widehat{H}^*_{S^1}(M)$ and $\widehat{H}^*_{S^1}(N)$ are finite dimensional vector spaces over the field $\mathbb{K}$. Lemma 4.4 tells us that this definition generalizes the definition for closed immersions. Furthermore, properties (4.2) and (4.3) for pushforwards along arbitrary maps are immediate corollaries of the definition. If f is a projection, then f_* will be given by integration along the fibers. Using the fact that every map factors through its graph as a closed immersion and a projection, we always have a geometric interpretation of the pushforward.

As an application, let us consider the manifold $M \times M$, along with the two projections π_1 and π_2, and the diagonal map $\mathrm{Diag} : M \to M \times M$. Suppose that we can write

$$\mathrm{Diag}_*(1) = \sum \pi_1^* a_i \cdot \pi_2^* b_i$$

for a finite collection of classes $a_i, b_i \in \widehat{H}^*_{S^1}(M)$. The following Proposition will be used in Section 5.1.

Proposition 4.7 *The set $\{b_i\}$ is an additive spanning set for $\widehat{H}^*_{S^1}(M)$.*

Proof: For any $\alpha \in \widehat{H}^*_{S^1}(M)$, we have

$$\begin{aligned}
\alpha &= \mathrm{id}_* \,\mathrm{id}^* \,\alpha \\
&= (\pi_2 \circ \mathrm{Diag})_* (\pi_1 \circ \mathrm{Diag})^* \alpha \\
&= \pi_{2*}\big(\, \mathrm{Diag}_* \,(1 \cdot \mathrm{Diag}^* \,\pi_1^* \alpha)\,\big) \\
&= \pi_{2*}\big(\pi_1^* \alpha \cdot \mathrm{Diag}_*(1)\big) \\
&= \pi_{2*}\left(\sum \pi_1^*(a_i \alpha) \cdot \pi_2^* b_i\right) \\
&= \sum \pi_{2*} \pi_1^*(a_i \alpha) \cdot b_i \\
&= \sum \langle a_i, \alpha \rangle \cdot b_i,
\end{aligned}$$

hence α is in the span of $\{b_i\}$. $\square$

4.2 Hyperkähler abelianization

Let M be a hyperkähler manifold with a circle action, and suppose that a compact Lie group G acts hyperhamiltonianly on M with hyperkähler moment map

$$\mu_G = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : M \to \mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}},$$

where $\mu_{\mathbb{C}}$ is holomorphic with respect to the distinguished complex structure I. We require that the action of G commute with the action of S^1, that $\mu_{\mathbb{R}}$ is S^1-invariant, and that $\mu_{\mathbb{C}}$ is S^1-equivariant with respect to the action of S^1 on $\mathfrak{g}^*_{\mathbb{C}}$ by complex multiplication.

Let $T \subseteq G$ be a maximal torus, and let $pr : \mathfrak{g}^* \to \mathfrak{t}^*$ be the natural projection. Then T acts on M with hyperkähler moment map

$$\mu_T = pr \circ \mu_{\mathbb{R}} \oplus pr_{\mathbb{C}} \circ \mu_{\mathbb{C}} : M \to \mathfrak{t}^* \oplus \mathfrak{t}^*_{\mathbb{C}}.$$

Let $\xi \in \mathfrak{g}^*$ be a central element such that $(\xi, 0)$ is a regular value of μ_G and $(pr(\xi), 0)$ is a regular value of μ_T. Assume further that G acts freely on $\mu_G^{-1}(\xi, 0)$, and T acts freely on $\mu_T^{-1}(pr(\xi), 0)$.[2] Let

$$M /\!\!/\!\!/\!\!/ G = \mu_G^{-1}(\xi, 0)/G \quad \text{and} \quad M /\!\!/\!\!/\!\!/ T = \mu_T^{-1}(pr(\xi), 0)/T$$

be the hyperkähler quotients of M by G and T, respectively. Because μ_G and μ_T are circle-equivariant, the action of S^1 on M descends to actions on the hyperkähler quotients. Note that $M /\!\!/\!\!/\!\!/ T$ also inherits an action of the Weyl group W of G.

Example 4.8 The main example to keep in mind is $M = T^*\mathbb{C}^n$, where S^1 acts on M by scalar multiplication on the fibers and the action of G on M is induced by a linear action of G on $\mathbb{C}^n$, as in Chapter 2.

Consider the Kirwan maps

$$\kappa_G : H^*_{S^1 \times G}(M) \to H^*_{S^1}(M /\!\!/\!\!/\!\!/ G) \quad \text{and} \quad \kappa_T : H^*_{S^1 \times T}(M) \to H^*_{S^1}(M /\!\!/\!\!/\!\!/ T),$$

induced by the inclusions of $\mu_G^{-1}(\xi, 0)$ and $\mu_T^{-1}(pr(\xi), 0)$ into M, along with their rationalizations

$$\hat{\kappa}_G : \widehat{H}^*_{S^1 \times G}(M) \to \widehat{H}^*_{S^1}(M /\!\!/\!\!/\!\!/ G) \quad \text{and} \quad \hat{\kappa}_T : \widehat{H}^*_{S^1 \times T}(M) \to \widehat{H}^*_{S^1}(M /\!\!/\!\!/\!\!/ T).$$

[2] We make this simplifying assumption in order to talk about manifolds, rather than orbifolds, which makes the integration formulae easier to state. In fact, Theorems 4.10 and 4.11 generalize easily to the orbifold case, as in [Ma, §6].

Let

$$r_T^G : \widehat{H}^*_{S^1\times G}(M) \to \widehat{H}^*_{S^1\times T}(M)^W$$

be the standard isomorphism.

Let $\Delta = \Delta^+ \sqcup \Delta^- \subset \mathfrak{t}^*$ be the set of roots of G. Let

$$e = \prod_{\alpha\in\Delta} \alpha(x-\alpha) \in (\mathrm{Sym}\,\mathfrak{t}^*)^W \otimes \mathbb{Q}[x] \cong H_{S^1\times G}(pt) \subseteq \widehat{H}_{S^1\times G}(pt),$$

and

$$e' = \prod_{\alpha\in\Delta^-} \alpha \cdot \prod_{\alpha\in\Delta} (x-\alpha) \in \mathrm{Sym}\,\mathfrak{t}^* \otimes \mathbb{Q}[x] \cong H_{S^1\times T}(pt) \subseteq \widehat{H}_{S^1\times T}(pt).$$

The following two theorems are analogues of Theorems B and A of [Ma], adapted to circle compact hyperkähler quotients. Our proofs follow closely those of Martin.

Theorem 4.9 *Suppose that $M/\!\!/\!\!/\!\!/G$ and $M/\!\!/\!\!/\!\!/T$ are both circle compact. If $\gamma \in \widehat{H}^*_{S^1\times G}(X)$, then*

$$\int_{X/\!\!/\!\!/\!\!/G} \hat{\kappa}_G(\gamma) = \frac{1}{|W|}\int_{X/\!\!/\!\!/\!\!/T} \hat{\kappa}_T \circ r_T^G(\gamma)\cdot e.$$

Theorem 4.10 *Suppose that $M/\!\!/\!\!/\!\!/G$ and $M/\!\!/\!\!/\!\!/T$ are both circle compact, and that the rationalized Kirwan map $\hat{\kappa}_G$ is surjective. Then*

$$\widehat{H}^*_{S^1}(M/\!\!/\!\!/\!\!/G) \cong \frac{\widehat{H}^*_{S^1}(M/\!\!/\!\!/\!\!/T)^W}{ann(e)} \cong \left(\frac{\widehat{H}^*_{S^1}(M/\!\!/\!\!/\!\!/T)}{ann(e')}\right)^W.$$

Proof of 4.9: Consider the following pair of maps:

$$\begin{array}{ccc} \mu_G^{-1}(\xi,0)/T & \xrightarrow{\ i\ } & \mu_T^{-1}(pr(\xi),0)/T \cong M/\!\!/\!\!/\!\!/T \\ \pi\downarrow & & \\ \mu_G^{-1}(\xi,0)/G \cong M/\!\!/\!\!/\!\!/G. & & \end{array}$$

Each of these spaces is a complex S^1-manifold with a compact, complex fixed point set, and therefore satisfies the hypotheses of Section 4.1. Let

$$b = \prod_{\alpha\in\Delta^+} \alpha \in H^*_{S^1\times T}(pt)$$

be the product of the positive roots of G, which we will think of as an element of $\widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)$. Martin shows that $\pi_* i^* b = |W|$, and that $i^* \circ \hat{\kappa}_T \circ r^G_T = \pi^* \hat{\kappa}_G$ [Ma], hence we have

$$\begin{aligned}
\int_{X/\!\!/\!\!/\!\!/G} \hat{\kappa}_G(\gamma) &= \frac{1}{|W|} \int_{X/\!\!/\!\!/\!\!/G} \hat{\kappa}_G(\gamma) \cdot \pi_* i^* b \\
&= \frac{1}{|W|} \int_{\mu_G^{-1}(\xi,0)/T} \pi^* \hat{\kappa}_G(\gamma) \cdot i^* b \qquad \text{by Definition 4.6} \\
&= \frac{1}{|W|} \int_{\mu_G^{-1}(\xi,0)/T} i^* \circ \hat{\kappa}_T \circ r^G_T(\gamma) \cdot i^* b \\
&= \frac{1}{|W|} \int_{X/\!\!/\!\!/\!\!/T} \hat{\kappa}_T \circ r^G_T(\gamma) \cdot b \cdot i_*(1) \qquad \text{by Lemma 4.4.}
\end{aligned}$$

It remains to compute $i_*(1) \in \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)$. For $\alpha \in \Delta$, let

$$L_\alpha = \mu_T^{-1}((pr(\xi), 0) \times_T \mathbb{C}_\alpha$$

be the line bundle on $M/\!\!/\!\!/\!\!/T$ with S^1-equivariant Euler class α. Similarly, let L_x be the (topologically trivial) line bundle with S^1-equivariant Euler class x. Following the idea of [Ma, 1.2.1], we observe that the restriction of $\mu_G - (\xi, 0)$ to $\mu_T^{-1}(pr(\xi), 0)$ defines an $S^1 \times T$-equivariant map

$$s : \mu_T^{-1}(pr(\xi), 0) \to V \oplus V_{\mathbb{C}},$$

where $V = pr^{-1}(0)$ and $V_{\mathbb{C}} = pr_{\mathbb{C}}^{-1}(0)$. This descends to an S^1-equivariant section of the bundle

$$E = \mu_T^{-1}(pr(\xi), 0) \times_T (V \oplus V_{\mathbb{C}})$$

with zero locus $\mu_G^{-1}(\xi, 0)/T$. The fact that $(\xi, 0)$ is a regular value implies that this section is generic, hence the equivariant Euler class $e(E) \in \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)$ is equal to $i_*(1)$.

The vector space V is isomorphic as a T-representation to $\bigoplus_{\alpha \in \Delta^-} \mathbb{C}_\alpha$, with S^1 acting trivially. Similarly, $V_{\mathbb{C}}$ is isomorphic to $V \otimes \mathbb{C} \cong V \oplus V^*$, with S^1 acting diagonally by scalars. Hence

$$\begin{aligned}
E &\cong \bigoplus_{\alpha \in \Delta^-} L_\alpha \oplus \bigoplus_{\alpha \in \Delta^-} (L_x \otimes L_\alpha) \oplus (L_x \otimes L_{-\alpha}) \\
&\cong \bigoplus_{\alpha \in \Delta^-} L_\alpha \oplus \bigoplus_{\alpha \in \Delta} L_x \otimes L_{-\alpha},
\end{aligned}$$

and therefore

$$i_*(1) = e(E) = \prod_{\alpha \in \Delta^-} \alpha \cdot \prod_{\alpha \in \Delta} (x - \alpha) = e'.$$

Multiplying by b we obtain e, and the theorem is proved. $\square$

Proof of 4.10: Observe that the restriction of π^* to the Weyl-invariant part $\widehat{H}^*_{S^1}\left(\mu_G^{-1}(\xi,0)/T\right)^W$ is given by the composition of isomorphisms

$$\widehat{H}^*_{S^1}\left(\mu_G^{-1}(\xi,0)/T\right)^W \cong \widehat{H}^*_{S^1\times T}\left(\mu_G^{-1}(\xi,0)\right)^W \cong \widehat{H}^*_{S^1\times G}\left(\mu_G^{-1}(\xi,0)\right) \cong \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/G),$$

hence we may define

$$i^*_W := (\pi^*)^{-1}\circ i^* : \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W \to \widehat{H}^*_{S^1}\left(\mu_G^{-1}(\xi,0)/T\right)^W.$$

Furthermore, we have $\hat{\kappa}_G = i^*_W \circ \hat{\kappa}_T \circ r^G_T$, hence i^*_W is surjective. As in [Ma, §3],

$$\begin{aligned}
i^*_W(a) = 0 \quad &\Leftrightarrow \quad \forall c \in \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W, \int_{X/\!\!/\!\!/\!\!/G} i^*_W(c)\cdot i^*_W(a) = 0 \quad \text{by 4.5 and surjectivity of } i^*_W \\
&\Leftrightarrow \quad \forall c \in \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W, \int_{X/\!\!/\!\!/\!\!/T} c\cdot a\cdot e = 0 \quad \text{by Theorem 4.9} \\
&\Leftrightarrow \quad \forall d \in \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T), \int_{X/\!\!/\!\!/\!\!/T} d\cdot a\cdot e = 0 \quad \text{by using } W \text{ to average } d \\
&\Leftrightarrow \quad a\cdot e = 0 \quad \text{by Lemma 4.5,}
\end{aligned}$$

hence $\operatorname{Ker} i^*_W = ann(e)$. By surjectivity of i^*_W,

$$\widehat{H}^*_{S^1}(M/\!\!/\!\!/\!\!/G) \;\cong\; \frac{\widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W}{\operatorname{Ker} i^*_W} \cong \frac{\widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W}{ann(e)}.$$

By a second application of Lemma 4.5, for any $a \in \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)$, we have

$$\begin{aligned}
i^*(a) = 0 \quad &\Rightarrow \quad \forall f \in \widehat{H}^*_{S^1}(\mu_G^{-1}(\xi,0)/T), \int_{\mu_G^{-1}(\xi,0)/T} f\cdot i^*(a) = 0 \\
&\Rightarrow \quad \forall c \in \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T), \int_{\mu_G^{-1}(\xi,0)/T} i^*(c)\cdot i^*(a) = 0 \\
&\Rightarrow \quad \forall c \in \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T), \int_{X/\!\!/\!\!/\!\!/T} c\cdot a\cdot i_*(1) = 0 \quad \text{by Lemma 4.4} \\
&\Rightarrow \quad a\cdot e' = a\cdot i_*(1) = 0 \quad \text{by Lemma 4.5,}
\end{aligned}$$

hence $\operatorname{Ker} i^* \subseteq ann(e')$. This gives us a natural surjection

$$\frac{\widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W}{ann(e)} = \frac{\widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W}{\operatorname{Ker} i^*_W} \cong \left(\frac{\widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)}{\operatorname{Ker} i^*}\right)^W \to \left(\frac{\widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)}{ann(e')}\right)^W,$$

which is also injective because e' divides e. This completes the proof of Theorem 4.10. $\square$

For the non-rationalized version of Theorem 4.10, we make the additional assumption that $M/\!\!/\!\!/\!\!/G$ and $M/\!\!/\!\!/\!\!/T$ are equivariantly formal S^1-manifolds, i.e. that $H^*_{S^1}(X/\!\!/\!\!/\!\!/G)$ and $H^*_{S^1}(X/\!\!/\!\!/\!\!/T)$ are free modules over $H^*_{S^1}(pt)$. Proposition 2.10 tells us that this is the case whenever the circle action is hamiltonian and its moment map is proper and bounded below.

Theorem 4.11 *Suppose that $M/\!\!/\!\!/\!\!/G$ and $M/\!\!/\!\!/\!\!/T$ are equivariantly formal, circle compact, and that the rationalized Kirwan map $\hat{\kappa}_G$ is surjective. Then*

$$H^*_{S^1}(M/\!\!/\!\!/\!\!/G) \supseteq \mathrm{Im}(\kappa_G) \cong \frac{(\mathrm{Im}\,\kappa_T)^W}{ann(e)} \cong \left(\frac{\mathrm{Im}\,\kappa_T}{ann(e')}\right)^W.$$

Remark 4.12 In the context of Example 4.8 with $pr \circ \mu$ proper, $M/\!\!/\!\!/\!\!/G$ and $M/\!\!/\!\!/\!\!/T$ are both circle compact and equivariantly formal (Proposition 2.10) and κ_T is always surjective (Theorem 3.16).

Proof of 4.11: Consider the following exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & H^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W \arrow[r, "i^*_W"] \arrow[d] & H^*_{S^1}(X/\!\!/\!\!/\!\!/G) \arrow[d] \\
0 \arrow[r] & \widehat{A} \arrow[r] & \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W \arrow[r, "i^*_W"] & \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/G).
\end{tikzcd}$$

Equivariant formality implies that the downward maps in the above diagram are inclusions, hence the map on top labeled i^*_W is simply the restriction of the map on the bottom to the subring $H^*_{S^1}(X/\!\!/\!\!/\!\!/T) \subseteq \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T)$. We therefore have

$$A = \widehat{A} \cap H^*_{S^1}(X/\!\!/\!\!/\!\!/T)^W = ann(e).$$

Just as in the rationalized case, we have $\kappa_G = i^*_W \circ \kappa_T \circ r^G_T$, hence

$$\mathrm{Im}(\kappa_G) \cong i^*_W \left(\mathrm{Im}\,\kappa_T \circ r^G_T\right) \cong \frac{(\mathrm{Im}\,\kappa_T)^W}{ann(e)}.$$

Now consider the analogous diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d] & H^*_{S^1}(X/\!\!/\!\!/\!\!/T) \arrow[r, "i^*"] \arrow[d] & H^*_{S^1}(\mu_G^{-1}(\xi,0)/T) \arrow[d] \\
0 \arrow[r] & \widehat{B} \arrow[r] & \widehat{H}^*_{S^1}(X/\!\!/\!\!/\!\!/T) \arrow[r, "i^*"] & H^*_{S^1}(\mu_G^{-1}(\xi,0)/T).
\end{tikzcd}$$

Since we have not assumed that $\mu_G^{-1}(\xi, 0)/T$ is equivariantly formal, we only know that the first two downward arrows are inclusions, and hence can only conclude that B is contained in the annihilator of e'. Since e' divides e, we have a series of natural surjections

$$\frac{(\operatorname{Im}\kappa_T)^W}{ann(e)} \cong \frac{(\operatorname{Im}\kappa_T)^W}{A} \cong \left(\frac{\operatorname{Im}\kappa_T}{B}\right)^W \to \left(\frac{\operatorname{Im}\kappa_T}{ann(e')}\right)^W \to \left(\frac{\operatorname{Im}\kappa_T}{ann(e)}\right)^W.$$

The composition of these maps is an isomorphism, hence so is each one. $\square$

Chapter 5

Hyperpolygon spaces

A hyperpolygon space is the hyperkähler analogue of a polygon space, which parameterizes n-sided polygons in $\mathbb{R}^3$ with fixed edge lengths. It is also an example of a quiver variety, introduced by Nakajima [N1, N2], and since studied by many authors. In Section 5.1 we give the basic constructions of quiver varieties and hyperpolygon spaces, and show that they satisfy all of the hypotheses of Chapter 4. Section 5.2 is devoted to understanding the components of the core of a hyperpolygon space; in particular, we show that they are all smooth (Theorem 5.7), and interpret them as moduli spaces of spatial polygons with certain properties (Theorem 5.11). Sections 5.3 and 5.4 contain computations of the S^1-equivariant cohomology rings of the hyperpolygon space as well as its core components, making use of the abelianization technique of Chapter 4.

This chapter is taken from [HP2] and [HP]. The reader is warned that our notation differs significantly from that of [HP2]; most glaring is the fact that our spaces $\mathfrak{X}$ and $\mathfrak{M}$ correspond to the spaces M and X (respectively) in [HP2]. This abrupt switch is necessary to conform with the conventions of Chapters 2-4.

5.1 Quiver varieties

Let Q be a quiver with vertex set I and edge set $E \subseteq I \times I$, where $(i, j) \in E$ means that Q has an arrow pointing from i to j. We assume that Q is connected and has no oriented cycles. Suppose given two collections of vector spaces $\{V_i\}$ and $\{W_i\}$, each indexed by I, and consider the affine space

$$\mathrm{Rep}(Q) = \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i\in I} \mathrm{Hom}(V_i, W_i).$$

The group $U(V) = \prod_{i\in I} U(V_i)$ acts on $\mathrm{Rep}(Q)$ by conjugation, and this action is hamiltonian. Given an element

$$(B, J) = \bigoplus_{(i,j)\in E} B_{ij} \oplus \bigoplus_{i\in I} J_i$$

of $\mathrm{Rep}(Q)$, the $\mathfrak{u}(V_i)^*$ component of the moment map is

$$\mu_i(B, J) = J_i^\dagger J_i + \sum_{(i,j)\in E} B_{ij}^\dagger B_{ij},$$

where $\dagger$ denotes adjoint, and $\mathfrak{u}(V_i)^*$ is identified with with the set of hermitian matrices via the trace form. Given a generic central element $\xi \in \mathfrak{u}(V)^*$, the Kähler quotient $\mathrm{Rep}(Q)/\!\!/_{\xi} U(V)$ parameterizes isomorphism classes of ξ-stable, framed representations of Q of fixed dimension [N2]. If $W_i = 0$ for all i, then the diagonal circle $U(1)$ in the center of $U(V)$ acts trivially, and we instead quotient by $PU(V) = U(V)/U(1)$.

Consider the hyperkähler quotient

$$\mathfrak{M} = T^* \mathrm{Rep}(Q)/\!\!/\!\!/\!\!/_{(\xi,0)} U(V),$$

or, if $W_i = 0$ for all i,

$$\mathfrak{M} = T^* \mathrm{Rep}(Q)/\!\!/\!\!/\!\!/_{(\xi,0)} PU(V).$$

As in Section 2.2, $\mathfrak{M}$ has a natural action of $\mathbb{C}^\times$ induced from scalar multiplication on the fibers of $T^* \mathrm{Rep}(Q)$. We now show that $M = T^* \mathrm{Rep}(Q)$ satisfies the hypotheses of Theorems 4.10 and 4.11.

Proposition 5.1 *Let $T(V) \subseteq U(V)$ be a maximal torus, and let $pr : \mathfrak{u}(V)^* \to \mathfrak{t}(V)^*$ be the natural projection. The moment maps $\mu = \bigoplus_{i\in I} \mu_i : \mathrm{Rep}(Q) \to \mathfrak{u}(V)^*$ and $pr \circ \mu : \mathrm{Rep}(Q) \to \mathfrak{t}(V)^*$ are each proper.*

Proof: To show that μ and $pr \circ \mu$ is proper, it suffices to find an element $t \in T(V) \subseteq U(V)$ such that the weights of the action of t on $\mathrm{Rep}(Q)$ are all strictly positive. Let $\lambda = \{\lambda_i \mid i \in I\}$ be a collection of integers, and let $t \in T(V)$ be the central element of $U(V)$ that acts on V_i with weight λ_i for all i. Then t acts on $\mathrm{Hom}(V_i, V_j)$ with weight $\lambda_j - \lambda_i$, and on

$\mathrm{Hom}(V_i, W_i)$ with weight $-\lambda_i$. Hence we have reduced the problem to showing that it is possible to choose λ such that $\lambda_i < 0$ for all $i \in I$ and $\lambda_i < \lambda_j$ for all $(i, j) \in E$.

We proceed by induction on the order of I. Since Q has no oriented cycles, there must exist a source $i \in I$; a vertex such that for all $j \in I$, $(j, i) \notin E$. Deleting i gives a smaller (possibly disconnected) quiver with no oriented cycles, and therefore we may choose $\{\lambda_j \mid j \in I \smallsetminus \{i\}\}$ such that $\lambda_j < 0$ for all $j \in I \smallsetminus \{i\}$ and $\lambda_j < \lambda_k$ for all $(j, k) \in E$. We then choose $\lambda_i < \min\{\lambda_j \mid j \in I \smallsetminus \{i\}\}$, and we are done. □

Proposition 5.2 *The rationalized Kirwan map* $\hat{\kappa}_{U(V)} : \widehat{H}^*_{S^1 \times U(V)}\Big(T^* \operatorname{Rep}(Q)\Big) \to \widehat{H}^*_{S^1}(\mathfrak{M})$ *is surjective.*

Proof: Nakajima [N2, §7.3] shows that there exist cohomology classes a_i, b_i in the image of $\hat{\kappa}_{U(V)}$ such that $\mathrm{Diag}_*(1) = \sum \pi_1^* a_i \cdot \pi_2^* b_i$. (Nakajima uses a slightly different circle action, but his proof is easily adapted to the circle action that we have defined.) It follows from Proposition 4.7 that the classes $\{b_i\}$ generate $\widehat{H}^*_{S^1}(\mathfrak{M})$. □

Remark 5.3 This Proposition shows that the assumptions of Theorems 4.9, 4.10, and 4.11 are satisfied for Nakajima's quiver varieties. Thus integration in equivariant cohomology yields a description of the rationalized S^1-equivariant cohomology, and also of the image of the non-rationalized Kirwan map κ_G. Therefore if we know that κ_G is surjective for a particular quiver variety, then we have a concrete description of the (S^1-equivariant) cohomology ring of that quiver variety. It is known that κ_G is surjective for Hilbert schemes of n points on an ALE space, so our theory applies and gives a description of the cohomology ring of these quiver varieties. More examples of quiver varieties with surjective Kirwan map, including hyperpolygon spaces, are discussed in Remark 5.16.

Remark 5.4 Another interesting application of Proposition 4.7 is to the moduli space $\mathcal{M}$ of stable rank n and degree 1 Higgs bundles on a genus $g > 1$ smooth projective algebraic curve C (see [H2]). It is an easy exercise to write down the cohomology class of the diagonal in $\mathcal{M} \times \mathcal{M}$ as an expression in a certain set of tautological classes. Proposition 4.7 implies that the rationalized S^1-equivariant cohomology ring $\widehat{H}^*_{S^1}(\mathcal{M})$ is generated by these classes. In fact the same result follows from the argument of [HT1]. There $\mathcal{M}$ was embedded into a circle compact manifold $\mathcal{M}_\infty$, whose cohomology is the free algebra on the tautological classes. The argument in [HT1] then goes by showing that the embedding of the S^1-fixed point set of $\mathcal{M}$ in that of $\mathcal{M}_\infty$ induces a surjection on cohomology. This already implies that $\widehat{H}^*_{S^1}(\mathcal{M}_\infty)$ surjects onto $\widehat{H}^*_{S^1}(\mathcal{M})$. In [HT1] it is shown that in the rank 2 case this embedding also implies the surjection on non-rationalized cohomology, and then a companion paper [HT2] describes the cohomology ring of $\mathcal{M}$ explicitly. However for higher rank this part of the argument of [HT1] breaks down. Later Markman [Mk] used similar diagonal arguments on certain compactifications of $\mathcal{M}$ and Hironaka's celebrated theorem on desingularization of algebraic varieties to deduce that the cohomology ring of $\mathcal{M}$ is generated by tautological classes for all n.

A *hyperpolygon space*, introduced in [K2], is a quiver variety associated to the following quiver (Figure 5.1), with $V_0 = \mathbb{C}^2$, $V_i = \mathbb{C}^1$ for $i \in \{1, \ldots, n\}$, and $W_i = 0$ for all i.

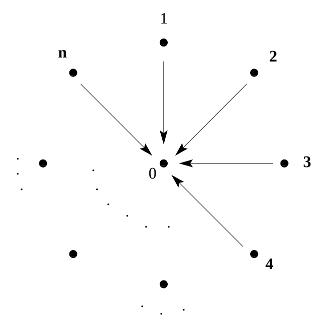

Figure 5.1: The quiver for a hyperpolygon space.

Let

$$G := PU(V) = \Big(SU(2) \times U(1)^n\Big)\Big/\mathbb{Z}_2,$$

and

$$G_{\mathbb{C}} := PGL(V) = \Big(SL(2,\mathbb{C}) \times (\mathbb{C}^\times)^n\Big)\Big/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts by multiplying each factor by -1. We represent an element of $\mathrm{Rep}(Q) \cong \mathbb{C}^{2n}$ by an n-tuple of column vectors

$$q = (q_1, \ldots, q_n).$$

Following the conventions in [K2], we consider the *right* action of $G \subseteq G_{\mathbb{C}}$ on $\mathrm{Rep}(Q)$ given explicitly by

$$q[\Theta; e_1, \ldots, e_n] = (\Theta^{-1} q_1 e_1, \ldots, \Theta^{-1} q_n e_n).$$

The compact group G acts with moment map $\mu : \mathbb{C}^{2n} \to \mathfrak{su}(2)^* \oplus (\mathfrak{t}^n)^*$ given by the equation

$$\mu(q) = \sum_{i=1}^{n} (q_i q_i^*)_0 \oplus \left(\frac{1}{2}|q_1|^2, \ldots, \frac{1}{2}|q_n|^2 \right),$$

where q_i^* denotes the conjugate transpose of q_i, $(q_i q_i^*)_0$ denotes the traceless part of $q_i q_i^*$, and $\mathfrak{su}(2)^*$ is identified with $i \cdot \mathfrak{su}(2)$ via the trace form. Given an n-tuple of real numbers $(\alpha_1, \ldots, \alpha_n)$, we define the *polygon space*

$$\mathfrak{X}(\alpha) := \mathbb{C}^{2n} /\!\!/_{\alpha} G,$$

where $\alpha = 0 \oplus (\alpha_1, \ldots, \alpha_n) \in \mathfrak{su}(2)^* \oplus (\mathfrak{t}^n)^*$. If we break the reduction into two steps, reducing first by $U(1)^n$ and then by $SU(2)$, we find that

$$\mathfrak{X}(\alpha) \cong \left\{ (v_1, \ldots, v_n) \in (\mathbb{R}^3)^n \;\middle|\; \|v_i\| = \alpha_i \text{ and } \sum v_i = 0 \right\} \Big/ SO(3) \tag{5.1}$$

(see Remark 5.12 and the proof of Theorem 5.11). Here $\mathfrak{su}(2)^*$ is being identified with $\mathbb{R}^3$, and the coadjoint action of $SU(2)$ on $\mathfrak{su}(2)^*$ is being replaced by the standard action of $SO(3)$ on $\mathbb{R}^3$ [HK]. This space, therefore, may be thought of as the moduli space of n-sided polygons in $\mathbb{R}^3$, with fixed edge lengths $(\alpha_1, \ldots, \alpha_n)$, up to rotation. In particular, $\mathfrak{X}(\alpha)$ is empty unless $\alpha_i \geq 0$ for all i.

We call α *generic* if there does not exist a subset $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} \alpha_i = \sum_{j \in S^c} \alpha_j$. Geometrically, this means that there is no element of $\mathfrak{X}(\alpha)$ represented by a polygon that is contained in a single line in $\mathbb{R}^3$. If α is generic, then $\mathfrak{X}(\alpha)$ is smooth [HK]. Throughout this paper we will assume that α is generic, and that $\alpha_i > 0$ for all i.

To define the hyperkähler analogue of $\mathfrak{X}(\alpha)$, we consider the induced action of G on $T^*\mathbb{C}^n$. Explicitly, we write an element of $T^* \operatorname{Rep}(Q)$ as (p, q), where $q = (q_1, \ldots, q_n)$ is an n-tuple of column vectors and $p = (p_1, \ldots, p_n)$ an n-tuple of row vectors, and we put

$$(p, q)[\Theta; e_1, \ldots, e_n] = \big((e_1^{-1} p_1 \Theta, \ldots, e_n^{-1} p_n \Theta), (\Theta^{-1} q_1 e_1, \ldots, \Theta^{-1} q_n e_n) \big).$$

The action of G on $T^* \operatorname{Rep}(Q)$ is hyperhamiltonian with hyperkähler moment map

$$\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : T^*\mathbb{C}^n \to \Big(\mathfrak{su}(2)^* \oplus (\mathfrak{t}^n)^* \Big) \oplus \Big(\mathfrak{sl}(2, \mathbb{C})^* \oplus (\mathfrak{u}(1)^n_{\mathbb{C}})^* \Big)$$

given by the equations

$$\mu_{\mathbb{R}}(p, q) = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} (q_i q_i^* - p_i^* p_i)_0 \oplus \left(\frac{1}{2} \left(|q_1|^2 - |p_1|^2 \right), \ldots, \frac{1}{2} \left(|q_n|^2 - |p_n|^2 \right) \right)$$

and
$$\mu_{\mathbb{C}}(p,q) = -\sum_{i=1}^{n} (q_i p_i)_0 \oplus \left(\sqrt{-1}p_1 q_1, \ldots, \sqrt{-1}p_n q_n\right).$$
We then define the hyperpolygon space to be the hyperkähler quotient

$$\mathfrak{M}(\alpha) := T^*\mathbb{C}^n /\!\!/\!\!/\!\!/_{(\alpha,0)} G = \left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right)\Big/ G,$$

a smooth, noncompact hyperkähler manifold of complex dimension $2(n-3)$. Recall that we also have
$$\mathfrak{X}(\alpha) \cong \left(\mathbb{C}^{2n}\right)^{st} / G_{\mathbb{C}} \qquad \text{and} \qquad \mathfrak{M}(\alpha) \cong \mu_{\mathbb{C}}^{-1}(0)^{st} / G_{\mathbb{C}},$$

where *st* means stable with respect to the weight α in the sense of geometric invariant theory (see Section 2.1).[1] Nakajima gives a stability criterion for general quiver varieties [N1, N2], which Konno interprets in the special case of hyperpolygon spaces. Call a subset $S \subseteq \{1, \ldots, n\}$ *short* if $\sum_{i \in S} \alpha_i < \sum_{j \in S^c} \alpha_j$, and call it *long* if its complement is short. (Assuming that α is generic is equivalent to assuming that every subset is either short or long.) Given a point $(p,q) \in T^*\mathbb{C}^n$ and a subset $S \subseteq \{1, \ldots, n\}$, we will say that S is *straight* in (p,q) if q_i is proportional to q_j for every $i, j \in S$. The terminology comes from Kähler polygon spaces, in which this condition is equivalent to asking that the vectors v_i and v_j be proportional over $\mathbb{R}_+$, or that the edges of lengths α_i and α_j (if they happen to be adjacent) line up to make a single edge of length $\alpha_i + \alpha_j$, as in Figure 5.2.

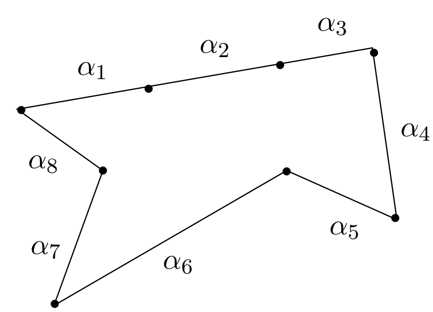

Figure 5.2: The subset $\{1, 2, 3\}$ is straight.

Theorem 5.5 [K2, 4.2] *Suppose that α is generic, and $\alpha_i > 0$ for all i. Then a point $(p,q) \in T^*\mathbb{C}^n$ is stable with respect to α if and only if the following two conditions are*

[1] Recall from Theorem 2.3 that the notions of stability and semistability agree for generic α.

satisfied:

1) $q_i \neq 0$ *for all* i, *and*

2) *if* S *is straight and* $p_j = 0$ *for all* $j \in S^c$, *then* S *is short.*

As in Chapter 2, we will use the notation $[p,q]$ to denote the $G_{\mathbb{C}}$-equivalence class of a point $(p,q) \in \mu_{\mathbb{C}}^{-1}(0)^{st}$, and $[p,q]_{\mathbb{R}}$ to denote the G-equivalence class of a point $(p,q) \in \mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)$. Recall that $\mathfrak{X}$ sits inside of $\mathfrak{M}$ as the locus of points $[p,q]$ with $p = 0$. This observation, along with Theorem 5.5, allows us to recover the stability condition for the action of G on $\mathbb{C}^{2n}$. A point $q \in \mathbb{C}^{2n}$ is stable if and only if $q_i \neq 0$ for all i, and no long subset is straight, as first shown in [Kl]. The polygonally-minded reader is warned that in the hyperpolygon space $\mathfrak{M}(\alpha)$, long subsets *can* be straight.

5.2 Moduli theoretic interpretation of the core

For the rest of the section we fix a generic $\alpha = 0 \oplus (\alpha_1, \ldots, \alpha_n) \in \mathfrak{su}(2)^* \oplus (\mathfrak{t}^n)^*$, with $\alpha_i > 0$ for all i, and write $\mathfrak{M} = \mathfrak{M}(\alpha)$, $\mathfrak{X} = \mathfrak{X}(\alpha)$. Following Konno, we define

$$\mathcal{S} = \big\{ S \subseteq \{1, \ldots, n\} \;\big|\; S \text{ is short} \big\}$$

and

$$\mathcal{S}' = \big\{ S \in \mathcal{S} \;\big|\; |S| \geq 2 \big\}.$$

Theorem 5.6 [K2] *The fixed point set* $\mathfrak{M}^{\mathbb{C}^\times} = \mathfrak{M}^{S^1} = \mathfrak{X} \cup \bigcup_{S \in \mathcal{S}'} \mathfrak{M}_S$, *where*

$$\mathfrak{M}_S = \big\{ [p,q] \;\big|\; S \textit{ and } S^c \textit{ are each straight, and } p_j = 0 \textit{ for all } j \in S^c \big\}.$$

Furthermore, $\mathfrak{M}_S$ *is diffeomorphic to* $\mathbb{C}P^{|S|-2}$.

For all $S \in \mathcal{S}'$, let $U_S = U_{\mathfrak{X}_S}$ be the piece of the core $\mathfrak{L} \subseteq \mathfrak{M}$ defined in Section 2.2. *A priori* we know only that U_S is an irreducible, isotropic subvariety of dimension at most $n-3$ (Proposition 2.8).

Theorem 5.7 *The core component* U_S *is smooth of complex dimension* $n-3$, *and we have*

$$U_S = \Big\{ [p,q] \;\Big|\; S \textit{ is straight, and } p_j = 0 \textit{ for all } j \in S^c \Big\}.$$

Before proving Theorem 5.7, we describe the way in which the various components of the core fit together. For all $S \in \mathcal{S}'$, let

$$\mathfrak{X}_S = U_S \cap \mathfrak{X} = \big\{[0,q] \;\big|\; S \text{ is straight}\big\}.$$

We call this space the *polygon subspace* of $\mathfrak{X}$ corresponding to the short subset S. Note that $\mathfrak{X}_S$ is itself a polygon space with $n - |S| + 1$ edges, of lengths $\{\alpha_j \mid j \in S^c\} \cup \{\sum_S \alpha_i\}$. In particular, it is smooth. Now suppose given two short subsets $S, T \in \mathcal{S}'$, and consider the intersection $U_S \cap U_T$.

- If $S \cap T = \emptyset$, then $U_S \cap U_T = \mathfrak{X}_S \cap X_T$, a polygon subspace both of $\mathfrak{X}_S$ and of X_T.
- If $S \cap T \neq \emptyset$ and $S \cup T$ is long, then $U_S \cap U_T = \emptyset$.
- If $S \cap T \neq \emptyset$ and $S \cup T$ is short, then

  $$U_S \cap U_T = \Big\{[p,q] \;\Big|\; S \cup T \text{ is straight, and } p_j = 0 \text{ for all } j \in (S \cap T)^c\Big\}.$$

  This is a subvariety of $U_{S\cup T}$ given by taking the closure inside of $U_{S\cup T}$ of a certain subbundle of the conormal bundle to $X_{S\cup T} \subseteq \mathfrak{X}$, defined by setting $p_j = 0$ for all $j \in (S \cap T)^c \supseteq (S \cup T)^c$.

Each of these descriptions generalizes to higher intersections without modification.

Finally, we compute the fixed point set $U_S^{\mathbb{C}^\times}$. If $[p,q] \in U_S^{\mathbb{C}^\times}$, then either $p = 0$ and $[p,q] \in \mathfrak{X}_S$, or $[p,q] \in \mathfrak{X}_T$ for some $T \in \mathcal{S}'$. If $[p,q] \in \mathfrak{X}_T$ then Theorem 5.6 tells us that T and T^c are each straight, hence $S \subseteq T$ or $S \subseteq T^c$. Since $p \neq 0$, we must have $S \subseteq T$. Indeed, $U_S \cap \mathfrak{X}_T$ is the linear subspace of $\mathfrak{X}_T \cong \mathbb{C}P^{|T|-2}$ given by the condition $p_j = 0$ for all $j \in T \smallsetminus S$. In particular, $U_S \cap \mathfrak{X}_T$ is isomorphic to $\mathbb{C}P^{|S|-2}$ for any $T \supseteq S$.

Example 5.8 Let $n = 5$, $\alpha_1 = \alpha_2 = 1$, and $\alpha_3 = \alpha_4 = \alpha_5 = 3$, and consider the short subset $S = \{1,2\}$. The fixed point set of U_S consists of $\mathfrak{X}_S \cong \mathbb{C}P^1$, and four points $\mathfrak{X}_S$, $U_S \cap \mathfrak{X}_{T_3}$, $U_S \cap \mathfrak{X}_{T_4}$, and $U_S \cap \mathfrak{X}_{T_5}$, where $T_j = \{1,2,j\}$ for $j = 3,4,5$. For each j, $U_S \cap U_{T_j}$ is isomorphic to $\mathbb{C}P^1$, and touches $\mathfrak{X}_S$ at the point X_{T_j}. In the following picture, an ellipse represents a copy of $\mathbb{C}P^1$ flowing between two fixed points, where the numbers or pairs of numbers indicate subsets that are straight on this $\mathbb{C}P^1$. (For example, $12, 45$ means that 1 and 2 are straight, as are 4 and 5.) We will revisit this example at the end of Section 5.4.

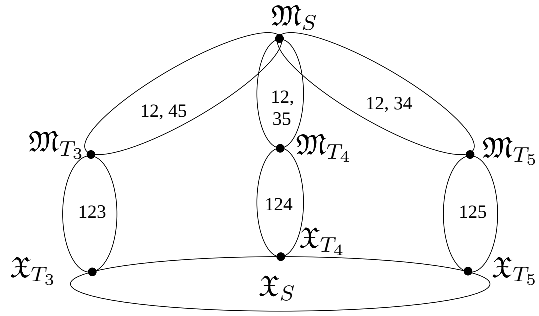

Figure 5.3: U_S, with $S = \{1, 2\}$

Proof of 5.7: Consider a point $[p, q] \in \mathfrak{M}$ with S straight, and $p_j = 0$ for all $j \in S^c$. By applying an element of G, we may assume that $q_i = \binom{1}{0}$ for all $i \in S$. Suppose further that there exists an $i \in S$ with $p_i \neq 0$, and that no strict superset of S is straight. In other words, if $q_j = \binom{a_j}{b_j}$ for $j \in S^c$, suppose that $b_j \neq 0$. For $t \in \mathbb{C}^\times$, let $\Theta(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, let $e_i(t) = t$ for all $i \in S$, and let $e_j(t) = t^{-1}$ for all $j \in S^c$. Then for $i \in S$, we have $e_i(t)^{-1} p_i \Theta(t) = t^{-2} p_i$ and $\Theta(t)^{-1} q_i e_i = q_i$. For $j \in S^c$, we have $\Theta(t)^{-1} q_j e_j = \binom{t^{-2} a_j}{b_j}$. Hence

$$\begin{aligned}
\lim_{t \to \infty} t \cdot [p, q] &= \lim_{t \to \infty} t^2 \cdot [p, q] \\
&= \lim_{t \to \infty} [t^2 p, q] \\
&= \lim_{t \to \infty} [t^2 e(t)^{-1} p \Theta(t), \Theta(t)^{-1} q e(t)] \\
&= [p, q'],
\end{aligned}$$

where $q'_i = q_i$ for $i \in S$, and $q'_j = \binom{0}{b_j}$ for $j \in S^c$. Since we have assumed that $b_j \neq 0$ for all $j \in S^c$ and that $p_i \neq 0$ for some $i \in S$, (p, q') is stable, and hence defines an element of $\mathfrak{M}_S$. Since U_S is defined to be the closure of the set of elements that flow up to $\mathfrak{M}_S$, it includes all $[p, q]$ with S straight and $p_j = 0$ for all $j \in S^c$. By dimension count, this containment is an equality, and we have $\dim U_S = n - 3$.

To see that U_S is smooth, it is sufficient to show that U_S is smooth at $[p, q]$ for all $[p, q] \in \mathfrak{M}^{\mathbb{C}^\times}$. First suppose that $[p, q] \in \mathfrak{M}_T$ for some $T \in \mathcal{S}'$ containing S. Suppose, without loss of generality, that $T = \{1, \ldots, l\}$ and $S = \{1, \ldots, m\}$ for some $l \leq m$. Konno computes an explicit local complex chart for $\mathfrak{M}$ at the point $[p, q]$, with coordinates $\{z_i, w_i \mid 3 \leq i \leq n-1\}$ [K2]. With respect to these coordinates, a point $[p', q']$ has the property that S is straight and $p'_j = 0$ for all $j \in S^c$ if and only if $w_i = 0$ for all $3 \leq i \leq l$ and $z_j = 0$ for all $l+1 \leq j \leq n-1$. Hence U_S is smooth at $[p,q]$.

It remains only to show that U_S is smooth at $\mathfrak{X}_S = U_S \cap \mathfrak{X}$. Let

$$E = \{(p,q) \mid S \text{ is straight}, \, p_j = 0 \text{ for all } j \in S^c, \text{ and } \mu_{\mathbb{C}}(p,q) = 0\},$$

and let $N = \{(p,q) \in E \mid p = 0\}$. The natural projection from E to N exhibits E as a vector bundle over N, because the equation $\mu_{\mathbb{C}}(p,q) = 0$ is linear in p. We have $U_S = E /\!\!/ G = E^{st}/G$, and $\mathfrak{X}_S = N /\!\!/ G = N^{st}/G$. The set $E|_{N^{st}}/G \subseteq E^{st}/G$ is an open neighborhood of $\mathfrak{X}_S$ inside of U_S, and is isomorphic to a vector bundle over $\mathfrak{X}_S$. Since $\mathfrak{X}_S$ is a polygon space it is smooth, hence U_S is smooth in a neighborhood of $\mathfrak{X}_S$. □

Corollary 5.9 *U_S is a compactification of the conormal bundle to $\mathfrak{X}_S$ in $\mathfrak{X}$.*

Proof: Choose a point $[q,0] \in \mathfrak{X}_S$, and a decomposition

$$T_{[q,0]}\mathfrak{M} = \nu_1 \oplus \nu_2 \oplus T_{[q,0]}\mathfrak{X}_S \oplus E,$$

where ν_1 is the normal space to $\mathfrak{X}_S$ inside of U_S, and ν_2 is the normal space to $\mathfrak{X}_S$ inside of $\mathfrak{X}$. Proposition 2.8 tells us that U_S and $\mathfrak{X}$ are both $\omega_{\mathbb{C}}$-lagrangian submanifolds of $\mathfrak{M}$, hence $\omega_{\mathbb{C}}$ gives a perfect pairing between $T_{[q,0]}\mathfrak{X}_S$ and E. It follows that $\omega_{\mathbb{C}}$ also gives a perfect pairing between ν_1 and ν_2, and therefore that the normal bundle to $\mathfrak{X}_S$ inside of U_S is dual to the normal bundle of $\mathfrak{X}_S$ inside of $\mathfrak{X}$. □

Remark 5.10 This argument generalizes to the smooth intersection of any two lagrangian submanifolds of a symplectic manifold.

We next describe U_S in polygon-theoretic terms, as a certain moduli space of pairs of polygons in $\mathbb{R}^3$.

Theorem 5.11 *The core component U_S is homeomorphic to the moduli space of $n+1$ vectors*

$$\{u_i, v_j, w \in \mathbb{R}^3 \mid i \in S, j \in S^c\},$$

taken up to rotation, satisfying the following conditions:

$$
\begin{aligned}
&1) \quad w + \sum_{j \in S^c} v_j = 0 \\
&2) \quad \sum_{i \in S} u_i = 0 \\
&3) \quad u_i \cdot w = 0 \quad \textit{for all } i \in S \\
&4) \quad \|v_j\| = \alpha_j \quad \textit{for all } j \in S^c \\
&5) \quad \|w\| = \sum_{i \in S} \sqrt{\alpha_i^2 + \|u_i\|^2}.
\end{aligned}
$$

Remark 5.12 In more descriptive terms, a point in U_S specifies two polygons in $\mathbb{R}^3$, as in Figure 5.4. The first is the $n - |S| + 1$ sided polygon consisting of the vectors $\{v_j \mid j \in S^c\}$

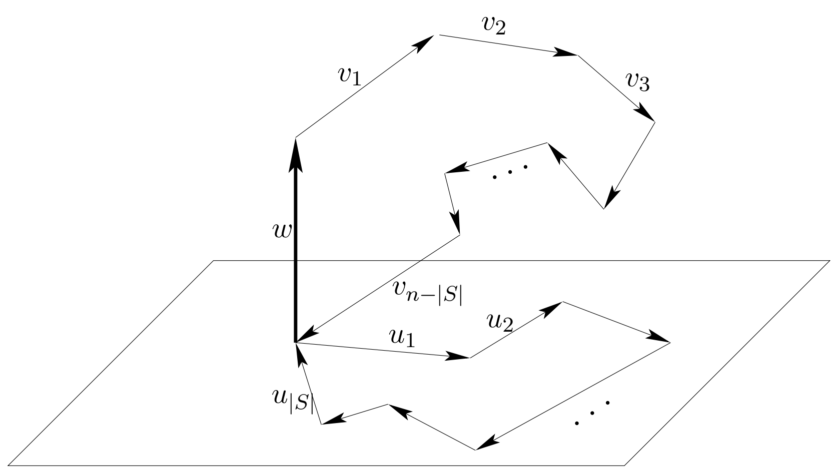

Figure 5.4: An element of U_S, represented by a spatial polygon with a distinguished edge, and a planar polygon perpendicular to that edge.

and w. Each vector v_j has length α_j, and w has a variable length, always greater than or equal to $\sum_{i \in S} \alpha_i$. This variable length is determined by the lengths of the edges in the second polygon, which consists of $|S|$ vectors $\{u_i \mid i \in S\}$, all contained in the plane perpendicular to w. Note that this description also applies to the Kähler polygon space $\mathfrak{X}$ by taking $S = \emptyset$.

By setting $u_i = 0$ for all i we get $\mathfrak{X}_S$, the minimum of the Morse-Bott function Φ on U_S. On the other hand, consider the submanifold of U_S obtained by imposing the extra condition that $\|w\| = \sum_{j \in S^c} \|v_j\|$. Then the first of the two polygons is forced to be linear, and we are left with $|S|$ vectors $\{u_i\}$ in the perpendicular plane satisfying a certain norm condition and adding to zero. Identifying this plane with $\mathbb{C}$ and dividing by the circle action rotating this plane, we obtain $\mathbb{C}P^{|S|-2} \cong \mathfrak{M}_S$, the maximum of Φ on U_S. Other critical points of Φ occur whenever the first polygon is linear, which is possible for finitely many values of $\|w\|$.

Proof of 5.11: Suppose given a point $[p,q]_{\mathbb{R}} \in U_S$, and let

$$u_i = q_i p_i + p_i^* q_i^* \quad \text{for all } i \in S,$$

$$v_j = (q_j q_j^*)_0 \quad \text{for all } j \in S^c,$$

$$w = \sum_{i \in S} (q_i q_i^*)_0 - (p_i^* p_i)_0.$$

These vectors live in $i \cdot \mathfrak{su}(2) \cong \mathfrak{su}(2)^* \cong \mathbb{R}^3$, which is endowed with the metric $u \cdot v = \frac{1}{2} \operatorname{tr} uv$, invariant under the coadjoint action. With respect to this metric, we have the equalities $\|(qq^*)_0\| = \frac{1}{2}|q|^2$ and $\|(p^*p)_0\| = \frac{1}{2}|p|^2$, hence conditions (1), (2), and (4) are immediate consequences of the moment map equations.

To verify condition (3), note that the vectors $\{q_i \mid i \in S\}$ are all proportional over $\mathbb{C}$, which implies that the vectors $(q_i q_i^*)_0$ are positive scalar multiples of each other. Furthermore, the moment map equation $p_i q_i = 0$ implies that $(p_i^* p_i)_0$ is a non-positive scalar multiple of $(q_i q_i^*)_0$, therefore $w = \sum (q_i q_i^*)_0 - (p_i^* p_i)_0$ is proportional over $\mathbb{R}_+$ to $(q_i q_i^*)_0$ for any $i \in S$. Then $u_i \cdot w = \frac{1}{2} \operatorname{tr} u_i w$ is a multiple of

$$\operatorname{tr} u_i (q_i q_i^*)_0 = \operatorname{tr} u_i q_i q_i^* = \operatorname{tr} p_i^* q_i^* q_i q_i^* = |q_i|^2 \operatorname{tr} p_i^* q_i^* = 0,$$

where the first equality comes from the fact that $q_i q_i^* - (q_i q_i^*)_0$ is a scalar multiple of the identity, and $\operatorname{tr} u_i = 0$.

To check condition (5), we first compute the norm of u_i:

$$\begin{aligned} \|u_i\|^2 &= \frac{1}{2} \operatorname{tr} u_i^2 \\ &= \frac{1}{2} \operatorname{tr}(q_i p_i p_i^* q_i^* + p_i^* q_i^* q_i p_i) \\ &= |q_i|^2 |p_i|^2 \\ &= |q_i|^2 (|q_i|^2 - 2\alpha_i). \end{aligned}$$

Since all of the vectors $\{(q_i q_i^*)_0 - (p_i^* p_i)_0 \mid i \in S\}$ point in the same direction, we have

$$\|w\| = \sum_{i \in S} \|(q_i q_i^*)_0\| + \|(p_i^* p_i)_0\| = \sum_{i \in S} \frac{1}{2}|q_i|^2 + \frac{1}{2}|p_i|^2 = \sum_{i \in S} |q_i|^2 - \alpha_i = \sum_{i \in S} \sqrt{\alpha_i^2 + \|u_i\|^2}.$$

We have defined a continuous map from U_S to the moduli space of vectors $\{u_i, v_j, w\}$ satisfying conditions (1)-(5), and we claim that this map is a homeomorphism. Since the source of this map is compact and the target is Hausdorff, it is sufficient to show that the map is bijective.

Suppose given a collection of vectors $\{u_i, v_j, w\} \subseteq \mathfrak{su}(2)$ satisfying conditions (1)-(5). Using the adjoint action of $SU(2)$, we may assume that w is a positive scalar multiple of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By condition (3), this implies that for all $i \in S$, there exists $t_i \in \mathbb{C}$ with $u_i = \begin{pmatrix} 0 & t_i \\ \bar{t_i} & 0 \end{pmatrix}$. For $j \in S^c$, we choose $q_j \in \mathbb{C}^2$ with $(q_j q_j^*)_0 = v_j$, and observe that q_j is unique up to the action of $U(1)^n$. We know that for all $i \in S$, $(q_i q_i^*)_0$ must be a positive multiple of w, hence there exist $a_i, b_i \in \mathbb{C}$ such that

$$q_i = \begin{pmatrix} a_i \\ 0 \end{pmatrix} \text{ and } p_i = (0 \; b_i)$$

for all $i \in S$. In order to have $u_i = q_i p_i + p_i^* q_i^*$ and $\frac{1}{2}|q_i|^2 - \frac{1}{2}|p_i|^2 = \alpha_i$, we must have

$$a_i b_i = t_i \text{ and } \frac{1}{2}|a_i|^2 - \frac{1}{2}|b_i|^2 = \alpha_i.$$

These equations uniquely define a_i and b_i up to the action of $U(1)^n$. It follows from conditions (1)-(5) that $(p, q) \in \mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)$ and that $w = \sum_{i \in S}(q_i q_i^*)_0 - (p_i^* p_i)_0$. This shows that our map is bijective, and thus completes the proof of Theorem 5.11. □

Remark 5.13 Suppose that S has only two elements; without loss of generality we will assume that $S = \{1, 2\}$. Then forgetting u_1 and u_2 gives a diffeomorphism from U_S to the "vertical polygon space" $VP(\alpha_3, \ldots, \alpha_n, \alpha_1 + \alpha_2)$ defined in [HK], shown to be diffeomorphic to a toric variety. More generally with $S = \{1, \ldots, k\}$, given any two-element subset $T \subseteq S$, the subvariety of U_S given by the equations $u_i = 0$ for all $i \in S \smallsetminus T$ is diffeomorphic to $VP(\alpha_{k+1}, \ldots, \alpha_n, \sum_T \alpha_i)$.

5.3 Cohomology rings

In this section we use Theorem 4.11 to compute the circle-equivariant cohomology of a hyperpolygon space $\mathfrak{M}$, thus reproducing (by different means) the results of [HP2, §3]. Recall that we have

$$\mathfrak{M} = T^*\mathbb{C}^{2n} /\!/\!/\!/ G,$$

where G is a quotient of $U(1)^n \times SU(2)$ by $\mathbb{Z}_2$. We will simplify our computations by dividing first by the torus $U(1)^n$. We have

$$\begin{aligned}\mathfrak{M} &= \left(T^*\mathbb{C}^{2n}\right) /\!/\!/\!/ G \\ &\cong \left(\left(T^*\mathbb{C}^2\right)^n /\!/\!/\!/ U(1)^n\right) /\!/\!/\!/ SU(2) \\ &\cong \prod_{i=1}^n T^*\mathbb{C}P^1 /\!/\!/\!/ SU(2),\end{aligned}$$

where the action of $SU(2)$ on each copy of $T^*\mathbb{C}P^1$ is induced by the rotation action on $\mathbb{C}P^1 \cong S^2$.

Proposition 5.14 *The non-rationalized Kirwan map $\kappa_{U(V)} : H^*_{S^1\times U(V)}(T^*\mathbb{C}^{2n}) \to H^*_{S^1}(\mathfrak{M})$ is surjective.*

Proof: The map $\kappa_{U(V)}$ factors as a composition

$$H^*_{S^1\times U(V)}(T^*\mathbb{C}^{2n}) \to H^*_{S^1\times SU(2)}\left(\prod_{i=1}^n T^*\mathbb{C}P^1\right) \overset{\kappa_{SU(2)}}{\Longrightarrow} H^*_{S^1}(\mathfrak{M}),$$

where the first map is the Kirwan map for a toric hyperkähler variety, and therefore surjective by [HP1]. Hence it suffices to show that $\kappa_{SU(2)}$ is surjective.

The level set $\mu_{\mathbb{C}}^{-1}(0)$ for the action of $SU(2)$ on $\prod_{i=1}^n T^*\mathbb{C}P^1$ is a subbundle of the cotangent bundle, given by requiring the n cotangent vectors to add to zero after being restricted to the diagonal $\mathbb{C}P^1$. In particular this set is smooth, and its $S^1 \times SU(2)$-equivariant cohomology ring is canonically isomorphic to that of $\prod_{i=1}^n T^*\mathbb{C}P^1$. Hence $\kappa_{SU(2)}$ factors as

$$H^*_{S^1\times SU(2)}\left(\prod_{i=1}^n T^*\mathbb{C}P^1\right) \cong H^*_{S^1\times SU(2)}\left(\mu_{\mathbb{C}}^{-1}(0)\right) \to H^*_{S^1}\left(\mu_{\mathbb{C}}^{-1}(0) /\!/ SU(2)\right) \cong H^*_{S^1}(\mathfrak{M}),$$

where the map in the middle is the Kähler Kirwan map. Surjectivity of this map follows from the following more general lemma, applied to the manifold $\mu_{\mathbb{C}}^{-1}(0)$.

Lemma 5.15 *Suppose given a hamiltonian action of $S^1 \times G$ on a symplectic manifold M, such that the S^1 component of the moment map is proper and bounded below with finitely many critical values. Then the Kähler Kirwan map $\kappa : H^*_{S^1\times G}(M) \to H^*_{S^1}(M/\!/G)$ is surjective.*

Proof: Extend the action of S^1 to an action on $M \times \mathbb{C}$ by letting S^1 act only on the left-hand factor. On the other hand, consider a second copy of the circle, which we will call $\mathbb{T}$ to avoid confusion, acting diagonally on $M \times \mathbb{C}$. Choose $r \in \mathrm{Lie}(\mathbb{T})^* \cong \mathbb{R}$ greater than the largest critical value of the $\mathbb{T}$-moment map, and consider the space

$$Cut(M/\!\!/G) := (M \times \mathbb{C})/\!\!/_r \mathbb{T} \times G \cong \big((M/\!\!/G) \times \mathbb{C}\big)/\!\!/_r \mathbb{T}.$$

This space, which is called the *symplectic cut* of $M/\!\!/G$ [Le], is an S^1-equivariant (orbifold) compactification of $M/\!\!/G$. We then have a commutative diagram

$$\begin{array}{ccc} H^*_{S^1 \times G \times \mathbb{T}}(M \times \mathbb{C}) & \longrightarrow & H^*_{S^1 \times G}(M) \\ \big\downarrow & & \big\downarrow \kappa \\ H^*_{S^1}(Cut(M/\!\!/G)) & \longrightarrow & H^*_{S^1}(M/\!\!/G). \end{array}$$

The vertical map on the left is surjective because the $G \times \mathbb{T}$ moment map is proper, and the map on the bottom is surjective because the long exact sequence in cohomology for $M/\!\!/G \subseteq Cut(M/\!\!/G)$ splits naturally, hence κ is surjective as well. $\square$

By applying Lemma 5.15 to $M = \mu_{\mathbb{C}}^{-1}(0)$, this completes the proof of Proposition 5.14. $\square$

Remark 5.16 The argument in Proposition 5.14 generalizes immediately to show that the hyperkähler Kirwan map for the quotient

$$\left(\prod_{i=1}^{n} T^* Flag(\mathbb{C}^k)\right) /\!\!/\!\!/\!\!/ SU(k)$$

is surjective. This is itself a quiver variety, and like the hyperpolygon space, it has a moduli theoretic interpretation. The Kähler quotient

$$\left(\prod_{i=1}^{n} Flag(\mathbb{C}^k)\right) /\!\!/ SU(k)$$

is isomorphic to the space of n-tuples of $k \times k$ hermitian matrices with fixed eigenvalues adding to zero, modulo conjugation. This space has been studied by many authors. The classical problem, due to Horn, of determining the values of the moment map for which it is nonempty, has only recently been solved [KT]. For a survey, see [Fu].

To compute the kernel of the hyperkähler Kirwan map for the hyperpolygon space, we first need to study the abelian quotient

$$\mathfrak{N} := \prod_{i=1}^{n} T^*\mathbb{C}P^1 /\!\!/\!\!/\!\!/ T,$$

where $T \cong U(1) \subseteq SU(2)$ is a maximal torus.[2] The space $\prod_{i=1}^{n} T^*\mathbb{C}P^1$ is a hypertoric variety given by an arrangement of $2n$ hyperplanes in $\mathbb{R}^n$, where the $(2i-1)^{\text{st}}$ and $(2i)^{\text{th}}$ hyperplanes are given by the equations $x_i = \pm\xi_i$. Taking the hyperkähler quotient by T corresponds on the level of arrangements to restricting this arrangement to the hyperplane $\{x \in \mathbb{R}^n \mid \sum x_i = 0\}$. By Theorem 3.18, we have

$$H^*_{S^1}(\mathfrak{N}) \cong \mathbb{Q}[a_1, b_1, \ldots, a_n, b_n, \delta, x] \Big/ \Big\langle a_i - b_i - \delta,\ a_i b_i \,\Big|\, i \leq n \Big\rangle + \Big\langle A_S, B_S \,\Big|\, S \text{ short} \Big\rangle,$$

where

$$A_S = \prod_{i \in S}(x - a_i) \prod_{j \in S^c} b_j \quad \text{and} \quad B_S = \prod_{i \in S}(x - b_i) \prod_{j \in S^c} a_j.$$

Here δ is the image in $H^*_{S^1}(\mathfrak{N})$ of the unique positive root of $SU(2)$. The Weyl group W of $SU(2)$, isomorphic to $\mathbb{Z}/2\mathbb{Z}$, acts on this ring by fixing x and switching a_i and b_i for all i. Let $c_i = a_i + b_i$, and let $C_S = A_S + B_S$. Let

$$P = \mathbb{Q}[c_1, \ldots, c_n, \delta, x] \Big/ \Big\langle c_i^2 - \delta^2 \,\Big|\, i \leq n \Big\rangle$$

and

$$Q = P^W = \mathbb{Q}[c_1, \ldots, c_n, \delta^2, x] \Big/ \Big\langle c_i^2 - \delta^2 \,\Big|\, i \leq n \Big\rangle.$$

Let

$$\mathcal{I} = \Big\langle A_S, B_S \,\Big|\, S \text{ short} \Big\rangle \subseteq P \quad \text{and} \quad \mathcal{J} = \Big\langle C_S \,\Big|\, S \text{ short} \Big\rangle \subseteq Q,$$

so that

$$H^*_{S^1}(\mathfrak{N}) \cong P/\mathcal{I} \quad \text{and} \quad H^*_{S^1}(\mathfrak{N})^W \cong Q/\mathcal{J}.$$

Note that all odd powers of δ in the expression for $C_S = A_S + B_S$ cancel out. Then by Theorem 4.11 and Remark 4.12,

$$H^*_{S^1}(\mathfrak{M}) \;\cong\; \frac{H^*_{S^1}(\mathfrak{N})^W}{ann(e)} \cong \frac{Q}{(e : \mathcal{J})},$$

[2]This is the hyperkähler analogue of the *abelian polygon space* from [HK].

where $e = \delta^2(x^2 - \delta^2)$, and $(e : \mathcal{J})$ is the ideal of elements of Q whose product with e lies in $\mathcal{J}$.

If S is a nonempty short subset, let m_S be the smallest element of S, n_S the smallest element of S^c, and

$$D_S = \prod_{m_S \neq i \in S} (c_i - x) \;\cdot\; \prod_{n_S \neq j \in S^c} (c_{n_S} + c_j) \in Q.$$

Theorem 5.17 *The circle-equivariant cohomology ring of the hyperpolygon space $\mathfrak{M}$ is isomorphic to*[3]

$$Q/\big\langle D_S \mid \emptyset \neq S \text{ short}\big\rangle.$$

Proof: We begin by proving that $e \cdot D_S \in \mathcal{J}$ for all nonempty short subsets $S \subseteq \{1, \ldots, n\}$. We will in fact prove the slightly stronger statement

$$e \cdot D_S \in \left\langle C_T \;\middle|\; T \subseteq S \text{ short} \right\rangle \subseteq \mathcal{J},$$

proceeding by induction on $|S|$. We will assume, without loss of generality, that $n \in S$. The base case occurs when $S = \{n\}$, and in this case we observe that

$$e \cdot D_S = 2^{n-3} \cdot (x + c_n) \cdot \Big((2x - c_n) \cdot C_\emptyset - c_n \cdot C_S \Big).$$

We now proceed to the inductive step, assuming that the proposition is proved for all short subsets of size less than $|S|$, and all values of n. For all $T \subseteq S \smallsetminus \{n\}$, we have

$$\frac{1}{2}\Big(C_T - C_{T \cup \{n\}} \Big) = (c_n - x) \cdot C'_T,$$

where C'_T is the polynomial in the variables $\{c_1, \ldots, c_{n-1}, \delta^2\}$ corresponding to the short subset $T \subseteq \{1, \ldots, n-1\}$. Since $S \smallsetminus \{n\}$ is a short subset of $\{1, \ldots, n-1\}$ of size strictly smaller than S, our inductive hypothesis tells us that $e \cdot D_S/(c_n - x)$ can be written as a linear combination of polynomials C'_T, where the coefficients are quadratic polynomials in $\{c_1, \ldots, c_{n-1}, \delta^2\}$. Replacing C'_T with $\frac{1}{2}\left(C_T - C_{T \cup \{n\}}\right) = (c_n - x) \cdot C'_T$, we have expressed $e \cdot D_S$ in terms of the appropriate polynomials. This completes the induction.

Suppose that $F \in Q$ is an element of degree less than $n - 2$ such that $e \cdot F \in \mathcal{J}$. By the second isomorphism of Theorem 4.11, this implies that $e' \cdot F \in \mathcal{I} \subseteq P$, where $e' = \delta(x^2 - \delta^2)$. Consider the quotient ring R of P obtained by setting $a_i^2 = b_i^2 = x = 0$

[3] The class denoted by c_i in [HP2] differs from our c_i by a sign, hence to recover the presentation of [HP2] we must replace $c_i - x$ with $c_i + x$ in the expression for D_S.

for all i. (Recall that $a_i = \frac{1}{2}(c_i + \delta)$ and $b_i = \frac{1}{2}(c_i - \delta)$.) Then element e' maps to zero in R, while the generators $\{A_S, B_S\}$ of $\mathcal{I}$ descend to a basis for the n^{th} degree part of R. This means that we must have $e' \cdot F = 0 \in P$. Using the additive basis for P consisting of monomials that are squarefree in the variables $c_1, \ldots, c_n$, it is easy to check that e' is not a zero divisor in P, and therefore that $F = 0$.

Finally, we must show that $\{D_S \mid \emptyset \neq S \text{ short}\}$ generates all elements of $(e : \mathcal{J})$ of degree at least $n - 2$. We obtain this fact from the following technical lemma, the proof of which we defer until the end of the section.

Lemma 5.18 *The set $\{D_S \mid \emptyset \neq S$ short$\}$ descends to a basis for the degree $n - 2$ part of the quotient ring $Q/\langle x \rangle$.*

Let F be an element of minimal degree $k \geq n - 2$ that is in $(e : \mathcal{J})$ but not $\langle D_S \mid \emptyset \neq S \text{ short}\rangle$. By Lemma 5.18, F differs from an element of $\langle D_S \mid \emptyset \neq S \text{ short}\rangle$ by $x \cdot F'$ for some F' of degree $k - 1$. By equivariant formality of $H^*_{S^1}(\mathfrak{M})$,

$$x \cdot F' = F \in (e : \mathcal{J}) \Rightarrow F' \in (e : \mathcal{J}),$$

which contradicts the minimality of $k = \deg F$. Hence $\langle D_S \mid \emptyset \neq S \text{ short}\rangle = (e : \mathcal{J})$, and the proposition is proved. □

Corollary 5.19 *The ordinary cohomology ring $H^*(\mathfrak{M})$ is isomorphic to*

$$\mathbb{Q}[c_1, \ldots, c_n] \Big/ \big\langle c_i^2 - c_j^2 \mid i, j \leq n \big\rangle + \langle \textit{all monomials of degree } n - 2 \rangle.$$

Proof: This follows from the fact that $H^*(\mathfrak{M}) \cong H^*_{S^1}(\mathfrak{M})/\langle x \rangle$ for any equivariantly formal space M, and the observation in [HP2] that $\{D_S \mid \emptyset \neq S \text{ short}\}$ descends to a basis for the degree $n - 2$ part of $Q/\langle x \rangle$. □

Proof of 5.18: Let $d_k = \frac{1}{2}(c_1 + c_k)$ for all k, so that $c_k = 2d_k - d_1$. The relations $c_k^2 = c_1^2$ translate to $d_k^2 = d_1 d_k$ for all k, and we have

$$Q/\langle x \rangle = \mathbb{Q}[d_1, \ldots, d_n] \Big/ \big\langle d_k^2 - d_1 d_k \;\big|\; k \in \{2, \ldots, n\} \big\rangle.$$

For all short subsets S, put

$$\bar{S} = S \smallsetminus \{m_S\} \quad \text{and} \quad \overline{S^c} = S^c \smallsetminus \{n_S\},$$

and let

$$v_S = (-1)^n \prod_{j \in \overline{S^c}} (d_j + d_{n_S} - d_1) \times \prod_{i \in \bar{S}} (2d_i - d_1).$$

For all $A \subseteq \{2, \ldots, n\}$, let

$$d_A = (-1)^{|A|} d_1^{n-2-|A|} \prod_{k \in A} d_k$$

for all $A \subsetneq \{2, \ldots, n\}$. Then $\{d_A\}$ is a basis for the $(n-2)^{\text{nd}}$ graded piece of $Q/\langle x \rangle$, and v_S is equal to $(-1)^n \cdot 2^{-|\overline{S^c}|}$ times the image of D_S in $Q/\langle x \rangle$. Hence our the statement of Lemma 5.18 is that for all A, d_A may be expressed as a linear combination of the elements $\{v_S \mid S \in \mathcal{S}\}$.

Notational Convention 5.20 The notation S^c refers to the complement of S inside of the set $\{1, \ldots, n\}$, while the notation A^c refers to the complement of A inside of the set $\{2, \ldots, n\}$.

Claim 5.21 *We have the following expression for v_S in terms of the basis $\{d_A\}$:*

$$v_S = \begin{cases} \displaystyle\sum_{\substack{\overline{S^c} \subseteq A \\ m_S \notin A}} 2^{|A \cap \bar{S}|}\, d_A & \textit{if} \quad 1 \in S^c; \\ \\ \displaystyle\sum_{S^c \nsubseteq A} 2^{|A \cap \bar{S}|}\, d_A & \textit{if} \quad 1 \in S. \end{cases}$$

Proof: Any degree $n-2$ monomial in $d_1, \ldots, d_n$ is equal to $(-1)^{|A|} d_A$, where A is the set of $k > 1$ such that d_k appears in the monomial. Expanding v_S, we need to count (with sign) the occurrence of d_A for each A. In most cases we find that there is no cancellation, and the claim is straightforward. The most difficult case occurs when $1 \in S$ (therefore $n_S = 1$) and $m_S \in A$; in this case the number of times (with multiplicity) that d_A occurs in v_S is

$$\begin{aligned}
& (-1)^n (-1)^{|A|} (-1)^{|A^c \cap \bar{S}|}\, 2^{|A \cap \bar{S}|} \sum_{E \subsetneq A^c \cap \overline{S^c}} (-1)^{|E|} \\
= \;& (-1)^n (-1)^{|A|} (-1)^{|A^c \cap \bar{S}|}\, 2^{|A \cap \bar{S}|} \left((1-1)^{|A^c \cap \overline{S^c}|} - (-1)^{|A^c \cap \overline{S^c}|} \right) \\
= \;& (-1)^{n + |A| + |A^c \cap \bar{S}| + |A^c \cap \overline{S^c}| + 1}\, 2^{|A \cap \bar{S}|} \\
= \;& (-1)^{2n}\, 2^{|A \cap \bar{S}|} \\
= \;& 2^{|A \cap \bar{S}|}.
\end{aligned}$$

We leave the remaining cases to be checked by the reader. □

Claim 5.22 *Suppose that* $1 \in S$*. Let* $S_0 = S$*, and for* $1 \leq k \leq |S|$*, let* $S_k = S_{k-1} \setminus \{m_{S_{k-1}}\}$*. (In other words, let* S_k *consist of the* $|S| - k$ *largest elements of* S*). Then*

$$v_S + \sum_{k=1}^{|S|-1} 2^{k-1}\, v_{S_k} = \sum_A 2^{|A \cap \bar{S}|}\, d_A.$$

Proof: We proceed by induction to show that

$$v_S + \sum_{k=1}^{l} 2^{k-1}\, v_{S_k} = \sum_A 2^{|A \cap \bar{S}|}\, d_A - \; 2^l \cdot \sum_{\overline{S_{l+1}^c} \subseteq A} 2^{|A \cap \bar{S}_l|}\, d_A.$$

The case $l = |S| - 1$ is the statement of the claim. The base case $l = 0$ follows from Claim 5.21, together with the observation that $\overline{S_1^c} = S^c$. More generally, for all $l \geq 1$, we have

$$\overline{S_{l+1}^c} = S^c \cup \{m_{S_1}, \ldots, m_{S_l}\}.$$

Then

$$\begin{aligned} v_S + \sum_{k=1}^{l+1} 2^{k-1}\, v_{S_k} &= v_S + \sum_{k=1}^{l} 2^{k-1}\, v_{S_k} + 2^l\, v_{S_{l+1}} \\ &= \sum_A 2^{|A \cap \bar{S}|}\, d_A - 2^l \cdot \sum_{\overline{S_{l+1}^c} \subseteq A} 2^{|A \cap \bar{S}_l|}\, d_A + 2^l \cdot \sum_{\substack{\overline{S_{l+1}^c} \subseteq A \\ m_{S_{l+1}} \notin A}} 2^{|A \cap \bar{S}_{l+1}|}\, d_A \end{aligned}$$

by the inductive hypothesis and Claim 5.21. Using the fact that $A \cap \bar{S}_{l+1} = A \cap \bar{S}_l$ when $m_{S_{l+1}} \notin A$, this is equal to

$$\sum_A 2^{|A \cap \bar{S}|}\, d_A - 2^l \cdot \sum_{\overline{S_{l+1}^c} \cup \{m_{S_{l+1}}\} \subseteq A} 2^{|A \cap \bar{S}_l|}.$$

Finally, since $|A \cap \bar{S}_{l+1}| = |A \cap \bar{S}_l| - 1$ when $m_{S_{l+1}} \in A$, this reduces to

$$\sum_A 2^{|A \cap \bar{S}|}\, d_A - 2^{l+1} \cdot \sum_{\overline{S_{l+2}^c} \subseteq A} 2^{|A \cap \bar{S}_{l+1}|}\, d_A,$$

thus proving our claim. □

For all short subsets T containing 1, let $w_T = \sum_A 2^{|A\cap\bar{T}|}\, d_A$, which by Claim 5.22 is expressible as a linear combination of elements of the set $\{v_S \mid \emptyset \neq S \in \mathcal{S}\}$. Let

$$x_S = \begin{cases} \displaystyle\sum_{1\in T\subseteq S} (-1)^{|S|+|T|} w_T & \text{if } 1 \in S, \\ v_S & \text{if } 1 \in S^c. \end{cases}$$

Our last task will be to prove that the transition matrix Υ taking the basis $\{d_A\}$ to the set $\{x_S\}$ is lower triangular with ones on the diagonal, and therefore invertible. In order to make sense of "the diagonal," we must first give an explicit bijection between the set of proper subsets of $\{2,\ldots,n\}$ and the set of nonempty short subsets of $\{1,\ldots,n\}$. We do this as follows: given $A \subsetneq \{2,\ldots,n\}$, let

$$S(A) = \begin{cases} A^c & \text{if } A^c \text{ is short,} \\ \{1,\ldots,n\} \setminus A^c = A \cup \{1\} & \text{if } A^c \text{ is long.} \end{cases}$$

The rows of Υ will be indexed by A, and the sets will appear in lexicographic order within cardinality class. For example, when $n = 4$, the order of the rows will be $\emptyset$, $\{2\}$, $\{3\}$, $\{4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$. The columns will be indexed by S according to the bijection described above.

Claim 5.23 *The matrix Υ is lower triangular with ones on the diagonal.*

Proof: First consider a column corresponding to a short subset S that does *not* contain 1. The entries in this column correspond to the coefficient of d_A in $x_S = v_S$. From Claim 5.21, we see that d_A appears in v_S only if $\overline{S^c} \subseteq A \subseteq \overline{S^c} \cup \bar{S}$, and if so it appears with a coefficient of $2^{|A\cap\bar{S}|}$. Since $1 \notin S$, we have $\overline{S^c} = S^c \setminus \{1\} = \{2,\ldots,n\} \setminus S$. The diagonal entry corresponds to the set $A = \{2,\ldots,n\} \setminus S = \overline{S^c}$, therefore in this row we get the number $2^{|A\cap\bar{S}|} = 2^{|\overline{S^c}\cap\bar{S}|} = 1$. Since the set A corresponding to a given row can never contain the set B corresponding to a lower row, the rows above the diagonal fail to satisfy the condition $\overline{S^c} \subseteq A$, and we get all zeros.

Now consider a column corresponding to a short subset S that *does* contain 1. In this case, the coefficient of d_A in x_S is

$$(-1)^{|S|} \sum_{1\in T\subseteq S} (-1)^{|T|} 2^{|A\cap\bar{T}|}.$$

The diagonal entry corresponds to the set $A = \bar{S}$, and we get

$$\begin{aligned}(-1)^{|S|}\sum_{1\in T\subseteq S}(-1)^{|T|}2^{|\bar{T}|} &= (-1)^{|\bar{S}|}\sum_{1\in T\subseteq S}(-2)^{|\bar{T}|}\\ &= (-1)^{|\bar{S}|}(1-2)^{|\bar{S}|} = 1.\end{aligned}$$

Any row above the diagonal corresponds to a set A which does not contain $\bar{S}$. Choose an element $l \in \bar{S} \setminus A$. Then

$$\begin{aligned}(-1)^{|S|}\sum_{1\in T\subseteq S}(-1)^{|T|}2^{|A\cap\bar{T}|} &= (-1)^{|S|}\sum_{l\in T}(-1)^{|T|}2^{|A\cap\bar{T}|} + (-1)^{|S|}\sum_{l\notin T}(-1)^{|T|}2^{|A\cap\bar{T}|}\\ &= (-1)^{|S|}\sum_{l\notin T}\left[(-1)^{|T|}2^{|A\cap\bar{T}|} + (-1)^{|T\cup\{l\}|}2^{|A\cap\bar{T}|}\right]\\ &= 0.\end{aligned}$$

Hence Υ is lower triangular. □

Claim 5.23 tells us that each d_A can be expressed as a linear combination of elements of the form x_S, and therefore of elements of the form v_S. This completes the proof of Lemma 5.18 □

5.4 Cohomology of the core components

In this section we compute the S^1-equivariant and ordinary cohomology rings of the core component U_S corresponding to a short subset $S \subseteq \{1,\ldots,n\}$. Since U_S is the closure of a cell in an even-dimensional equivariant cellular decomposition of $\mathfrak{M}$, the restriction map $H^*_{S^1}(\mathfrak{M}) \to H^*_{S^1}(U_S)$ is surjective. In particular, $H^*_{S^1}(U_S)$ is generated by restrictions of the Kirwan classes $c_1,\ldots,c_n,x$. For our presentation, it will be convenient to assume that $1 \in S$, and to work with the classes $d_k = \frac{1}{2}(c_1 + c_k)$ introduced in the proof of Lemma 5.18. With respect to these generators, we obtain the following result.

Theorem 5.24 *The equivariant cohomology ring $H^*_{S^1}(U_S)$ is isomorphic to $\mathbb{Q}[d_1,\ldots,d_n,x]/\mathcal{J}_S$,* *where $\mathcal{J}_S$ is generated by the following four families:*

$$
\begin{aligned}
&1) \quad d_1 - d_i \quad \textit{for all} \quad i \in S \\
&2) \quad d_j(d_1 - d_j) \quad \textit{for all} \quad j \in S^c \\
&3) \quad \prod_{j \in R} d_j \quad \textit{for all} \quad R \subseteq S^c \textit{ such that } R \cup S \textit{ is long} \\
&4) \quad (d_1 + x)^{|S|-1} \cdot \frac{1}{d_1} \left(\prod_{j \in L} (d_j - d_1) - \prod_{j \in L} d_j \right) \quad \textit{for all long subsets } L \subseteq S^c.
\end{aligned}
$$

Corollary 5.25 *The ordinary cohomology ring $H^*(U_S)$ is isomorphic to $\mathbb{Q}[d_1, \ldots, d_n]/\mathcal{I}_S$, where $\mathcal{I}_S$ is generated by the following four families:*

$$
\begin{aligned}
&1) \quad d_1 - d_i \quad \textit{for all} \quad i \in S \\
&2) \quad d_j(d_1 - d_j) \quad \textit{for all} \quad j \in S^c \\
&3) \quad \prod_{j \in R} d_j \quad \textit{for all} \quad R \subseteq S^c \textit{ such that } R \cup S \textit{ is long} \\
&4) \quad d_1^{|S|-2} \prod_{j \in L} (d_j - d_1) \quad \textit{for all long subsets } L \subseteq S^c.
\end{aligned}
$$

Remark 5.26 Each of these relations has a geometric interpretation. For $i \in \{1, \ldots, n\}$, it is possible to construct a line bundle on $\mathfrak{M}$ with equivariant Euler class $d_i - d_1$ which has a section supported on the locus where $q_1 q_1^*$ and $q_i q_i^* \in \mathbb{R}^3$ point in opposite directions. Since this locus is disjoint from U_S when $i \in S$, we have

$$1) \;\; d_i = d_1 \in H^*_{S^1}(U_S) \;\; \text{for all} \;\; i \in S.$$

Similarly, $-d_j = -\frac{1}{2}(c_1 + c_j)$ is represented by the divisor $Z_{1j} \subseteq \mathfrak{M}$ of points on which $q_1 q_1^*$ and $q_i q_i^* \in \mathbb{R}^3$ point in the same direction [HP2, §3]. Then by the previous reasoning, we obtain

$$2) \;\; d_j(d_1 - d_j) = 0 \in H^*_{S^1}(U_S) \;\; \text{for all} \;\; j \in S^c.$$

For any $R \subseteq S^c$, we may intersect the divisors $Z_{ij} \subseteq \mathfrak{M}$ (defined in the analogous way) for all $j \in R$ to find that the cohomology class $(-1)^{|R|} \prod_{j \in R} d_j$ is represented by the subvariety $Z_R \subseteq \mathfrak{M}$ of points with q_j proportional to q_1 for all $j \in R$. When restricted to U_S, this becomes $U_S \cap U_{R \cup S}$, the closure of the unstable manifold for the critical locus $\mathfrak{M}_{R \cup S} \cap U_S$ of the Morse-Bott function $\Phi|_{U_S}$. In particular, we have

$$3) \;\; \prod_{j \in R} d_j = 0 \in H^*_{S^1}(U_S) \text{ if } R \cup S \;\text{ is long.}$$

To understand the fourth family of relations, we note that the class

$$d_1 + x = 2d_i - d_1 + x = c_i + x \in H^*_{S^1}(U_S)$$

is represented by the divisor W_i of points with $p_i = 0$ for any $i \in S$ [HP2, §3]. In particular, $(d_1 + x)^{|S|-1}$ is represented by the subvariety of points in U_S on which $p_i = 0$ for all $i \in \bar{S}$, which is equal to $\mathfrak{X}_S$ by the complex moment map condition. Hence the fourth family of generators of $\mathcal{J}_S$ (or of $\mathcal{I}_S$) can be interpreted geometrically as $(d_1 + x)^{|S|-1}$ (respectively $d_1^{|S|-1}$ in the nonequivariant case) times classes that vanish in $H^*_{S^1}(\mathfrak{X}_S)$ (see Lemma 5.28).

Proof of 5.24: Let $\phi : \mathbb{Q}[d_1, \ldots, d_n, x] \to H^*_{S^1}(U_S)$ denote the composition of the Kirwan map with restriction to U_S. Our claim is that $\operatorname{Ker}\phi = \mathcal{J}_S$. For every short subset T containing S, let

$$\phi_T : \mathbb{Q}[d_1, \ldots, d_n, x] \to H^*_{S^1}(X_T \cap U_S)$$

denote the composition of the Kirwan map with restriction to $X_T \cap U_S$, and let

$$J_T = \operatorname{Ker}\phi_T.$$

Similarly, let

$$\phi_\emptyset : \mathbb{Q}[d_1, \ldots, d_n, x] \to H^*_{S^1}(\mathfrak{X}_S)$$

be the natural map, and let

$$J_\emptyset = \operatorname{Ker}\phi_\emptyset.$$

The kernel of the restriction map $H^*_{S^1}(U_S) \to H^*_{S^1}(U_S^{S^1})$ to the fixed point set of U_S is a torsion module over $H^*_{S^1}(pt)$ [AB, 3.5], and Proposition 2.10 tells us that $H^*_{S^1}(U_S)$ is a free $H^*_{S^1}(pt)$-module. Hence the restriction map is injective, and we have

$$\operatorname{Ker}\phi = \operatorname{Ker}\phi_\emptyset \cap \bigcap_{T \supseteq S} \operatorname{Ker}\phi_T.$$

We know that $X_T \cap U_S \cong \mathbb{C}P^{|S|-2}$ for all short $T \supseteq S$, therefore

$$H^*_{S^1}(X_T \cap U_S) \cong \mathbb{Q}[h, x]/h^{|S|-1}.$$

Furthermore, we know that for all $i \in T$, the restriction of $d_i + x$ to $H^*_{S^1}(U_T)$ is represented by the divisor $W_i \cap U_T$ (see Remark 5.26), and therefore restricts to the class of a hyperplane on $X_T \cap U_S$. Hence $\phi_T(d_i + x) = h$ for all $i \in T$. On the other hand, for $j \in T^c$, the class

d_j is represented by the divisor Z_{1j} on $\mathfrak{M}$, which is disjoint from $X_T \cap U_S$, hence $\phi(d_j) = 0$ for all $j \in T^c$. Thus we conclude that

$$\operatorname{Ker} \phi_T = \left\langle d_1 - d_i,\ d_j, (d_1 + x)^{|S|-1} \mid i \in T, j \in T^c \right\rangle.$$

Lemma 5.27 *The intersection* $\bigcap_{T \supseteq S} \operatorname{Ker} \phi_T$ *is equal to*

$$\left\langle d_1 - d_i,\ d_j(d_1 - d_j), \prod_{j \in R} d_j, (d_1 + x)^{|S|-1} \;\middle|\; i \in S, j \in S^c, R \cup S \text{ long} \right\rangle.$$

Proof: First, since the variable x appears only in the generator $(d_1 + x)^{|S|-1}$, which is contained in every ideal on both sides of the equation, we may reduce the problem to showing that

$$\bigcap_{T \supseteq S} \left\langle d_1 - d_i,\ d_j \;\middle|\; i \in T, j \in T^c \right\rangle = \left\langle d_1 - d_i,\ d_j(d_1 - d_j), \prod_{j \in R} d_j \;\middle|\; i \in S, j \in S^c, R \cup S \text{ long} \right\rangle \tag{5.2}$$

in the ring $\mathbb{Q}[d_1, \ldots, d_n]$. Both ideals cut out the (reducible) variety

$$\bigcup_{T \supseteq S} Y_T \subseteq \operatorname{Spec} \mathbb{Q}[d_1, \ldots, d_n],$$

where

$$Y_T = \left\{ (z_1, \ldots, z_n \;\middle|\; z_i = z_1\ \forall i \in S, z_j = 0\ \forall j \in S^c \right\}.$$

The left hand side of Equation (5.2) is an intersection of prime ideals, and is therefore radical. Thus by Hilbert's Nullstellensatz, it is sufficient to prove that the right hand side of Equation (5.2) is radical. This involves showing that the ideal is saturated, with Hilbert polynomial equal to the constant $\#\{\text{short } T \supseteq S\}$.

The degree k piece of the quotient

$$\mathbb{Q}[d_1, \ldots, d_n] / \left\langle d_1 - d_i,\ d_j(d_1 - d_j) \mid i \in S, j \in S^c \right\rangle$$

has a basis of elements of the form

$$d_1^{e_1} \prod_{j \in S^c} d_j^{e_j},$$

where $e_j \in \{0, 1\}$ for all $j > 0$, and $e_1 + \sum_{j \in S^c} e_j = k$. The subset of these elements with the property that $S \cup \{j \mid e_j = 1\}$ is short descends to a basis for the degree k part of the ring

$$\mathbb{Q}[d_1,\ldots,d_n]\Big/\left\langle d_1-d_i,\ d_j(d_1-d_j),\prod_{j\in R}d_j\ \middle|\ i\in S, j\in S^c, R\cup S \text{ long}\right\rangle,$$

hence our ideal has the desired Hilbert polynomial. It is also clear from this description that if an element a of the quotient ring is nonzero, so is $d_1^d\cdot a$ for any $d\geq 0$, hence our ideal is saturated. □

It now remains to show that

$$\mathcal{J}_S=\left\langle d_1-d_i,\ d_j(d_1-d_j),\prod_{j\in R}d_j,\ (d_1+x)^{|S|-1}\ \middle|\ i\in S, j\in S^c, R\cup S \text{ long}\right\rangle\cap \operatorname{Ker}\phi_\emptyset.$$

The fact that $\mathcal{J}_S$ is contained in the intersection is clear. To show the opposite containment, consider an element

$$a+\eta\cdot(d_1+x)^{|S|-1}\in\left\langle d_1-d_i,\ d_j(d_1-d_j),\prod_{j\in R}d_j,\ (d_1+x)^{|S|-1}\ \middle|\ i\in S, j\in S^c, R\cup S \text{ long}\right\rangle,$$

with

$$a\in\left\langle d_1-d_i,\ d_j(d_1-d_j),\prod_{j\in R}d_j\ \middle|\ i\in S, j\in S^c, R\cup S \text{ long}\right\rangle,$$

and suppose that we also have

$$a+\eta\cdot(d_1+x)^{|S|-1}\in\operatorname{Ker}\phi_\emptyset.$$

Lemma 5.28 [HK] *The kernel of $\phi_\emptyset$ is equal to*

$$\left\langle d_1-d_i,\ d_j(d_1-d_j),\prod_{j\in R}d_j,\ (d_1+x)^{|S|-1}d_1^{-1}\left(\prod_{j\in L}(d_j-d_1)-\prod_{j\in L}d_j\right)\right\rangle,$$

where $i\in S, j\in S^c$, and $R,L\subseteq S^c$, with $R\cup S$ and L both long.

Lemma 5.28 tells us that $a\in\operatorname{Ker}\phi_\emptyset$, therefore

$$\eta\cdot(d_1+x)^{|S|-1}\in\operatorname{Ker}\phi_\emptyset.$$

But $(d_1+x)^{|S|-1}$ is represented in $H^*_{S^1}(U_S)$ by the subvariety $\mathfrak{X}_S$ (see Remark 5.26), hence

$$0=\phi_\emptyset(\eta\cdot(d_1+x)^{|S|-1})=\phi_\emptyset(\eta)\cdot e(\mathfrak{X}_S),$$

where $e(\mathfrak{X}_S)$ is the equivariant Euler class of the normal bundle to $\mathfrak{X}_S$ inside of U_S. Since the equivariant Euler class of the normal bundle to a component of the fixed point set is never a zero-divisor, we have $\eta \in \operatorname{Ker}\phi_\emptyset$. Then by Equation 5.28,

$$a + \eta \cdot (d_1 + x)^{|S|-1} \in \mathcal{J}_S.$$

This completes the proof of Theorem 5.24. $\square$

Example 5.29 For arbitrary n and α, suppose that S is a maximal short subset. Then Corollary 5.25 tells us that $H^*(U_S) \cong \mathbb{Q}[d_1]/\langle d_1^{n-2}\rangle$. We conjecture that in this case we in fact have $U_S \cong \mathbb{C}P^{n-3}$.

Example 5.30 Consider the core component pictured in Example 5.8. By Theorem 5.25 and Remark 5.26,

$$H^*(U_S) \cong \mathbb{Q}[d_1, d_3, d_4, d_5] \Big/ \left\langle \begin{array}{c} d_3(d_1-d_3),\ d_4(d_1-d_4),\ d_5(d_1-d_5),\ d_3d_4,\ d_3d_5,\ d_4d_5, \\ d_1(d_1-d_3-d_4),\ d_1(d_1-d_3-d_5),\ d_1(d_1-d_4-d_5) \end{array} \right\rangle,$$

where d_1 is the fundamental class of $\mathfrak{X}_S$, and d_3, d_4, and d_5 are the negatives of the fundamental classes of the curves labeled 123, 124, and 125, respectively. Because the transverse intersection of two complex varieties is positive, we know that $-d_1d_3[U_S] = 1$. With respect to the basis

$$\{d_1 - d_3 - d_4 - d_5,\ d_3,\ d_4,\ d_5\},$$

the intersection form on $H^2(U_S)$ is represented by the matrix

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

It is likely that U_S is isomorphic to the blow-up of $\mathbb{C}P^2$ at three points.

Example 5.31 Using the same $\alpha = (1,1,3,3,3)$, consider the short subset $S = \{1,3\}$. In this case, Theorem 5.25 tells us that

$$H^*(U_S) \cong \mathbb{Q}[d_1, d_2]/\left\langle d_1^2, d_2(d_1-d_2)\right\rangle.$$

With respect to the basis $\{d_1 - d_2,\ d_2\}$, the intersection form on $H^2(U_S)$ is represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, hence U_S is homeomorphic to the blow-up of $\mathbb{C}P^2$ at a single point.

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