The equivariant Orlik-Solomon algebra

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Abstract. Given a real hyperplane arrangement $\mathcal{A}$, the complement $\mathcal{M}(\mathcal{A})$ of the complexification of $\mathcal{A}$ admits an action of the group $\mathbb{Z}_2$ by complex conjugation. We define the equivariant Orlik-Solomon algebra of $\mathcal{A}$ to be the $\mathbb{Z}_2$-equivariant cohomology ring of $\mathcal{M}(\mathcal{A})$ with coefficients in the field $\mathbb{F}_2$. We give a combinatorial presentation of this ring, and interpret it as a deformation of the ordinary Orlik-Solomon algebra into the Varchenko-Gelfand ring of locally constant $\mathbb{F}_2$-valued functions on the complement $\mathcal{M}_{\mathbb{R}}(\mathcal{A})$ of $\mathcal{A}$ in $\mathbb{R}^n$. We also show that the $\mathbb{Z}_2$-equivariant homotopy type of $\mathcal{M}(\mathcal{A})$ is determined by the oriented matroid of $\mathcal{A}$. As an application, we give two examples of pairs of arrangements $\mathcal{A}$ and $\mathcal{A}'$ such that $\mathcal{M}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A}')$ have the same nonequivariant homotopy type, but are distinguished by the equivariant Orlik-Solomon algebra.

1   Introduction

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of n hyperplanes in $\mathbb{C}^d$, with $H_i = \omega_i^{-1}(0)$ for some affine linear map $\omega_i : \mathbb{C}^d \to \mathbb{C}$. Let $\mathcal{M}(\mathcal{A})$ denote the complement of $\mathcal{A}$ in $\mathbb{C}^d$. It is a fundamental problem in the study of hyperplane arrangements to study the extent to which the topology of $\mathcal{M}(\mathcal{A})$ is determined by the combinatorics of $\mathcal{A}$.

Let $\mathcal{C}\mathcal{A}$ denote the central arrangement of hyperplanes in $\mathbb{C}^{d+1}$ given by first adding a "hyperplane at infinity" to $\mathcal{A}$ to produce an arrangement of hyperplanes in $\mathbb{C}P^d$, and then taking its cone. The *pointed matroid* of $\mathcal{A}$ is defined to be the matroid of dependence relations among the hyperplanes of $\mathcal{C}\mathcal{A}$, along with a specified basepoint corresponding to the cone over the hyperplane at infinity [F2]. Geometrically, the pointed matroid encodes two types of data:

1. which subsets $S \subseteq \{1, \ldots, n\}$ have the property that $\bigcap_{i \in S} H_i = \emptyset$, and

2. which subsets $S \subseteq \{1, \ldots, n\}$ have the property that $\operatorname{codim} \bigcap_{i \in S} H_i < |S|$.

Definition 1.1 The *Orlik-Solomon algebra* $A(\mathcal{A}; R)$ is the cohomology ring $H^*(\mathcal{M}(\mathcal{A}); R)$ of the complement of the complexified arrangement with coefficients in the ring R.

For each $i \leq n$, let $e_i = \omega_i^*[\mathbb{R}^+] \in A(\mathcal{A}; R)$ be the pullback of the generator $[\mathbb{R}^+] \in H^1(\mathbb{C}^*; R)$ under the map $\omega_i : \mathcal{M}(\mathcal{A}) \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The following theorem, due to Orlik and Solomon, states that the elements $e_1, \ldots, e_n$ generate $A(\mathcal{A}; R)$, and gives explicit relations in terms of the pointed matroid of $\mathcal{A}$. We give here a simplified version by working only with the coefficient ring $R = \mathbb{F}_2$, the field with two elements, because this is the version that will extend well to the equivariant setting.

Theorem 1.2 [OS] *Consider the linear map $\partial = \sum_{i=1}^n \frac{\partial}{\partial e_i}$ from $\mathbb{F}_2[e_1, \ldots, e_n]$ to itself, lowering degree by 1. The Orlik-Solomon algebra $A(\mathcal{A}; \mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2[e_1, \ldots, e_n]/\mathcal{I}$, where $\mathcal{I}$ is generated by the following three families of relations:*

$$1) \quad e_i^2 \quad for \quad i \in \{1, \ldots, n\}$$

$$2) \quad \prod_{i \in S} e_i \quad if \quad \bigcap_{i \in S} H_i = \emptyset$$

$$3) \quad \partial \prod_{i \in S} e_i \quad if \quad \bigcap_{i \in S} H_i \ is \ nonempty \ with \ codimension \ less \ than \ |S|.$$

Now suppose that our arrangement $\mathcal{A}$ is defined over the real numbers. More precisely, suppose that ω_i restricts to a map $\omega_i : \mathbb{R}^d \to \mathbb{R}$ for all i. Let

$$H_i^+ = \{p \mid \omega_i(p) > 0\} \quad \text{and} \quad H_i^- = \{p \mid \omega_i(p) < 0\},$$

both *open* half-spaces in $\mathbb{R}^d$ with boundary H_i. The *pointed oriented matroid* of $\mathcal{A}$ is defined to be the oriented matroid with basepoint given by the dependence relations of $\mathcal{CA}$. Like the pointed matroid, the pointed oriented matroid also encodes two types of geometrical data:

1. which subsets $S \subseteq \{1, \ldots, n\}$ have the property that $\bigcap_{i \in S} H_i = \emptyset$, and

2. which pairs of subsets $S^+, S^- \subseteq \{1, \ldots, n\}$ have the property that $\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \emptyset$.

Note that this data is stronger than the data of the oriented matroid; if $\cap_{i \in S} H_i$ is nonempty, then its codimension is less than $|S|$ if and only if there exists a decomposition $S = S^+ \cup S^-$ satisfying the second condition above.

In this paper we study the action of $\mathbb{Z}_2 = Gal(\mathbb{C}/\mathbb{R})$ on $\mathcal{M}(\mathcal{A})$ by complex conjugation, with fixed point set $\mathcal{M}_{\mathbb{R}}(\mathcal{A}) \subseteq \mathbb{R}^d$ equal to the complement of the real loci of the hyperplanes. This is an enhancement of the topological data of $\mathcal{A}$, just as the pointed oriented matroid is an enhancement of the combinatorial data. It is therefore natural to make the following definition. We will attempt to avoid confusion by using distinguishing between the group $\mathbb{Z}_2$ and the field $\mathbb{F}_2$.

Definition 1.3 The *equivariant Orlik-Solomon algebra* $\mathcal{A}_2(\mathcal{A}, \mathbb{F}_2)$ of a hyperplane arrangement defined over $\mathbb{R}$ is the equivariant cohomology ring $H^*_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}); \mathbb{F}_2)$.

In Section 3 we give a presentation of the equivariant Orlik-Solomon algebra in terms of the pointed oriented matroid of $\mathcal{A}$, analogous to Theorem 1.2.[1] Moreover, we interpret $\mathcal{A}_2(\mathcal{A}, \mathbb{F}_2)$ as a deformation from the ordinary Orlik-Solomon algebra $A(\mathcal{A}; \mathbb{Z})$ to the *Varchenko-Gelfand ring* $VG(\mathcal{A}; \mathbb{F}_2)$, which is defined to be the ring of locally constant functions from $\mathcal{M}_{\mathbb{R}}(\mathcal{A})$ to $\mathbb{F}_2$. We thus recover by independent means the presentation of $VG(\mathcal{A}; \mathbb{F}_2)$ given in [VG], and provide a topological explanation for the parallels that Varchenko and Gelfand observe between the the rings $A(\mathcal{A}; \mathbb{Z})$ and $VG(\mathcal{A}; \mathbb{F}_2)$. Note that, while the Orlik-Solomon algebra is super-commutative and the Varchenko-Gelfand ring is commutative, these two notions agree in characteristic 2.

A celebrated theorem of Salvetti [Sa] says that if $\mathcal{A}$ is central and essential, then $\mathcal{M}(\mathcal{A})$ is homotopy equivalent to a simplicial complex that can be constructed from the oriented matroid[2] of $\mathcal{A}$ (see [Sa], [Pa], and [GR]). In Section 4, we show that this simplicial complex has a natural, combinatorially defined action of $\mathbb{Z}_2$, and that the homotopy equivalence is equivariant with respect to this action. Hence the oriented matroid of $\mathcal{A}$ in fact determines the equivariant homotopy type of $\mathcal{M}(\mathcal{A})$. This observation provides an explanation for the recent discovery of Huisman that the equivariant fundamental group of a line arrangement is determined by its pointed oriented matroid [Hu].

In Example 5.2 we consider the first example of two real arrangements with different pointed matroids, but with homotopy equivalent complements [F1]. We show that these two arrangements are distinguished by the equivariant Orlik-Solomon algebra, hence the homotopy equivalence cannot be made equivariant. In Example 5.3, we consider two arrangements whose pointed oriented matroids are related by a flip [F1]. This implies that their complements are homotopy equivalent, and that their unoriented pointed matroids are isomorphic, but once again their equivariant homotopy types are distinguished by the equivariant Orlik-Solomon algebra. We conclude with a conjecture regarding the relationship between the combinatorial data and the equivariant topology of a real arrangement.

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[1] A special case of this presentation first appeared in [HP, 5.5], using the geometry of hypertoric varieties.

[2] If $\mathcal{A}$ is central, the oriented matroid and pointed oriented matroid encode the same data.

2   Equivariant cohomology

In this section we review some basic definitions and results from [Bo]. Let X be a topological space equipped with an action of a group G.

Definition 2.1 Let EG be a contractible space with a free G-action. Then we put

$$X_G := X \times_G EG = (X \times EG)/G$$

(well-defined up to homotopy equivalence), and define the G-equivariant cohomology of X

$$H_G^*(X) := H^*(X_G).$$

The G-equivariant map from X to a point induces a map on cohomology in the other direction, hence $H_G^*(X)$ is a module over $H_G^*(pt) \cong H^*(BG)$, where $BG = EG/G$ is the classifying space for G. Indeed, H_G^* is a contravariant functor from the category of G-spaces to the category of $H_G^*(pt)$-modules.

Example 2.2 If $G = \mathbb{Z}_2$, then we may take $EG = S^\infty$ and $BG = S^\infty/\mathbb{Z}_2 = \mathbb{R}P^\infty$. Then $H_{\mathbb{Z}_2}^*(pt; \mathbb{F}_2) = H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$.

Suppose that X is a finite-dimensional manifold, and let $Y \subseteq X$ be a G-invariant submanifold. We denote by $[Y] \in H_G^*(X)$ the cohomology class represented in Borel-Moore homology by the finite-codimension submanifold $Y_G \subseteq X_G$. This will be our principal means of understanding specific equivariant cohomology classes in this paper. We will need two technical theorems about equivariant cohomology, both of which we state below. Let X be a $\mathbb{Z}_2$-space, and let $F \subseteq X$ be the fixed point set.

Theorem 2.3 [Bo, §XII, 3.5] *Suppose that F is nonempty, the induced action of $\mathbb{Z}_2$ on $H^*(X; \mathbb{F}_2)$ is trivial, and $H^*(X; \mathbb{F}_2)$ is generated in degree 1. Then the Leray-Serre spectral sequence for the fiber bundle $X \hookrightarrow X_{\mathbb{Z}_2} \to \mathbb{R}P^\infty$ collapses at the E_2 term.*

Corollary 2.4 *Under the hypotheses of Theorem 2.3, any additive basis from $H^*(X; \mathbb{F}_2)$ lifts to a $\mathbb{F}_2[x]$-basis for $H_{\mathbb{Z}_2}^*(X; \mathbb{F}_2)$ (and any set of degree-preserving lifts will do). In particular, $H_{\mathbb{Z}_2}^*(X; \mathbb{F}_2)$ is a free module over $\mathbb{F}_2[x]$.*

Theorem 2.5 [Bo, §IV, 3.7(b)] *The restriction map $H_{\mathbb{Z}_2}^*(X; \mathbb{F}_2) \to H_{\mathbb{Z}_2}^*(F; \mathbb{F}_2) \cong H^*(F; \mathbb{F}_2)[x]$ is an isomorphism in all degrees greater than the dimension of X.*

Corollary 2.4 says that we may interpret $H_{\mathbb{Z}_2}^*(X; \mathbb{F}_2)$ as a flat family of rings over the $\mathbb{F}_2$ affine line. The following corollary says that this family is a deformation of $H^*(X; \mathbb{F}_2)$ into $H^*(F; \mathbb{F}_2)$.

Corollary 2.6 *Under the hypotheses of Theorem 2.3,*

$$H^*(X; \mathbb{F}_2) \cong H_{\mathbb{Z}_2}^*(X; \mathbb{F}_2)/\langle x \rangle$$

and

$$H^*(F; \mathbb{F}_2) \cong H_{\mathbb{Z}_2}^*(X; \mathbb{F}_2)/\langle x - 1 \rangle.$$

Proof: The first statement follows immediately from Corollary 2.4. For the second statement, consider the ring $H_{\mathbb{Z}_2}^*(X; \mathbb{F}_2)[x^{-1}]$ obtained by formally inverting x. Theorem 2.5 tells us that the restriction map

$$H_{\mathbb{Z}_2}^*(X; \mathbb{F}_2)[x^{-1}] \to H_{\mathbb{Z}_2}^*(F; \mathbb{F}_2)[x^{-1}] \cong H^*(F; \mathbb{F}_2)[x, x^{-1}]$$

is an isomorphism in high degree. But this map commutes with multiplication by x and x^{-1}, so it must be an isomorphism in every degree. Setting x equal to 1, we obtain the desired result. $\square$

The following example will be fundamental to our applications.

Example 2.7 Let $X = \mathbb{C}^*$, with $\mathbb{Z}_2$ acting by complex conjugation. Since X deformation-retracts equivariantly onto the compact space S^1, Theorem 2.3 applies. The image of x under the standard map $\mathbb{F}_2[x] = H^*_{\mathbb{Z}_2}(pt, \mathbb{F}_2) \to H^*_{\mathbb{Z}_2}(X; \mathbb{F}_2)$ is the $\mathbb{Z}_2$-equivariant Euler class of the topologically trivial real line bundle with a nontrivial $\mathbb{Z}_2$ action. This bundle has a $\mathbb{Z}_2$-equivariant section, transverse to the zero section, vanishing exactly on the real points of X, and is therefore represented by the submanifold $\mathbb{R}^* \subseteq \mathbb{C}^*$. Abusing notation, we will write $x = [\mathbb{R}^*] \in H^*_{\mathbb{Z}_2}(X; \mathbb{F}_2)$. Let $y = [\mathbb{R}^+] \in H^*_{\mathbb{Z}_2}(X; \mathbb{F}_2)$. Then $x - y$ is represented by $\mathbb{R}^-$, therefore $y(x - y) = 0$. Corollary 2.4 says that $H^*_{\mathbb{Z}_2}(X; \mathbb{F}_2)$ is additively generated by x and y. Since $\mathbb{F}_2[x, y]/\langle y(x - y) \rangle$ is already a free module of rank 2 over $\mathbb{F}_2[x]$, Corollary 2.4 tells us that there can be no more relations.

3   The equivariant Orlik-Solomon algebra

We now give a combinatorial presentation of the equivariant Orlik-Solomon algebra.

Theorem 3.1 *The ring $A_2(\mathcal{A}; \mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2[e_1, \ldots, e_n, x]/\mathcal{J}$, where $\mathcal{J}$ is generated by the following three families of relations:*[3]

$$1) \quad e_i(x - e_i) \quad for \quad i \in \{1, \ldots, n\}$$

$$2) \quad \prod_{i \in S^+} e_i \times \prod_{j \in S^-} (x - e_j) \quad if \quad \bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \emptyset$$

$$3) \quad x^{-1}\left(\prod_{i \in S^+} e_i \times \prod_{j \in S^-} (x - e_j) - \prod_{i \in S^+} (x - e_i) \times \prod_{j \in S^-} e_j \right)$$

$$if \quad \bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \emptyset \quad and \quad \bigcap_{i \in S} H_i \neq \emptyset, \quad where \quad S = S^+ \cup S^-.$$

Proof: Let $y = [\mathbb{R}^+] \in H^*_{\mathbb{Z}_2}(\mathbb{C}^*; \mathbb{F}_2)$, and let

$$e_i = \omega_i^*(y) \in A_2(\mathcal{A}; \mathbb{F}_2),$$

represented by the submanifold

$$Y_i^+ = \omega_i^{-1}(\mathbb{R}^+).$$

Let $x \in A_2(\mathcal{A}; \mathbb{F}_2)$ be the image of the generator of $H^*_{\mathbb{Z}_2}(pt; \mathbb{F}_2)$; by functoriality, we have $x = \omega_i^*(x)$ for all i. Recall from Example 2.7 that $[\mathbb{R}^-] = x - y \in H^*_{\mathbb{Z}_2}(\mathbb{C}^*; \mathbb{F}_2)$, hence

$$x - e_i = \omega^*(x - y) \in A_2(\mathcal{A}; \mathbb{F}_2)$$

is represented by the submanifold

$$Y_i^- = \omega^{-1}(\mathbb{R}^-).$$

Theorem 1.2 tells us that $e_1, \ldots, e_n$ are lifts of ring generators for the ordinary Orlik-Solomon algebra $A(\mathcal{A}; \mathbb{F}_2)$. Since the manifolds Y_i^+ are stable under the action of $\mathbb{Z}_2$, the induced action of $\mathbb{Z}_2$ on $A(\mathcal{A}; \mathbb{F}_2)$ is trivial. The space $\mathcal{M}(\mathcal{A})$ has a compact $\mathbb{Z}_2$-equivariant deformation retract, therefore Corollary 2.4 tells us that $A_2(\mathcal{A}; \mathbb{F}_2)$ is generated as a ring by the classes e_i and x. We must now check that each of the three families of generators of $\mathcal{J}$ do indeed vanish in $A_2(\mathcal{A}; \mathbb{F}_2)$, and that they generate all of the relations.

The first family of relations follows from the fact that $Y_i^+ \cap Y_i^- = \emptyset$ for all $i \in \{1, \ldots, n\}$. For the second family, we must show that if

$$\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \emptyset,$$

then

$$\bigcap_{i \in S^+} Y_i^+ \cap \bigcap_{j \in S^-} Y_j^- = \emptyset.$$

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[3]Note that all of these relations are polynomial; the x^{-1} in the third relation cancels.

Suppose that
$$p \in \bigcap_{i \in S^+} Y_i^+ \cap \bigcap_{j \in S^-} Y_j^-,$$
in other words $\omega_i(p) \in \mathbb{R}^+$ for all $i \in S^+$ and $\omega_j(p) \in \mathbb{R}^-$ for all $j \in S^-$. Then the real part
$$\mathrm{Re}(p) \in \bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^-,$$
hence the intersection is not empty.

For the third family, note that since $A_2(\mathcal{A}; \mathbb{F}_2)$ is free over $\mathbb{F}_2[x]$, it is sufficient to show that

$$\prod_{i \in S^+} e_i \times \prod_{j \in S^-} (x - e_j) - \prod_{i \in S^+} (x - e_i) \times \prod_{j \in S^-} e_j = 0. \tag{1}$$

By hypothesis,
$$\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \emptyset \quad \text{and} \quad \bigcap_{i \in S} H_i \neq \emptyset,$$
where $S = S^+ \cup S^-$. Choose a point $p \in \bigcap_{i \in S} H_i$. The involution of $\mathbb{R}^d$ given by reflection through p takes H_i^+ to H_i^- and vice versa for all $i \in S$, hence we also have

$$\bigcap_{i \in S^+} H_i^- \cap \bigcap_{j \in S^-} H_j^+ = \emptyset.$$

Thus both of the terms in Equation (1) are in fact contained in the second family of relations, and are therefore equal to zero in the ring $A_2(\mathcal{A}; \mathbb{F}_2)$.

Now we must show that we have found all of the relations. Let

$$\psi : \mathbb{F}_2[e_1, \ldots, e_n, x] \to \mathbb{F}_2[e_1, \ldots, e_n]$$

be the map given by sending x to zero, and note that $\psi(\mathcal{J}) = \mathcal{I}$. Now suppose that $\alpha \in \mathbb{F}_2[e_1, \ldots, e_n, x]$ is a relation in $A_2(\mathcal{A}; \mathbb{F}_2)$ that is *not* in the ideal $\mathcal{J}$, and choose α of minimal degree. By Corollary 2.6 we must have $\psi(\alpha) \in \mathcal{I}$, hence there exists $\beta \in \mathcal{J}$ with $\psi(\alpha - \beta) = 0$. This implies that $\alpha - \beta = x\gamma$ for some $\gamma \in \mathbb{F}_2[e_1, \ldots, e_n, x]$. Since α and β are both relations in $A_2(\mathcal{A}; \mathbb{F}_2)$, and $A_2(\mathcal{A}; \mathbb{F}_2)$ is free over $\mathbb{F}_2[x]$, γ must also be a relation. Since β is in $\mathcal{J}$ and α is not, γ cannot be in $\mathcal{J}$. Since $\deg \gamma = \deg \alpha - 1$, we have reached a contradiction. $\square$

By Corollary 2.6, $A_2(\mathcal{A}; \mathbb{F}_2)$ is a flat family of rings parameterized by the affine line $\mathrm{Spec}\, \mathbb{F}_2[x]$, specializing at $x = 0$ to $H^*(\mathcal{M}(\mathcal{A}); \mathbb{F}_2) = A(\mathcal{A}; \mathbb{F}_2)$, and at $x = 1$ to $H^*(\mathcal{M}_\mathbb{R}(\mathcal{A}); \mathbb{F}_2) = VG(\mathcal{A}; \mathbb{F}_2)$. In particular, this provides a topological explanation for the fact that the dimension of the Orlik-Solomon algebra is equal to the number of connected components of $\mathcal{M}_\mathbb{R}(\mathcal{A})$. By setting $x = 1$ in Theorem 3.1 we obtain a nontrivial presentation of $VG(\mathcal{A}; \mathbb{F}_2)$, first given in the central case (over the integers) in [VG]. Varchenko and Gelfand interpret $e_i \in VG(\mathcal{A}; \mathbb{F}_2)$ as the ith *Heaviside function* $\mathcal{M}_\mathbb{R}(\mathcal{A}) \to \mathbb{R}$, restricting to 1 on $\mathcal{M}_\mathbb{R}(\mathcal{A}) \cap H_i^+$ and 0 on $\mathcal{M}_\mathbb{R}(\mathcal{A}) \cap H_i^-$. These functions are easily seen to generate the ring $VG(\mathcal{A}; \mathbb{F}_2)$, and the three families of relations are clear, but the proof that there are no other relations is nontrivial. Varchenko and Gelfand observe that this presentation defines a filtration on $VG(\mathcal{A}; \mathbb{F}_2)$ with $A(\mathcal{A}; \mathbb{F}_2)$ as its associate graded. This is also a consequence of Corollaries 2.4 and 2.6; this phenomenon is explored in greater detail in [Ca].

Remark 3.2 Our presentations of $VG(\mathcal{A}; \mathbb{F}_2)$ and $A_2(\mathcal{A}; \mathbb{F}_2)$ depend on the coorientations of the hyperplanes, while the isomorphism classes of the rings themselves do not. Reversing the orientation of the hyperplane H_i corresponds to changing every appearance of e_i to $x - e_i$ in the generators of $\mathcal{J}$.

4 The Salvetti complex

Let $\mathcal{A}$ be an essential central arrangement in $\mathbb{R}^d$. Salvetti [Sa] has constructed a simplicial complex from a poset $\mathrm{Sal}(\mathcal{A})$, depending only on the oriented matroid of $\mathcal{A}$, which is homotopy equivalent to the complement $\mathcal{M}(\mathcal{A})$ of the complexification of $\mathcal{A}$. In this section we define a combinatorial action of $\mathbb{Z}_2$ on $\mathrm{Sal}(\mathcal{A})$, and show that the homotopy equivalence is equivariant.

The hyperplanes of $\mathcal{A}$ subdivide $\mathbb{R}^d$ into faces, open in their supports, which form a poset $\mathcal{F}$ ordered by reverse inclusion. The minimal elements of $\mathcal{F}$ are the connected components of $\mathcal{M}_{\mathbb{R}}(\mathcal{A})$, and $\{0\}$ is the unique maximal element. The *Salvetti poset* $\mathrm{Sal}(\mathcal{A})$ is a poset consisting of elements of the form

$$\{(F, C) \mid C \text{ minimal and } C \leq F\}.$$

The partial order is determined by putting $(F', C') \leq (F, C)$ if and only if $F' \leq F$ and $C' = F'C$, where the latter equality means that C and C' lie on the same side of every hyperplane containing F'. The *Salvetti complex* $|\mathrm{Sal}(\mathcal{A})|$ is defined to be the order complex of this poset.

The poset $\mathrm{Sal}(\mathcal{A})$ admits an action of $\mathbb{Z}_2$ given by setting $(F, C)^* = (F, \tilde{C})$, where $\tilde{C}$ is obtained from C by reflecting it over all of the hyperplanes that contain F. In [GR], $\mathrm{Sal}(\mathcal{A})$ is defined as a subset of the set of all functions from the ground set of the oriented matroid to the set $\{\pm 1, \pm i\}$. In this language, our $\mathbb{Z}_2$-action is simply complex conjugation, and is easily seen to be an invariant of the oriented matroid. This action induces an action of $\mathbb{Z}_2$ on the Salvetti complex $|\mathrm{Sal}(\mathcal{A})|$.

Theorem 4.1 *The complex $|\mathrm{Sal}(\mathcal{A})|$ is equivariantly homotopy equivalent to $\mathcal{M}(\mathcal{A})$. In particular, the equivariant homotopy type of $\mathcal{M}(\mathcal{A})$ is determined by the oriented matroid associated to $\mathcal{A}$.*

Proof: Though it is possible to adapt the original argument of [Sa] to the equivariant context, we will instead follow the proof given in [Pa]. For every $F \in \mathcal{F}$, choose a point $x(F) \in F \subseteq \mathbb{R}^d$. Each element of $\mathrm{Sal}(\mathcal{A})$ determines a vertex in the complex $|\mathrm{Sal}(\mathcal{A})|$. For all $(F, C) \in \mathrm{Sal}(\mathcal{A})$, let

$$V(F, C) = \begin{cases} \left\{ \left. \sum_{C' \leq F} \lambda_{C'} x(C') \; \right| \; \lambda_{C'} > 0 \right\} & \text{if } F \neq \{0\} \\ \mathbb{R}^d & \text{if } F = \{0\}, \end{cases}$$

and let

$$W(F, C) = \{x \in \mathbb{R}^d \mid x \text{ and } C \text{ lie on the same side of every hyperplane containing } F\}.$$

Paris [Pa] shows that

$$\mathcal{U} = \left\{ V(F, C) + iW(F, C) \; \middle| \; (F, C) \in \mathrm{Sal}(\mathcal{A}) \right\}$$

is an open cover of $\mathcal{M}(\mathcal{A})$ with nerve $|\mathrm{Sal}(\mathcal{A})|$, and that any nonempty intersection of open sets from $\mathcal{U}$ is contractible, hence concluding that $\mathcal{M}(\mathcal{A})$ is homotopy equivalent to $|\mathrm{Sal}(\mathcal{A})|$. To extend this proof to the equivariant context, we need only observe that $W(F, \tilde{C}) = W(F, C)$, and $V(F, \tilde{C}) = -V(F, C)$. Both of these equalities are clear from the definitions. $\qquad\square$

Remark 4.2 The Salvetti complex may be defined for an arbitrary oriented matroid, which may not be realizable by a hyperplane arrangement (see for example [BLSWZ]). We can then define the equivariant Orlik-Solomon algebra of an arbitrary oriented matroid to be the $\mathbb{Z}_2$-equivariant cohomology ring of its Salvetti complex. Theorem 4.1 implies that this definition agrees with our original one if the oriented matroid is realizable.

5 Examples

In this section we discuss two examples in which the equivariant Orlik-Solomon algebra distinguishes two arrangements with (nonequivariantly) homotopy equivalent complements. In both, we work with affine arrangements to keep dimensions as low as possible. The analogous central examples can be understood via the following proposition.

Proposition 5.1 *There is a $\mathbb{Z}_2$-equivariant diffeomorphism $\mathcal{M}(\mathcal{CA}) \cong \mathcal{M}(\mathcal{A}) \times \mathbb{C}^*$, and*

$$A_2(\mathcal{CA}; \mathbb{F}_2) \cong A_2(\mathcal{A}; \mathbb{F}_2) \otimes_{\mathbb{F}_2[x]} \mathbb{F}_2[x,y]/y(x-y).$$

Proof: The standard diffeomorphism $\mathcal{M}(\mathcal{CA}) \cong \mathcal{M}(\mathcal{A}) \times \mathbb{C}^*$, found for example in [OT], is $\mathbb{Z}_2$-equivariant. The second half of the proposition is simply the statement of the equivariant Künneth theorem [Se, 7.4], combined with Example 2.7. $\qquad\qquad\square$

Example 5.2 The example of Figure 1 was introduced by Falk [F1, 3.1]. The arrangements $\mathcal{A}$ and $\mathcal{A}'$ have
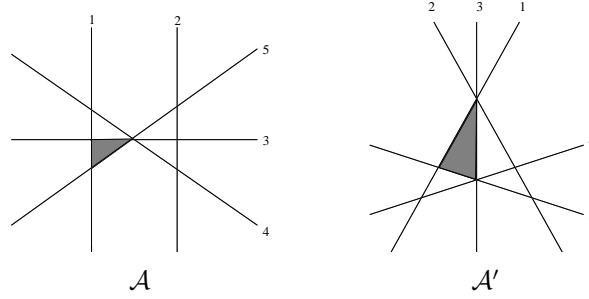

Figure 1: Two arrangements whose complements are homotopy equivalent, but not equivariantly.

nonisomorphic pointed matroids, but their complements are homotopy equivalent. In particular, they cannot be distinguished by their Orlik-Solomon algebras. We show that their equivariant Orlik-Solomon algebras are nonisomorphic, therefore the homotopy equivalence between their complements cannot be $\mathbb{Z}_2$-equivariant. Choose coorientations so that the intersections $\cap_{i \leq 5} H_i^-$ are equal to the shaded regions. Then

$$A_2(\mathcal{A}; \mathbb{F}_2) = \mathbb{F}_2[e_1, \ldots, e_5, x]/\mathcal{J} \quad \text{and} \quad A_2(\mathcal{A}'; \mathbb{F}_2) = \mathbb{F}_2[e_1, \ldots, e_5, x]/\mathcal{J}',$$

where

$$\mathcal{J} = \left\langle \begin{array}{c} e_1(x-e_1), \ldots, e_5(x-e_5), e_1e_2, e_1(x-e_3)e_4, e_1e_3e_5, e_1e_4e_5, e_2e_3(x-e_4), \\ e_2(x-e_4)(x-e_5), e_2(x-e_3)(x-e_5), e_3e_4 + e_3e_5 + e_4e_5 + e_4x \end{array} \right\rangle$$

and

$$\mathcal{J}' = \left\langle \begin{array}{c} e_1(x-e_1), \ldots, e_5(x-e_5), e_1e_2e_4, e_1e_2e_5, e_1e_3e_4, e_1e_3e_5, e_1e_4(x-e_5), e_2(x-e_3)e_4, \\ e_2(x-e_3)e_5, e_2(x-e_4)e_5, e_1e_2 + e_1e_3 + e_2e_3 + e_2x, e_3e_4 + e_3e_5 + e_4e_5 + e_4x \end{array} \right\rangle.$$

Using Macaulay 2 [M2], we can compute the number of elements in each degree that annihilate a given linear form. By comparing the lists that we obtain for the two rings, we conclude that the rings are not isomorphic.

These two arrangements are generic rank 2 truncations of a pair of rank 3 arrangements $\mathcal{A}_3$ and $\mathcal{A}_3'$ which have diffeomorphic complements by a general construction relating parallel connections to direct sums (see [EF, Theorem 2] and [F2, 3.8]). The first arrangement $\mathcal{A}_3$ is given by the equation

$$(x+1)(x-1)y(y+z)(y-z) = 0,$$

with $\mathcal{A}$ obtained from $\mathcal{A}_3$ by setting $z = x$. The second arrangement $\mathcal{A}_3'$ is given by the equation

$$(2x+y-z)(2x-y+z)x(x-y)(x+y) = 0,$$

with $\mathcal{A}'$ obtained from $\mathcal{A}_3'$ by setting $z = 1$. The diffeomorphism between $\mathcal{M}(\mathcal{A}_3)$ and $\mathcal{M}(\mathcal{A}_3')$ given in [EF] is easily seen to be $\mathbb{Z}_2$-equivariant, as it is essentially derived from repeated applications of the diffeomorphism of Proposition 5.1. Furthermore, it is not hard to produce an explicit isomorphism between $A_2(\mathcal{A}_3; \mathbb{F}_2)$ and $A_2(\mathcal{A}_3'; \mathbb{F}_2)$. This shows that a theorem of Pendergrass [F2, 3.11], which states that truncation of matroids preserves isomorphisms of Orlik-Solomon algebras, does not extend to the equivariant setting.
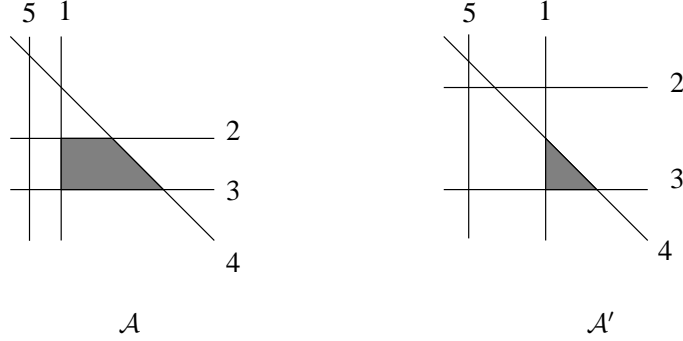
Figure 2: Two arrangements related by a flip with nonisomorphic equivariant Orlik-Solomon algebras.

Example 5.3 Consider the two line arrangements shown in Figure 2.[4]

The pointed oriented matroids corresponding to these two arrangements are related by a flip. Geometrically, this means that $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by translating one of the hyperplanes from one side of a vertex to another. (For a precise definition of flips, see [BLSWZ, §7.3].) Falk [F1] has shown that any two real line arrangements related by a flip have homotopy equivalent complements; in this example we show that Falk's theorem does not extend to the equivariant setting. We have

$$A_2(\mathcal{A};\mathbb{F}_2) \cong \mathbb{F}_2[\vec{e},x] \Big/ \left\langle \begin{array}{c} e_1(x-e_1), e_2(x-e_2), e_3(x-e_3), e_4(x-e_4), \\ e_5(x-e_5), e_2e_3, (x-e_1)e_5, e_1(x-e_2)e_4, \\ e_1e_3e_4, (x-e_2)e_4e_5, e_3e_4e_5 \end{array} \right\rangle$$

and

$$A_2(\mathcal{A}';\mathbb{F}_2) \cong \mathbb{F}_2[\vec{e},x] \Big/ \left\langle \begin{array}{c} e_1(x-e_1), e_2(x-e_2), e_3(x-e_3), e_4(x-e_4), \\ e_5(x-e_5), e_2e_3, (x-e_1)e_5, (x-e_1)e_2(x-e_4), \\ e_1e_3e_4, (x-e_2)e_4e_5, e_3e_4e_5 \end{array} \right\rangle.$$

Using Macaulay 2 [M2], we find that the annihilator of the element $e_2 \in A_2(\mathcal{A};\mathbb{F}_2)$ is generated by two linear elements (namely e_3 and $x-e_2$) and nothing else, while none of the (finitely many) elements of $A_2(\mathcal{A}';\mathbb{F}_2)$ has this property. Hence the two rings are not isomorphic, and $\mathcal{M}(\mathcal{A})$ is not equivariantly homotopy equivalent to $\mathcal{M}(\mathcal{A}')$. From this example we conclude that the equivariant Orlik-Solomon algebra of an arrangement is *not* determined by the pointed *un*oriented matroid.

Both of the arrangements that we have discussed have connected pointed matroids. Eschenbrenner and Falk [EF] conjecture that if $\mathcal{A}$ is a complex central arrangement with connected matroid, then the matroid of $\mathcal{A}$ is determined by the homotopy type of $\mathcal{M}(\mathcal{A})$. The analogous conjecture for real arrangements may be stated as follows.

Conjecture 5.4 *If $\mathcal{A}$ is a real central arrangement with connected matroid, then the oriented matroid of $\mathcal{A}$ is determined by the equivariant homotopy type of $\mathcal{M}(\mathcal{A})$.*

References

[Bo] A. Borel. *Seminar on Transformation Groups*. Annals of Mathematical Studies 46, Princeton, 1960.

[BLSWZ] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler. *Oriented Matroids*. Cambridge University Press, 1991.

[Ca] J. Carrell. Torus actions and cohomology. *Encyclopaedia Math. Sci.*, 131, Springer, Berlin, 2002.

[EF] C. Eschenbrenner and M. Falk. Orlik-Solomon algebras and Tutte polynomials. *J. Algebr. Comb.* 10 (1999), 189–199.

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[4]This example appeared first in [HP].

[F1]   M. Falk. Homotopy types of line arrangements. *Inv. Math.* 111 (1993), 139–150.

[F2]   M. Falk. Combinatorial and algebraic structure in Orlik-Solomon algebras. *Europ. J. Combinatorics* 22 (2001), 687–698.

[GR]   I.M. Gelfand and G.L. Rybnikov. Algebraic and topological invariants of oriented matroids. *Soviet Math. Dokl.* 40 (1990), no. 1, 148–152.

[HP]   M. Harada and N. Proudfoot. Properties of the residual circle action on a hypertoric variety. To appear in *Pacific J. of Math.* http://math.berkeley.edu/∼proudf/ht.ps

[Hu]   J. Huisman. Real line arrangements and fundamental groups. http://name.math.univ-rennes1.fr/johannes.huisman/recherche/publications/rlafg.html

[M2]   D. Grayson and M. Stillman. Macaulay 2: a computer algebra system for algebraic geometry. Version 0.8.56, www.math.uiuc.edu/macaulay2, 1999.

[OS]   P. Orlik and L. Solomon. Combinatorics and topology of complements of hyperplanes. *Invent. Math.* 56 (1980), 167–189.

[OT]   P. Orlik and H. Terao. *Arrangements of Hyperplanes.* Grundlehren der mathematischen Wissenschaften 300, Springer-Verlag, 1992.

[Pa]   L. Paris. Universal cover of Salvetti's complex and topology of simplicial arrangements of hyperplanes. *Trans. Am. Math. Soc.* 340 (1993), no. 1, 149–178.

[Sa]   M. Salvetti. Topology of the complement of real hyperplanes in $\mathbb{C}^N$. *Invent. Math.* 88 (1987), 603–618.

[Se]   R.M. Seymour. On G-cohomology theories and Künneth formulae. *Current trends in algebraic topology, Part 1* (London, Ontario, 1981), 257–271, CMS Conf. Proc., 2, Amer. Math. Soc., Providence, R.I., 1982.

[VG]   A.N. Varchenko and I.M. Gelfand. Heaviside functions of a configuration of hyperplanes. *Functional Anal. Appl.* 21 (1987), no. 4, 255–270.