PROBLEMS FOR JUNE 13 – HALL ALGEBRAS AND HOPF ALGEBRAS

General notation If V is a representation of a path algebra or preprojective algebra, we write $[V]$ for the constructible function which is 1 on representations isomorphic to V and 0 elsewhere.

1. Consider the Hall algebra of finite dimensional vector spaces over $\mathbb{C}$. Write $[n]$ for the class of $\mathbb{C}^n$. Show that $[n] = [1] * [1] * \cdots * [1]/n!$, where the numerator is n repetitions of the Hall product.

2 Let Q be the quiver A_2, oriented as $\bullet_1 \to \bullet_2$. Bruce never actually finished checking the Serre relations. Let's do so!

2.a List the isomorphism types of indecomposable representations of Q. (Hint: There are 3.)

2.b Let S_1 and S_2 be the simple representations of Q, and let $*$ be the Hall product. Compute $[S_1] * [S_1] * [S_2]$, $[S_1] * [S_2] * [S_1]$ and $[S_2] * [S_1] * [S_1]$.

2.c Check the Serre relation $[e_1, [e_1, e_2]]$.

2.d Repeat the above when the quiver is oriented the other way.

3 Let Q be as above and let $R(Q)$ be the corresponding pre-projective algebra. The goal of this exercise is to repeat exercise 3 for the preprojective algebra.

3.a List the isomorphism types of indecomposable representations of $R(Q)$. (Hint: There are 4.)

3.b Repeat the other parts of exercise 2.

4 The goal of this problem is to demonstrate that David was telling the truth when he told you the semicanonical basis for $\mathfrak{sl}_3$.

4.a Given nonnegative integers $i \leq j$, what are the isomorphism classes of representations of $R(Q)$ with dimension vector (i, j)?

4.b Of the isomorphism classes you found in the previous part, which of them occur on open strata of the space of representations. (Hint: There should be $i + 1$ such in total.)

4.c For $0 \leq a \leq i$, compute the constructible function $[S_1]^a [S_2]^j [S_1]^{i-a}$ on these strata. Verify that David got the semicanonical basis right.

5.a Let A be a $\mathbb{C}$-algebra and let M and N be A-modules. For $\zeta \in \mathrm{Ext}^1(M, N)$, let $E(\zeta)$ be the corresponding extension $0 \to N \to E(\zeta) \to M \to 0$. For any $\zeta \in \mathrm{Ext}^1(M, N)$ and $\alpha \in \mathbb{C}^*$, show that $E(\zeta) \cong E(\alpha\zeta)$ as an A-module.

5.b Let A be a $\mathbb{C}$-algebra. Let M be an indecomposable A-module of finite $\mathbb{C}$-dimension. Show that $\Delta([M]) = [M] \otimes 1 + 1 \otimes [M]$, where Δ is the Green coproduct, defined by Euler characteristics of subspaces of Ext^1.

5.c Let A be as above and let M and N be nonisomorphic A-modules of finite $\mathbb{C}$-dimension. Compute $\Delta([M \oplus N])$ and $\Delta([M \oplus M])$.

If R is a Hopf algebra over a field $\mathbb{C}$, then an element v of R is called ***primitive*** if $\epsilon(v) = 0$ (where ϵ is the coproduct) and $\Delta(v) = v \otimes 1 + 1 \otimes v$.

5.d Show that, if $[M]$ is primitive, then M is indecomposable.

5.e Show that the primitive elements of R form a Lie algebra, under the vector space structure and commutator from R.

6. The point of this exercise is to show that Hall's multiplication and bi-multiplication do not always form a bi-algebra. Note that the definition of Δ is in terms of subsets of Ext^1. We will be working over finite fields, so we will be counting points rather than computing Euler characteristics.

Let k be the field with q elements. Let A be the preprojective algebra of A_2 over k. (The preprojective algebra in characteristic p is not a natural object to consider, but we can do so anyway.) Define $*$ and Δ to be the product and co-product on linear combinations of A-modules, defined as in the Hall bi-algebra.

Let S_1 and S_2 be the simple modules $(k \quad 0)$ and $(0 \quad k)$ for A, and let P_1 and P_2 be the projective modules $(k \to k)$ and $(k \leftarrow k)$, where undrawn arrows are the zero map and present arrows are the identity.

6.a Compute $[S_1] * [P_1]$. Show that $[S_1] \otimes [P_2]$ has coefficient 0 in $\Delta([S_1] * [P_1])$.

6.b Show that $\Delta([P_1]) = [P_1] \otimes 1 + (q-1)[S_1] \otimes [S_2] + 1 \otimes [P_1]$ and $\Delta(S_1) = [S_1] \otimes 1 + 1 \otimes [S_1]$. (Warning: David tends to mix up left and right. Make sure he hasn't done so!)

6.c Compute the coefficient of $[S_1] \otimes [P_2]$ in $\Delta([S_1]) * \Delta([P_1])$. If you have done everything right, you should be able to conclude that $\Delta([S_1]) * \Delta([P_1]) \neq \Delta([S_1] * [P_1])$.

6.d (**David doesn't know the answer to this one**) If you define Δ as AJ did in his talk, by $\Delta(f)([M], [N]) = f([M \oplus N])$, do you always get a bi-algebra?

7. (**David wasn't able to finish this computation on the plane ride, and would love to try to work it through with some people who understand perverse sheaves better than he does.**) Consider the Kronecker quiver: Two vertices joined by two arrows. Consider representations of dimension $(2, 2)$. Write A and B for the two maps in the representation.

7.a Show that a generic representation of dimension vector $(2, 2)$ breaks up as a direct sum of two representations of dimension vectors $(1, 1)$. Show, however, that indecomposable representations of dimension vector $(2, 2)$ do exist.

Let U be the space of indecomposable representations; and let $\overline{U}$ be its closure.

7.b Describe $\overline{U}$ explicitly. (Hint: It is a certain hypersurface.)

7.c Let V be the perverse sheaf on $\overline{U}$, pushed forward from U. Compute the singular support of V. If David understands correctly the analogies in Kashiwara-Saito, then this should have more than one component.

7.d If David is wrong about the above, go back to the Kashiwara-Sato paper and understand their example.

8. The point of this problem is to get straight how the gradings on $U(\mathfrak{n}_+)$ and $\mathbb{C}[N_+]$ relate. Let $\mathfrak{g}$ be a reductive lie algebra, $\mathfrak{h}$ the Cartan, $\mathfrak{n}_+$ the unipotent radical of the Borel and G, H and N_+ the corresponding Lie groups. Let Λ be the character lattice of H.

8.a Show that $U(\mathfrak{n}_+)$ is Λ-graded. How does this grading relate to the grading of the grading of the Hall algebra by dimension vector?

8.b The group H acts on N_+ by conjugation. Suppose that $\phi \in \mathbb{C}[N]$ is homogenous of degree χ for this action – that is to say, for some $\chi \in \Lambda$, we have $\phi(n) = e^{\chi(h)}\phi(e^h n e^{-h})$ for $h \in \mathfrak{h}$ and $n \in \mathfrak{n}_+$. Let δ in $U(\mathfrak{n}_+)$ be homogenous of degree λ. Thinking of δ as a differential operator, show that $\delta(\phi)$ is homogenous. What degree does it live in?

8.c Show that, if $\phi \in \mathbb{C}[N_+]$ is homogenous of degree χ, for $\chi \neq 0$, then $\phi(\mathrm{Id}) = 0$.

8.d Suppose we have a basis for $U(\mathfrak{n}_+)$, all of whose elements are homogenous with respect to the Λ grading. Explain the following statement: "We can compute the dual basis of $\mathbb{C}[N_+]$ while thinking only about finite dimensional vector spaces."

8.e Let $\mathfrak{g}$ be $\mathfrak{gl}_n$. Explicitly, what is the grading on $\mathbb{C}[N]$?

9 This problem fixes a potentially confusing issue in Matt's talk. Let Q be a quiver with vertices v_1, v_2, ..., v_r. The correct definition of the path algebra contains r generators, called e_1, e_2, ..., e_r where e_i is the "path of length 0 from v_i to v_i". If γ is a path from v_j to v_k, then $e_i\gamma$ is 0 if $i \neq j$, and $e_i\gamma = \gamma$ if $i = j$. Similarly, γe_i is 0 or γ according to whether or not $i = k$.

Show that a module for the path algebra is automatically I-graded, and a map of path algebra modules automatically preserves this grading.