Let G be a reductive algebraic group over the complex numbers and V a finite dimensional linear representation of G. Let X be the **Coulomb space** obtained via the construction of Braverman, Finkelberg, and Nakajima [BFN] and let $X^!$ be the **Higgs space** obtained as a symplectic quotient of T^*V by G. We assume that X and $X^!$ are both conical symplectic singularities, and that there exists a cocharacter of G that induces a Hamiltonian $\mathbb{G}_m$-action on $X^!$ with a unique fixed point along with a conical symplectic resolution $\tilde{X}^!$ of $X^!$. We refer to $X^!$ as the **symplectic dual** of $X^!$; see [BLPW16] and [Web] for more on this notion.

We first focus on a purely on the Coulomb side. Let $\mathscr{A}$ be the canonical quantization of the universal filtered Poisson deformation of a $\mathbb{Q}$-factorial terminalization of X; roughly speaking, this is an algebra with a large center whose central quotients give all possible quantizations of $\mathbb{C}[X]$. Let T be a maximal torus of the automorphism group of X. Using the algebra $\mathscr{A}$, we construct in [KMP] a D-module on an affine T-toric variety which we call the **D-module of traces** because it serves as a universal source for certain types of graded trace maps. The fiber over the identity of T is isomorphic to the degree zero Hochschild homology of $\mathscr{A}$, while the fiber over the T-fixed point is isomorphic to the degree zero Hochschild homology of the B-algebra of $\mathscr{A}$, a gadget which is useful for studying category $\mathcal{O}$.

We now move over to the Higgs side. Assuming some conjectures of Okounkov [Oko, §2.3.4], we define the **quantum D-module** for $\tilde{X}^!$, which is a D-module over an open subset of the same affine toric variety that appeared on the Coulomb side. We then pass to the Calabi-Yau specialization by setting the Rees parameter equal to the $\mathbb{G}_m$-equivariant parameter. This allows us to formulate the **quantum Hikita conjecture** [KMP].

Conjecture 1. *The D-module of graded traces for X, after restricting to the appropriate open subset, becomes isomorphic to the specialized quantum D-module for $\tilde{X}^!$.*

Remark 2. By taking fibers over the T-fixed point, Conjecture 1 specializes to a version of Nakajima's extension of the Hikita conjecture [KTW$^+$, 8.9], which relates the B-algebra of $\mathscr{A}$ to the equivariant cohomology of $\tilde{X}^!$. If we further kill the equivariant parameters, we obtain the original conjecture of Hikita [Hik17], which relates the B-algebra of $\mathbb{C}[X]$ to the ordinary cohomology of $\tilde{X}^!$.

Remark 3. Conjecture 1 is proved for hypertoric varieties (which are dual to other hypertoric varieties) and for the Springer resolution (which is self-dual) [KMP]. Note that the Springer resolution only arises via the Coulomb/Higgs construction described above in type A, so we are actually using a slightly more flexible notion of symplectic duality here.

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