PROBLEMS FOR JUNE 15 – CLUSTER ALGEBRAS AND DOUBLE BRUHAT CELLS

1. Compute all the clusters and cluster variables for the cluster algebras with B-matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

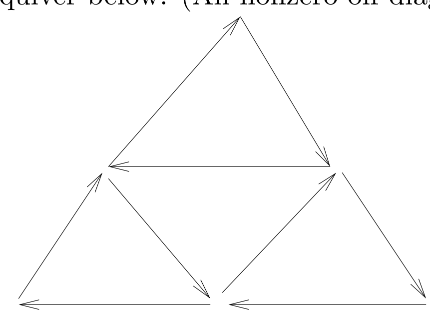
2. Let $\mathcal{A}$ be the cluster algebra of rank 2 with initial exchange matrix

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

So any cluster variable is a Laurent polynomial in the two initial cluster variables x_1 and x_2.

Compute the first few cluster variables in $\mathcal{A}$. If you do this right, the denominator vectors should be of the form $(k, k+1)$ and $(k+1, k)$, for $k \geq 0$, corresponding to the real positive roots of Kac-Moody type $\tilde{A}_1$.

3. The point of this exercise is to give you practice in mutation. Start with the skew-symmetric B-matrix corresponding to the quiver below. (All nonzero off-diagonal entries are ± 1.)



This is of finite type. Is it A_6, D_6 or E_6?

4. Let w_0 be the element $i \mapsto n+1-i$ in S_n.

4.a Write down a reduced word for w_0. Hint: It should have length $\binom{n}{2}$.

4.b Write down the B-matrix for the initial cluster for GL_n^{e,w_0}, where $n = 2, 3, 4, 5$.

4.c For $n = 2, 3$ and (if you want to put in some effort) 4, compute all the cluster variables.

5 (A problem repeated from Monday) The goal of this problem is to demonstrate that David was telling the truth when he told you the semicanonical basis for $\mathfrak{sl}_3$. Let Q be the quiver of type A_2.

5.a Given nonnegative integers $i \leq j$, what are the isomorphism classes of representations of $R(Q)$ with dimension vector (i, j)?

5.b Of the isomorphism classes you found in the previous part, which of them occur on open strata of the space of representations. (Hint: There should be $i+1$ such in total.)

5.c For $0 \leq a \leq i$, compute the constructible function $[S_1]^a [S_2]^j [S_1]^{i-a}$ on these strata. Verify that David got the semicanonical basis right.

6. The tools to do this problem efficiently will be developed on Thursday. The best strategy might be to get as far as you can; then come back to it later.

6.a List the isomorphism classes of indecomposable representations for the preprojective algebra of type A_3. (Hint: There are 12 of them.)

6.b For which pairs of representations is there nontrivial Ext^1? Draw the graph with vertices for the indecomposable representations and with edges where there are nontrivial extensions. (Hint: There are 3 isolated vertices.)

7. Let R be the preprojective algebra for A_5, so the doubled quiver is

$$\bullet \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \bullet$$

Find a dimension vector for which R has infinitely many nonisomorphic representations in that dimension. If you want a hint, apply `ROT13` to the following: Gur qvzrafvba irpgbe lbh jnag vf bar, gjb, qrhk, qbf, hab.

8. (**The point of this exercise is to explain the issue discussed on Monday – why graded bi-algebras tend to have antipodes. We won't actually be using this; it's just for fun.**) Let A be a ***graded bi-algebra*** over $\mathbb{C}$. Here ***graded*** means that $A \cong A_0 \oplus A_1 \oplus A_2 \oplus \cdots$, where A_i are $\mathbb{C}$-vector spaces; that $A_i * A_j \subseteq A_{i+j}$; and that $\Delta(A_k) \subseteq \bigoplus A_i \otimes A_{k-i}$. Suppose furthermore that $A_0 = \mathbb{C} \cdot 1$, where 1 is the multiplicative identity of A, and that the kernel of the co-unit is $\bigoplus_{j>0} A_j$.

We will show that A is a Hopf algebra.

8.a For n a positive integer, define the vector-space map $p_n : A \to A$ by $A \longrightarrow A^{\otimes n} \longrightarrow A$, where the two maps are the n-fold co-product and n-fold product. (If A is a group algebra, this is the "n-th power map".) Show that $p_n(A_k) \subseteq A_k$.

8.b Show that $p_m * p_n = p_{m+n}$, where $*$ is the convolution product from Theo's talk.

8.c Show that the map $p_n : A_k \to A_k$ is given by a polynomial of degree k, and that the coefficients of this polynomial are polynomials in n.

8.d Using part b, we can talk about p_n for n any element of $\mathbb{C}$, by just plugging into the polynomial. Show that p_{-1} is the antipode.

By the way, this proof also basically works over a field of finite characteristic, but you have to be a little more careful with your language. The details are left to you!