TYPE III FACTORS AND INDEX THEORY

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1. Introduction

These notes are based on a series of lectures given at Seoul National University in February of 1993. The Jones theory on index ([33]) has brought a revolutionary change to the theory of operator algebras, and since the appearance of the theory tremendous progress has been made for the subject matter and related subfactor analysis by many authors. The paragroup theory due to Ocneanu ([54, 55]) and classification results for subfactors due to Popa ([58, 59]) should be particularly mentioned. Details on the former can be found in Kawahigashi's work [36]. In the literature on the index theory type II_1 factors are mainly emphasized (although there are many important exceptions). The main purpose here is to give an account on the index theory and related topics with main emphasis on type III factors. The author believes (and hopes) that structure analysis of type III factors will be further enriched by the index theory (and vice versa).

In §2 we recall main ingredients of the modular theory and structure analysis on type III factors (see [63] for example). However, instead of going into technical details we consider typical examples arising from ergodic theory. In §3, after briefly explaining Jones' work [33] of index for II_1 factors, we describe a notion of the index for a normal conditional expectation onto a subfactor ([38]), Longo's approach to the index theory ([51]), a notion of the minimal index due to Havet, Hiai, and Longo ([26, 27, 51]), and the important work [56] by Pimsner and Popa. The importance of graphs (as well as other combinatorial invariants) derived from successive basic extensions was recognized from the early stage of the index theory (see [16]), and in §4 those graphs are described for factor-subfactor pairs arises from group-subgroup pairs (the approach based on bimodules can be found in [45]). In §5 inclusions of type III factors are considered, and we begin by comparing the flows of weights of two factors in question ([23, 39]). Classification results for certain subfactors of the Powers factor due to Loi and Popa are briefly explained ([48, 49, 60]). Factor-subfactor pairs arising from relation-subrelation pairs or factor maps via the Krieger construction (see [13, 46]) is very useful to describe typical inclusions of type III_0 factors ([25]), and brief explanation on related topics is also given. The sector theory originally occurred in [9], and its usefulness for the index theory was first noticed by Longo ([51]). It is closely related to bimodule theory whose relevance in subfactor analysis was first noticed by Ocneanu. The importance of this technique was further confirmed in a series of papers by Izumi. Basic facts on the sector theory as well as typical applications are explained in §6 (see [29, 30, 31, 32, 52, 53] for further applications). For the reader's convenience basic facts on conditional expectations and operator valued weights used here are summarized in Appendix.

The author thanks Professor Sa Ge Lee and all the members of the operator algebra seminar at Seoul National University for giving him the opportunity to give lectures. The author is also indebted to Professor Jeong Hee Hong for typing a part of the manuscript and various useful comments on materials. The main emphasis here being type III factors, the author was forced to omit many important topics on subfactor analysis in the type II_1 setting. Also very recent works are not listed in our references. Quite a complete updated list of references on the index theory and related topics (especially those on type II_1 factors) can be found in the recent textbook [11].

CONTENTS

2. Structure of type III factors

A von Neumann algebra $\mathcal{M}$ ($\subseteq \mathcal{B}(\mathcal{H})$) means a strongly closed $*$-algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. A von Neumann algebra $\mathcal{M}$ is called a factor if $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}' = \mathbf{C}I$, where

$$\mathcal{M}' = \{x \in \mathcal{B}(\mathcal{H}) \,|\, yx = xy,\, y \in \mathcal{M}\} \quad \text{(commutant)}.$$

The reduction theory says that every von Neumann algebra (with separable predual) can be expressed as the direct integral (over the center) of factors

$$\mathcal{M} = \int_X^{\oplus} \mathcal{M}(\omega) d\omega.$$

Factors are classified into the following types:

type I_n factor (i.e., $M_n(\mathbf{C})$) for $n = 1, 2, \cdots$,

type I_∞ factor (i.e., $\mathcal{B}(\mathcal{H})$),

type II_1 factor,

type II_∞ factor ($=$ (type II_1 factor) $\otimes\, \mathcal{B}(\mathcal{H})$),

type III factor.

A trace $tr : \mathcal{M}_+ \longrightarrow \mathbf{R}_+ \cup \{\infty\}$ is a linear functional with the tracial property $tr(xx^*) = tr(x^*x)$ for $x \in \mathcal{M}$. The pair $(\mathcal{M}_+, tr)$ can be considered as a "non-commutative" integral, and integration theory without Fatou's lemma is meaningless. Therefore, we always assume the normality of tr, i.e., tr is lower semi-continuous with respect to the σ-weak topology. It is said to be faithful when the condition $tr(x) = 0$ ($x \in \mathcal{M}_+$) implies $x = 0$. Finally, it is said to be semi-finite if $\{x \in \mathcal{M}_+;\ tr(x) < \infty\}$ is σ-weakly dense in $\mathcal{M}_+$.

Let $\mathcal{M}$ be a factor. It is a type II_1 factor if there exists a (unique) trace tr with $tr(I) = 1$ (the normalized trace). In this case, we have $tr(\mathcal{M}_p) = [0, 1]$, i.e. the continuous dimensions for the projection lattice $\mathcal{M}_p$, and the trace tr can be extended to a linear functional $tr : \mathcal{M} \longrightarrow \mathbf{C}$ with $tr(xy) = tr(yx)$. The factor $\mathcal{M}$ is of type II_∞ if there exists a (semi-finite) trace tr with $tr(I) = \infty$. Here we have $tr(\mathcal{M}_p) = [0, \infty]$. As in a type II_1 case one can extend a trace to a linear functional on $\mathcal{M}$, but in this case the domain of tr is just a dense part in $\mathcal{M}$. For example, let $\mathcal{M}$ be the abelian von Neumann algebra $L^\infty(X, \mu)$ with $\mu(X) = \infty$ (although it is not a factor). The trace $tr = \int_X \cdot d\mu$ defined on $L^\infty(X,\mu)_+$ is an infinite trace ($tr(I) = \infty$), and it is linearly extended to $L^1(X,\mu) \cap L^\infty(X,\mu)$.

Detailed analysis on type II factors can be carried out by making use of traces while traces are not available for type III factors. For analysis on the latter the modular theory is required.

2.1. Modular Theory and Structure of Type III Factors. After recalling the modular theory (the Tomita-Takesaki theory), we will briefly explain main ingredients of the structure analysis of type III factors. Most works were done in the 70's, and details (and quite complete references) can be found in standard textbooks such as [63].

Let $\mathcal{M}$ be a von Neumann algebra with a faithful state (or weight) $\varphi \in \mathcal{M}_+^*$. By the GNS construction, we may and do assume that there exists a cyclic and separating vector $\xi \in \mathcal{H}$ with $\varphi(x) = \omega_\xi(x) = \langle x\xi, \xi\rangle$. Then

$$S_\varphi : x\xi_\varphi \in \mathcal{M}\xi_\varphi \to x^*\xi_\varphi \in \mathcal{M}_\varphi$$

is a conjugate-linear closable operator. Let $\bar{S}_\varphi = J\Delta_\varphi^{1/2}$ be the polar decomposition of the closure. We have

$$\begin{aligned} \Delta_\varphi &= S_\varphi^* \bar{S}_\varphi, \\ J\Delta_\varphi J &= \Delta_\varphi^{-1}, \end{aligned}$$

and J is a unitary involution with $J^2 = I$. (When ξ is a trace vector, S_φ is isometric so that we have $\Delta_\varphi = I$.) The fundamental theorem of the modular theory states

$$J\mathcal{M}J = \mathcal{M}' \quad \text{and} \quad \Delta_\varphi^{it}\mathcal{M}\Delta_\varphi^{-it} = \mathcal{M} \quad (\text{for } t \in \mathbf{R}).$$

Hence $\{\sigma_t^\varphi = Ad\Delta_\varphi^{it}|_{\mathcal{M}}\}_{t\in\mathbf{R}}$ is a one-parameter group of automorphisms of $\mathcal{M}$, called the modular automorphism group (associated with φ). The modular automorphism group satisfies the following condition: For $x, y \in \mathcal{M}$ we set $f(t)$ $(= f_{x,y}(t)) = \varphi(x\sigma_t^\varphi(y))$. Then, f extends to a bounded continuous function on the strip $-1 \leq \Im z \leq 0$ analytic in the interior satisfying $f(-i+t) = \varphi(\sigma_t^\varphi(y)x)$. This condition is known as the KMS condition, and it actually characterizes the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbf{R}}$.

Example 2.1. Let $\mathcal{H} = M_n(\mathbf{C})$ equipped with the inner product $\langle x, y\rangle = Tr(y^*x)$. Then $\mathcal{M} = M_n(\mathbf{C})$ acts on $\mathcal{H} = M_n(\mathbf{C})$ as left multiplications. A positive linear functional on $\mathcal{M}$ is of the form $\varphi(x) = Tr(h_\varphi x)$ with a unique density matrix $h_\varphi \geq 0$. When φ is faithful, h_φ is invertible and $\xi = h_\varphi^{1/2} \in \mathcal{H}$ is a cyclic and separating vector. Then we compute

$$\langle x\xi, \xi\rangle = \langle xh_\varphi^{1/2}, h_\varphi^{1/2}\rangle = Tr((h_\varphi^{1/2})^* xh_\varphi^{1/2}) = Tr(h_\varphi x) = \varphi(x),$$

showing $\varphi = \omega_\xi$. It is an easy (and amusing) exercise to check

$$\Delta x = h_\varphi x h_\varphi^{-1}, \quad Jx = x^*, \quad \sigma_t(y) = h_\varphi^{it} y h_\varphi^{-it}$$

(for a vector $x \in \mathcal{H} = M_n(\mathbf{C})$ and for an operator $y \in \mathcal{M} = M_n(\mathbf{C})$). Note that the function f $(= f_{x,y})$ appearing in the KMS condition becomes $f(t) = Tr(xh^{it}yh^{1-it})$.

Recall that $C_1(\mathcal{H})$, the trace class operators, is the predual $\mathcal{B}(\mathcal{H})_*$ and $\mathcal{B}(\mathcal{H})$ acts on the Hilbert space $C_2(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}$ of Hilbert-Schmidt class operators as left multiplications (or as $x \otimes 1$ in the picture $\mathcal{H} \otimes \mathcal{H}$). The reader is strongly encouraged to do the above game for $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

Definition 2.2. *The crossed product $\widetilde{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ is the von Neumann algebra acting on the Hilbert space $\mathcal{K} = \mathcal{H} \otimes L^2(\mathbf{R}) = L^2(\mathbf{R}, \mathcal{H})$ generated by the operators $\pi_{\sigma^\varphi}(x)$ $(x \in \mathcal{M})$ and $\lambda(t)$ $(t \in \mathbf{R})$ defined by*

$$(\pi_{\sigma^\varphi}(x)\xi)(s) = \sigma_{-t}^\varphi(x)\xi(s),$$

$$(\lambda(t)\xi)(s) = \xi(s - t).$$

Note that $x \in \mathcal{M} \to \pi_{\sigma^\varphi}(x)$ is a normal representation, and we have the covariance relation $\lambda(t)\pi_{\sigma^\varphi}(x)\lambda(t)^* = \pi_{\sigma^\varphi}(\sigma_t^\varphi(x))$, which shows that the $*$-algebra (algebraically) generated by the two kinds of generators is exactly $\{\sum_{i=1}^n \pi_{\sigma^\varphi}(x_i)\lambda(t_i)\}$. By passing to the closure, we can imagine that any $x \in \widetilde{\mathcal{M}}$ can be expressed as $x = \int_{-\infty}^{\infty} \pi_{\sigma^\varphi}(x(t))\lambda(t)dt$ (and it is obvious that elements of this form form a dense subalgebra). Although the above expression can be justified for any x by making use of $\mathcal{M}$-valued distributions, we do not need it.

On the other hand, the crossed product $\mathcal{M} \rtimes_\alpha G$ by a discrete group G (relative to an action $\alpha : G \longrightarrow Aut\mathcal{M}$) is much easier to handle. In fact, any $x \in \mathcal{M} \rtimes_\alpha G$ can be written as $x = \sum_{g \in G} \pi_\alpha(x_g)\lambda_g$. But the reader is still warned that the convergence here is in the L^2-sense (not in the strong operator topology, etc.). If we further assume that G is a finite group, there is no subtle problem on convergence at all. For example, let us assume that α is an automorphism of period n (i.e. $\alpha^n = Id$). Then, it gives rise to a $\mathbf{Z}_n$-action on $\mathcal{M}$. Since $\mathcal{K} = \mathcal{H} \otimes \ell^2(\mathbf{Z}_n) = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n-\text{times}}$, $\pi_\sigma(x)$ and $\lambda(g)$ are indeed $\mathcal{B}(\mathcal{H})$-valued $n \times n$ matrices

$$\pi_\sigma(x) = \sum_{g \in \mathbf{Z}_n} \alpha_{g^{-1}}(x) \otimes e_{g,g}, \quad \lambda(g) = \sum_{h \in \mathbf{Z}_n} I \otimes e_{gh,h},$$

where $\{e_{g,g'}\}_{g,g' \in \mathbf{Z}_n}$ denote the obvious matrix units. In this case it is easy to see that the crossed product ($= \{\sum_{g \in \mathbf{Z}_n} \pi_\alpha(x_g)\lambda_g\}$) consists precise of $\mathcal{M}$-valued $n \times n$ matrices $[x_{g,g'}]_{g,g' \in \mathbf{Z}_n}$ satisfying the condition $\alpha_h(x_{g,g'}) = x_{hg,hg'}$.

Let us now go back to $\widetilde{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$. Identifying $\mathcal{M}$ with $\pi_{\sigma^\varphi}(\mathcal{M})$, we always regard $\mathcal{M}$ as a subalgebra in $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ and hence we write $x = \int_{-\infty}^{\infty} x(t)\lambda(t)dt \in \widetilde{\mathcal{M}}$ (i.e., we suppress π_{σ^φ}). A crucial fact is that the crossed product $\widetilde{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ is a von-Neumann algebra of type II_∞. In fact, it admits the natural (infinite) trace tr given (at least formally) by

$$tr\left(\int_{-\infty}^{\infty} x(t)\lambda(t)dt\right) = \varphi(x(-i)) = \varphi\left(\int_{\infty}^{\infty} \hat{x}(t)e^{-t}dt\right),$$

where $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(s)e^{its}ds$. The reason behind the fact that the above is a trace is the KMS condition.

We define the unitary operator U_s $(s \in \mathbf{R})$ by

$$(U_s\xi)(t) = e^{-ist}\xi(t).$$

It is plain to observe $[\pi_{\sigma^\varphi}(x), U_s] = 0$ and $U_s\lambda(t)U_s^* = e^{-ist}\lambda(t)$. Therefore, AdU_s leaves $\widetilde{\mathcal{M}}$ ($= \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$) invariant, and $\{Ad\, U_s|_{\widetilde{\mathcal{M}}}\}_{s \in \mathbf{R}}$ gives rise to a one-parameter automorphism group of $\widetilde{\mathcal{M}}$. This is denoted by $\{\theta_s\}_{s \in \mathbf{R}}$ and called the dual action (of $\{\sigma_t^\varphi\}_{t \in \mathbf{R}}$). From the construction, the dual action satisfies

$$\theta_s(x) = x \ \ (x \in \mathcal{M}), \quad \theta_s(\lambda(t)) = e^{-ist}\lambda(t).$$

The construction so far depends on the choice of φ, but the Connes Radon-Nikodym theorem guarantees that (up to equivalence) the pair $(\widetilde{\mathcal{M}}, \theta_s)$ does not depend on the choice of φ, and hence it is a very canonical object (attached to $\mathcal{M}$). Very important properties are:

(i) the fixed-point subalgebra $(\widetilde{\mathcal{M}})^{\theta}$ under the dual action is exactly $\mathcal{M}$,
(ii) the dual action is centrally ergodic (as long as $\mathcal{M}$ is a factor), i.e., $\{\theta_s|_{\mathcal{Z}(\tilde{\mathcal{M}})}\}_{s\in\mathbf{R}}$ is an ergodic action on $\mathcal{Z}(\widetilde{\mathcal{M}})$.

For $x = \int_{-\infty}^{\infty} x(t)\lambda(t)dt \in \widetilde{\mathcal{M}}$, we see

$$\theta_s(x) = \int_{-\infty}^{\infty} (e^{-ist}x(t))\lambda(t)dt,$$
$$\widehat{\theta_s(x)}(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ist}x(t)e^{irt}dt = \hat{x}(r-s).$$

With these, we conclude

$$\begin{aligned} tr\circ\theta_s(x) &= \varphi\left(\int_{\infty}^{\infty} \widehat{\theta_s(x)}(r)e^{-r}dr\right) = \varphi\left(\int_{-\infty}^{\infty} \hat{x}(r-s)e^{-r}dr\right) \\ &= \varphi\left(\int_{\infty}^{\infty} \hat{x}(r)e^{-(r+s)}dr\right) = e^{-s}tr(x). \end{aligned} \tag{2.1}$$

Thus, we have seen the trace-scaling property of the dual action, i.e., $tr\circ\theta_s = e^{-s}tr$.

Consider $x = \int_{-\infty}^{\infty} x(t)\lambda(t)dt \in \widetilde{\mathcal{M}}$ with $x(\cdot)$ regular enough in the sense that the integral $\int_{-\infty}^{\infty}\theta_s(x)ds$ converges. Then, this integral is a θ-invariant element and hence falls into the fixed-point subalgebra $\mathcal{M}$. Therefore, the map

$$x\in\widetilde{\mathcal{M}} \longrightarrow \varphi\left(\int_{\infty}^{\infty}\theta_s(x)ds\right)$$

gives rise to a (normal semi-finite faithful) weight on $\widetilde{\mathcal{M}}$. This is called the dual weight of φ and denoted by $\hat{\varphi}$. We point out that the value $\hat{\varphi}(x)$ is (formally) given by

$$\varphi\left(\int_{-\infty}^{\infty}\left(e^{-its}x(t)\lambda(t)dt\right)ds\right) = \varphi(x(0)).$$

Also note that the invariance $\hat{\varphi}\circ\theta_s = \hat{\varphi}$ is obvious from the definition.

Since $\{\lambda(t)\}_{t\in\mathbf{R}}$ is a one-parameter group of unitaries, the Stone theorem guarantees $\lambda(t) = e^{itH_0}$ with a self-adjoint operator H_0. We set $H = e^{H_0}$ (≥ 0) so that $\lambda(t) = H^{it}$. This turns out to be a Radon-Nikodym derivative of $\hat{\varphi}$ relative to the canonical trace tr with the scaling property, i.e., $\hat{\varphi} = tr(H\cdot)$. The following (formal) arguments clarify the meaning of the factor e^{-t} introduced to define tr: Let $x = \int_{-\infty}^{\infty} x(t)\lambda(t)dt$ be as usual, and we note

$$x\lambda(\alpha) = \int_{-\infty}^{\infty} x(t)\lambda(t+\alpha)dt = \int_{-\infty}^{\infty} x(t-\alpha)\lambda(t)dt.$$

Considering the both sides as functions of α, we differentiate them at $\alpha = 0$ and then divide the results by i. In this way one gets $y = xH_0 = \int_{\infty}^{\infty} y(t)\lambda(t)dt$ with

$$y(t) = \frac{1}{i}\frac{d}{d\alpha}\bigg|_{\alpha=0} x(t-\alpha) = \frac{1}{i}\frac{d}{d\alpha}\bigg|_{\alpha=0} \int_{-\infty}^{\infty} \hat{x}(s)e^{-is(t-\alpha)}ds = \int_{-\infty}^{\infty} s\hat{x}(s)e^{-ist}ds,$$

showing $\hat{y}(t) = t\hat{x}(t)$. Since $H = e^{H_0}$, we have $z = xH = \int_{\infty}^{\infty} z(t)\lambda(t)dt$ with $\hat{z}(t) = e^t\hat{x}(t)$. Now the trace value is computed by

$$tr(xH) = \varphi\left(\int_{-\infty}^{\infty} \hat{z}(t)e^t dt\right) = \varphi\left(\int_{-\infty}^{\infty} \hat{x}(t)dt\right) = \varphi(x(0)) = \hat{\varphi}(x).$$

Note that $\hat{\varphi} = tr(H\cdot)$ implies

$$\sigma_t^{\hat{\varphi}} = AdH^{it} = Ad\lambda(t), \tag{2.2}$$

which is of course inner.

The preceding arguments are very formal in the sense that we did not worry about convergence, etc. (although everything can be justified based on theories of weights, left Hilbert algebras, and so on). However, in the next §2.2 the objects appeared so far will be dealt with in a very concrete form. Indeed they are written in terms of measure-theoretic data. We also point out that the map

$$x \in \widetilde{\mathcal{M}}_+ \longrightarrow \int_{-\infty}^{\infty} \theta_s(x)ds \in \mathcal{M}_+$$

is a typical (and probably the most important) example of an operator valued weight (see Appendix **4**). Note that the convergence of the integral is not guaranteed here so that the value "$+\infty$" could be possible. (For example, when $x \in \mathcal{M} = (\widetilde{\mathcal{M}})^{\theta}$, we get $\int_{-\infty}^{\infty} \theta_s(x)ds = (+\infty) \times x$.) Therefore, more precisely we have to consider the extended positive part $\hat{\mathcal{M}}_+$ as the "range" of an operator valued weight (see Appendix **2**).

Definition 2.3. (Connes-Takesaki, [8]) *The above-mentioned pair* $(\mathcal{Z}(\widetilde{\mathcal{M}}), \theta_s|_{\mathcal{Z}(\tilde{\mathcal{M}})})$ *is called the* (*smooth*) *flow of weights of* $\mathcal{M}$.

Note $\mathcal{Z}(\widetilde{\mathcal{M}}) = L^{\infty}(X)$ and $\theta_s|_{\mathcal{Z}(\tilde{\mathcal{M}})}$ is realized as a point-map F_s. Therefore, in what follows $(X, \{F_s\}_{s\in\mathbf{R}})$ will be referred to as the flow of weights. The flow $\{F_s\}_{s\in\mathbf{R}}$ is a (non-singular) ergodic action on X as was pointed out, and it is an extremely important invariant for the study of type III factors.

In his thesis A. Connes classified type III factors further into type III_0, III_λ $(0 < \lambda < 1)$ and III_1 factors. In terms of flows of weights, this finer classification means

(i) $\mathcal{M}$ is of type III_1 if and only if X is a singleton (i.e., $\widetilde{\mathcal{M}}$ is a type II_∞ factor),
(ii) $\mathcal{M}$ is of type III_λ $(0 < \lambda < 1)$ if and only if $\{F_s\}_{s\in\mathbf{R}}$ has period $-\log\lambda$,
(iii) Otherwise $\mathcal{M}$ is of type III_0.

Theorem 2.4. (Takesaki Duality) *The crossed product* $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}) \rtimes_\theta \mathbf{R}$ *relative to the dual action is isomorphic to the tensor product* $\mathcal{M} \otimes \mathcal{B}(L^2(\mathbf{R}))$.

Since $\mathcal{M} \cong \mathcal{M} \otimes \mathcal{B}(L^2(\mathbf{R}))$ for type III factors, the duality shows that $\mathcal{M}$ admits the continuous decomposition

$$\mathcal{M} = \widetilde{\mathcal{M}} \rtimes_\theta \mathbf{R}$$

with the type II_∞ algebra $\widetilde{\mathcal{M}}$ and a (centrally ergodic) trace-scaling ($tr \circ \theta_s = e^{-s} tr$) action $\{\theta_s\}_{s\in\mathbf{R}}$.

It is also possible to express a type III_λ $(0 \le \lambda < 1)$ factor as a discrete crossed product. A type III_λ $(0 < \lambda < 1)$ factor can be written as $\mathcal{M} = \mathcal{M}_0 \rtimes_{\theta_0} \mathbf{Z}$ with a type II_∞ factor $\mathcal{M}_0$ and $tr \circ \theta_0 = \lambda\, tr$. In the type III_0 case, such a discrete decomposition is $\mathcal{M} = \mathcal{M}_0 \rtimes_{\theta_0} \mathbf{Z}$ with a von Neumann algebra $\mathcal{M}_0$ of type II_∞ and $tr \circ \theta_0 = tr(e^{-f} \cdot)$ with $f \in \mathcal{Z}(\mathcal{M}_0)^+ = L^\infty(X_0)^+$. In this picture, the flow of weights of $\mathcal{M}$ is the one built under the ceiling function f together with the base transformation (X_0, F), where F is a point-map realization of $\theta_0|_{\mathcal{Z}(\mathcal{M}_0)}$.

2.2. Krieger Construction and Examples of Factors. Murray and von Neumann used the group-measure-space construction to produce many non-trivial (i.e., non-type I) factors. Here, we will explain the Krieger construction. The difference here is: the former requires the freeness and the ergodicity to get a factor while the freeness is not needed for the latter. The description presented here is due to Feldman-Moore ([13]), and it is slightly different from the original description in [46].

Let (X, μ) be a Lebesgue space, G a countable group of non-singular transformations acting (ergodically) on X. We introduce an equivalence relation $\sim$ on X as follows:

$$x \sim y \iff y = gx \text{ for some } g \in G.$$

Denote the graph of $\sim$ by

$$\mathcal{R}_G = \{(x,y) \in X \times X;\ x \sim y\}\ (= \cup_{g \in G}\{(x,gx);\ x \in X\}).$$

Since the graph $\{(x,gx);\ x \in X\}$ of each $x \mapsto gx$ is Borel in $X \times X$ (equipped with the usual product structure), the countability of G guarantees that so is $\mathcal{R}_G$. Hence, (by restricting the structure) we see that $\mathcal{R}_G$ itself is a Lebesgue space. Let μ_l be the left counting measure on $\mathcal{R}_G$ defined by

$$\mu_l(C) = \int \#(C \cap \pi_l^{-1}(y))d\mu(y)$$

for each $C \in \mathcal{R}_G$. Here, $\pi_l : (x,y) \in \mathcal{R}_G \mapsto y \in X$ is the left projection, and the above definition justifies the symbolic notation $d\mu_l(x,y) = d\mu(y)$. We have

$$\int_{\mathcal{R}_G} f(x,y)d\mu_l(x,y) = \int_X \sum_{y \sim x} f(x,y)d\mu(y).$$

We can also define the right counting measure μ_r ($d\mu_r(x,y) = d\mu(x)$) on $\mathcal{R}_G$ analogously (by using the right projection π_r). Then, both of $d\mu_l, d\mu_r$ are σ-finite measures (if μ is) on $\mathcal{R}_G$, and they are equivalent in the sense of absolute continuity. Let $\delta = \frac{d\mu_r}{d\mu_l}$ (the module of the equivalence relation $\mathcal{R}_G$) be the Radon-Nikodym derivative. Note

$$\delta(x,y) = \frac{d\mu_r(x,y)}{d\mu_l(x,y)} = \frac{d\mu(x)}{d\mu(y)}, \tag{2.3}$$

from which we see

$$\delta(x,y) = \delta(x,z)\delta(z,y)$$

for $(x,z),(z,y) \in \mathcal{R}_G$ (and hence $(x,y) \in \mathcal{R}_G$). Note that for $(x,y) \in \mathcal{R}$ with $x = g^{-1}y$ ($g \in G$) we have

$$\delta(x,y) = \delta(g^{-1}y,y) = \frac{d\mu(g^{-1}y)}{d\mu(y)},$$

showing that the module is the Jacobian (relative to the measure μ) of the non-singular transformation $x \mapsto gx$. In particular, we have $\delta(x,y) = 1$ in the case that G is measure preserving.

Let $f, g \in \mathcal{R}_G$ be *nice* functions on $\mathcal{R}_G$ (bounded and "very small" supports). We define the convolution product by

$$(f * g)(x,y) = \sum_{z \sim x} f(x,z)g(z,y),$$

and we set $L_f(g) = f * g$ by regarding g as an element in $L^2(\mathcal{R}_G, d\mu_l)$. Then L_f (the left convolution operator) is a bounded operator, and we consider the von Neumann algebra $\mathcal{W}^*(\mathcal{R}_G)$ (acting on $L^2(\mathcal{R}_G, d\mu_l)$) generated by L_f's.

We can prove that an arbitrary element in $\mathcal{W}^*(\mathcal{R}_G)$ is of the form L_f, i.e.,

$$(L_f\xi)(x,y) = \sum_{z\sim x} f(x,z)\xi(z,y) \quad (\text{for } \xi \in L^2(\mathcal{R}_G, d\mu_l)).$$

The precise meaning here is that the right side is absolutely convergent for a.e. $(x,y) \in \mathcal{R}_G$ and the sum is equal to the left side. Moreover, we have

$$L_f L_g = L_{f*g}, \quad \textit{and } (L_f)^* = L_{f^*} \quad \text{with } f^*(x,y) = \overline{f(x,y)}.$$

For example we compute

$$\begin{aligned}
\langle \xi_1, f^* * \xi_2 \rangle &= \int_X \sum_{x\sim y} \xi_1(x,y)\overline{(f^* * \xi_2)(x,y)}d\mu(y) \\
&= \int_X \sum_{x\sim y} \xi_1(x,y) \left(\sum_{z\sim y} \overline{f^*(x,z)\xi_2(z,y)} \right) d\mu(y) \\
&= \int_X \sum_{x\sim y} \xi_1(x,y) \left(\sum_{z\sim y} f(z,x)\overline{\xi_2(z,y)} \right) d\mu(y) \\
&= \int_X \sum_{z\sim y} \left(\sum_{x\sim y} f(z,x)\xi_1(x,y) \right) \overline{\xi_2(z,y)}d\mu(y) \\
&= \int_X \sum_{z\sim y} (f * \xi_1)(z,y)\overline{\xi_2(z,y)}d\mu(y) \ = \ \langle f * \xi_1, \xi_2 \rangle.
\end{aligned}$$

We now assume that $Supp f \subseteq D = \{(x,x);\ x \in X\}$, the diagonal. We set

$$f(x,y) = \begin{cases} F(x) & \text{when } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

For another f' (supported on D) corresponding to F' we compute

$$(f * f')(x,x) = \sum_{z\sim x} f(x,z)f'(z,x) = F(x)F'(x)$$

and $(f * f')(x,y) = 0$ when $x \neq y$. This computation means

$$\mathcal{W}^*(\mathcal{R}_G) \supseteq \{L_f \in \mathcal{W}^*(\mathcal{R}_G);\ Supp f \subseteq D\} \cong L^\infty(X,\mu)$$

with the correspondence $f \leftrightarrow F$. The abelian algebra $\{L_f \in \mathcal{W}^*(\mathcal{R}_G);\ Supp f \subseteq D\}$ ($\cong L^\infty(X)$) will be denoted by $\mathcal{A}$.

Lemma 2.5. *The abelian subalgebra* $\mathcal{A}$ *is a MASA (maximal abelian subalgebra) in* $\mathcal{W}^*(\mathcal{R}_G)$*, i.e.,* $\mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{A}' = \mathcal{A}$.

Proof. Since $\mathcal{A}$ is abelian, it is enough to show $\mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{A}' \subseteq \mathcal{A}$. Assume $x = L_f \in \mathcal{W}^*(\mathcal{R}_G)$ commutes with L_g, where g is supported on D and corresponding to G as above. We notice $f(x,y)G(y) = (f * g)(x,y) = (g * f)(x,y) = G(x)f(x,y)$. Since g (and hence G) is arbitrary, we must have $f(x,y) = 0$ if $x \neq y$. □

We point out that the equality $f(x,y)G(y) = G(x)f(x,y)$ in the above proof is actually for a.e. $(x,y) \in \mathcal{R}_G$. Therefore, to conclude $f(x,y) = 0$ for a.e. (x,y) with $x \neq y$, one is not allowed to use uncountably many G's. But we have a countable family of separating subsets here, which does the job. This kind of arguments on measurability, etc. is everywhere in the theory.

Let $\Gamma(g) = \{(x, gx) : x \in X\} \subseteq \mathcal{R}_G$ be the graph of g (and $\chi_{\Gamma(g)}$ denotes its characteristic function). For a function f on $\mathcal{R}_G$ we compute

$$(\chi_{\Gamma(g)} * f)(x,y) = \sum_{z \sim x} \chi_{\Gamma(g)}(x,z) f(z,y) = f(gx,y)$$

so that the convolution operator $L_{\chi_{\Gamma(g)}}$ (in $\mathcal{W}^*(\mathcal{R}_G)$) is a unitary because of

$$||\chi_{\Gamma(g)} * f||^2 = \int_X \sum_{x \sim y} |f(gx,y)|^2 d\mu(y) = \int_X \sum_{x \sim y} |f(gx,y)|^2 d\mu(y) = ||f||^2.$$

It is straight-forward to check

$$\chi^*_{\Gamma(g)} = \chi_{\Gamma(g^{-1})} \quad \text{and} \quad \chi_{\Gamma(g_1)} * \chi_{\Gamma(g_2)} = \chi_{\Gamma(g_1 g_2)}.$$

Theorem 2.6. *When the action of* G *on* X *is ergodic,* $\mathcal{W}^*(\mathcal{R}_G)$ *is a factor.*

Proof. If $x \in \mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{W}^*(\mathcal{R}_G)'$, then it belongs to $\mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{A}' = \mathcal{A}$ by Lemma 2.5. Therefore, x is of the form L_f with a function f supported on D (corresponding to F). Notice

$$\begin{aligned}
(\chi_{\Gamma(g)} * f)(x,gx) &= \sum_{z \sim x} \chi_{\Gamma(g)}(x,z) f(z,gx) = \chi_{\Gamma(g)}(x,gx) f(gx,gx) = F(gx), \\
(f * \chi_{\Gamma(g)})(x,gx) &= \sum_{z \sim x} f(x,z) \chi_{\Gamma(g)}(z,gx) = f(x,x) = F(x).
\end{aligned}$$

Since x must commute with all $L_{\chi_{\Gamma(g)}}$'s, the above commutation shows the invariance $F(gx) = F(x)$. Since G acts ergodically, $F(x)$ must be a constant function. □

In the rest we assume $\mu(X) = 1$ and set $\xi_0 = \chi_D$. Note

$$||\xi_0||^2 = \int \sum_{x \sim y} |\chi_D(x,y)|^2 d\mu(y) = \mu(X) = 1$$

so that ξ_0 is a unit vector (in $L^2(\mathcal{R}_G; \mu_l)$). Since $L_f \xi_0 = f$, it is a cyclic and separating vector for $\mathcal{W}^*(\mathcal{R}_G)$. Let $\varphi = \omega_{\xi_0}$ be the corresponding vector state on $\mathcal{W}^*(\mathcal{R}_G)$. The value of this state is computed by

$$\begin{aligned} \varphi(L_f) &= \langle L_f \xi_0, \xi_0 \rangle = \langle f * \chi_D, \chi_D \rangle = \langle f, \chi_D \rangle \\ &= \int_X \sum_{y \sim x} f(x,y) \overline{\chi_D(x,y)} d\mu(y) = \int_X f(x,x) d\mu(x). \end{aligned}$$

We set

$$(J\xi)(x,y) = \delta(x,y)^{\frac{1}{2}} \overline{\xi(y,x)},$$

and note that J is a unitary involution. In fact, we compute

$$\begin{aligned} ||J\xi||^2 &= \int_X \sum_{x \sim y} |(J\xi)(x,y)|^2 d\mu(y) \\ &= \int_X \sum_{x \sim y} \delta(x,y) |\xi(y,x)|^2 d\mu(y) \\ &= \int_{\mathcal{R}_G} \delta(x,y) |\xi(y,x)|^2 d\mu_r(y,x) \\ &= \int_{\mathcal{R}_G} |\xi(y,x)|^2 d\mu_l(y,x) = ||\xi||^2. \end{aligned}$$

Here, the third equality follows from $\frac{d\mu_r}{d\mu_l}(y,x) = \delta(y,x) = \delta(x,y)^{-1}$ (see (2.3)). Let Δ be the multiplication operator defined by the module δ. Then, Δ and J are the modular operator (i.e., Δ_φ) and the modular conjugation respectively. In fact, we compute

$$(J\Delta^{\frac{1}{2}} f)(x,y) = \delta(x,y)^{\frac{1}{2}} \overline{(\Delta^{\frac{1}{2}} f)(y,x)} = \delta(x,y)^{\frac{1}{2}} \delta(y,x)^{\frac{1}{2}} \overline{f(y,x)} = \overline{f(y,x)}.$$

This formal argument is a bit dangerous, and actually more careful arguments (with consideration of a core for the modular operator Δ_φ, etc.) are needed. However, the argument is quite standard, and the above description of Δ shows

$$\sigma_t^\varphi(L_f) = L_{\delta^{it} f},$$

where $\delta^{it} f$ denotes the point-wise product (i.e, $(\delta^{it} f)(x,y) = \delta(x,y)^{it} f(x,y)$).

Example 2.7. Let $X = \{1, 2, \cdots, n\}$ with the counting measure. When $G = \mathfrak{S}_n$, the symmetric group, acts on X in the usual way, the action is transitive and hence $\mathcal{R}_G = X^2$. We also note that μ_l is the counting measure on $X^2 = \{1, 2, \cdots, n\}^2$. The Hilbert space $L^2(\mathcal{R}_G, \mu_l)$ in this case can be identified with $M_n(\mathbf{C})$ equipped with the Hilbert-Schmidt norm via $f \leftrightarrow [f(i,j)]_{i,j}$. We observe $\mathcal{W}^*(\mathcal{R}_G) = M_n(\mathbf{C})$, acting on $L^2(\mathcal{R}_G, \mu_l) = M_n(\mathbf{C})$ as left multiplications.

It is worth pointing out that what is relevant for the structure of $\mathcal{W}^*(\mathcal{R}_G)$ is the graph $\mathcal{R}_G$ (i.e., the orbits) rather than the G-action itself. For instance in the above (somewhat simple-minded) example let us consider just the action of the subgroup $\mathbf{Z}_n$ generated by the n-cycle $(1, 2, \cdots, n)$. The subgroup also acts transitively ($\mathcal{R}_G = \mathcal{R}_{\mathbf{Z}_n} = X^2$), and hence the resulting factor is the same $M_n(\mathbf{C})$. Therefore, we will often write $\mathcal{R}$ instead of $\mathcal{R}_G$. Our $\mathcal{R} = \mathcal{R}_G$ is a most typical measured groupoid ([4], see also [20, 21]), i.e., a principal measured groupoid. Furthermore, note that each orbit $\mathcal{R}_x = \{y \in X;\ x \sim y\} = Gx$ is countable. It is known that an (abstract) principal measured groupoid with countable orbits is always of the form $\mathcal{R}_G$ for some countable group G (and its action).

The case when the action of G is ergodic and (X, μ) is non-atomic is important, and in this case $\mathcal{W}^*(\mathcal{R}_G)$ is a non-type I factor. When G admits an invariant measure μ, then $\varphi(L_f) = \int_X f(x,x)d\mu(x)$ is a trace. Therefore, we get
(i) $\mathcal{W}^*(\mathcal{R}_G)$ is a type II_1 factor if there is a finite invariant measure on X equivalent to μ,
(ii) $\mathcal{W}^*(\mathcal{R}_G)$ is a type II_∞ factor if there is an invariant (but not finite) measure on X equivalent to μ,
(iii) $\mathcal{W}^*(\mathcal{R}_G)$ is a type III factor if there is no invariant measure on X equivalent to μ.

Example 2.8. Let $X = [0, 1)$. For a fixed irrational number θ, define $Tx = x + \theta\ (mod\ 1)$ for $x \in X$. Then T is a $\mathbf{Z}$-action on X and $\mathcal{W}^*(\mathcal{R}_{\mathbf{Z}})$ is a type II_1 factor. When $X = \mathbf{R}$ and $\mathbf{Q}$ acts on $\mathbf{R}$ as translations, $\mathcal{W}^*(\mathcal{R}_{\mathbf{Q}})$ is a type II_∞ factor. Indeed, there exists a unique invariant measure up to scalar multiple on X (i.e., the Lebesgue measure), but it is not finite.

Let us assume that G acts ergodically and freely on X (i.e., $\{x \in X;\ gx = x\}$ is a null set for $g \neq e$). Since $x = gy$ for a unique $g \in G$, we can define the map

$$\Phi : \xi(\cdot,\cdot) \in L^2(\mathcal{R}, d\mu_l) \longmapsto \int_X^{\oplus} \xi_y(\cdot) d\mu(y) \in \int_X^{\oplus} \ell^2(\mathcal{R}_y) d\mu(y),$$

which is a surjective isometry. Here, $\mathcal{R}_y$ $(= Gy)$ means the orbit of y, and $\xi_y(x) = \xi(x, y)$. Thanks to the freeness we can identify $\ell^2(\mathcal{R}_y)$ with $\ell^2(G)$ via $\xi_y(x = gy) \leftrightarrow \xi_y(g)$, and hence

$$\int_X^{\oplus} \ell^2(\mathcal{R}_y) d\mu(y) \cong L^2(X, \ell^2(G); \mu)\ (\cong L^2(X, \mu) \otimes \ell^2(G)).$$

In this picture we have

$$\Phi : \xi(\cdot,\cdot) \in L^2(\mathcal{R}, d\mu) \longmapsto \{\xi_y(\cdot)\}_{y \in X} \in L^2(X, \ell^2(G); \mu)$$

with $\xi_y(g) = \xi(gy, y)$. It is an amusing exercise to see that $\Phi \mathcal{W}(\mathcal{R})^* \Phi^{-1} = L^\infty(X, \mu) \rtimes G$, the crossed product, i.e., the group-measure-space construction by Murray and von Neumann. In the usual group-measure-space construction both of ergodicity and freeness are assumed to guarantee that $L^\infty(X, \mu) \rtimes G$ is a factor. On the other hand, the only ergodicity was required in Theorem 2.6. Notice that the ℓ^2-space over the orbit $\mathcal{R}_y$ is smaller than $\ell^2(G)$ (unless the action is free). This means that $\mathcal{W}^*(\mathcal{R}_G)$ acts on a smaller Hilbert space than $L^2(X, \mu) \otimes \ell^2(G)$, i.e., the standard Hilbert space of $L^\infty(X, \mu) \rtimes G$, and the above-mentioned difference is caused by this fact.

Recall that a MASA $\mathcal{A}$ in a von Neumann algebra $\mathcal{M}$ is called a Cartan subalgebra if there exists a (unique) normal conditional expectation $E : \mathcal{M} \longrightarrow \mathcal{A}$ (automatic in the type II_1 case) and there are enough normalizers (i.e., unitary normalizers for $\mathcal{A}$ generate $\mathcal{M}$).

Lemma 2.9. *The abelian subalgebra $\mathcal{A} \cong L^\infty(X, \mu)$ is a Cartan subalgebra in $\mathcal{W}^*(\mathcal{R}_G)$.*

Proof. We have already known that $\mathcal{A}$ is a MASA. Since $\sigma^\varphi(L_f) = L_{\delta^{it} f}$, we see $\sigma^\varphi|_{\mathcal{A}} = id$ and there exists a normal conditional expectation $E : \mathcal{W}^*(\mathcal{R}) \rightarrow \mathcal{A}$ (conditioned by φ) by Takesaki's theorem. In fact, E is simply the map cutting "off-diagonal components", i.e., $E(L_f) = L_{\chi_D f}$.

The unitaries $L^*_{\chi_{\Gamma(g)}}$ (see the paragraph before Theorem 2.6) normalize $\mathcal{A}$. In fact, when f is supported on D, it is plain to see $L_{\chi_{\Gamma(g)}} L_f L^*_{\chi_{\Gamma(g)}} = L_{f'}$ with f' supported on D (and $f'(x,x) = f(gx, gx)$), which shows $L_{\chi_{\Gamma(g)}} \mathcal{A} L^*_{\chi_{\Gamma(g)}} = \mathcal{A}$.

Notice $\mathcal{R}_G = \cup_{g \in G} \Gamma(g)$, and it is easy to see that any function supported on $\Gamma(g)$ is of the form $f * \chi_{\Gamma(g)}$ with a function f supported on the diagonal D. Therefore, each convolution operator is a (possibly infinite) linear combination of unitaries $L_{\chi_{\Gamma(g)}}$ with coefficients coming from $\mathcal{A}$. Therefore, $L_{\chi_{\Gamma(g)}}$'s together with $\mathcal{A}$ generate $\mathcal{W}^*(\mathcal{R}_G)$. □

Conversely, a factor admitting a Cartan subalgebra arises from a relation $\mathcal{R}$. Rough idea for a proof is: $\mathcal{A} \cong L^\infty(X)$ and unitary normalizers give use to transformations on X, which generate an equivalence relation. Since we have enough normalizers, the resulting $\mathcal{W}^*(\mathcal{R}_G)$ recovers the original algebra. Strictly speaking, in this proof a 2-cocyle enters. This is something one cannot avoid. In fact, in the famous construction of a factor $\mathcal{M}$ not anti-isomorphic to itself a certain 2-cocycle plays an essential role ([3]). However, in the AFD case a 2-cocycle vanishes so that we will not worry about 2-cocycles in what follows.

The complete classification of AFD factors is known, and in all the cases model factors can be constructed by using equivalence relations (see Examples 2,8, 2.12, 2,13, and 2.14). Therefore, every AFD factor admits a Cartan subalgebra. A very deep result on conjugacy for Cartan subalgebras can be found in the Connes-Feldman-Weiss paper [7].

2.3. Poincaré Suspension. In this section, for $\mathcal{M} = \mathcal{W}^*(\mathcal{R}_G)$ we try to express the algebra $\widetilde{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ in terms of an equivalence relation, which will enable us to compute the flow of weights of $\mathcal{W}^*(\mathcal{R}_G)$ obtained via the Krieger construction.

We at first find a Cartan subalgebra in $\widetilde{\mathcal{M}}$. Since $\mathcal{M} = \mathcal{W}^*(\mathcal{R}_G)$ is generated by $L_{\chi_{\Gamma(g)}}$ $(g \in G)$ and $\mathcal{A} = L^\infty(X, \mu)$, $\widetilde{\mathcal{M}}$ is generated by $\pi_{\sigma^\varphi}(L_{\chi_{\Gamma(g)}})$ $(g \in G)$, $\pi_{\sigma^\varphi}(\mathcal{A})$ and $\lambda(t)$ $(t \in \mathbf{R})$. At first we notice

$$\lambda(t)\pi_{\sigma^\varphi}(L_f)\lambda(t)^* = \pi_{\sigma^\varphi}(\sigma^\varphi_t(L_f)) = \pi_{\sigma^\varphi}(L_f) \quad (\text{for } L_f \in \mathcal{A})$$

because of $\delta|_D = 1$. We set $\mathcal{B} = \langle \pi_{\sigma^\varphi}(\mathcal{A}), \lambda(\mathbf{R})\rangle'' \cong \mathcal{A} \otimes \lambda(\mathbf{R})''$.

Lemma 2.10. *The abelian algebra $\mathcal{B}$ is a Cartan subalgebra in $\widetilde{\mathcal{M}}$.*

Proof. First let $x \in \widetilde{\mathcal{M}} \cap \mathcal{B}'$. Recall that the modular automorphism group of the dual weight $\hat{\varphi}$ is $\sigma_t^{\hat{\varphi}} = Ad\lambda(t)$ (see (2.2)). Since $\mathcal{B}$ contains $\lambda(t)$, we get

$$x \in \widetilde{\mathcal{M}}_{\hat{\varphi}} = \langle \mathcal{M}_{\varphi}, \lambda(t) \rangle'' = \mathcal{M}_{\varphi} \otimes \lambda(\mathbf{R})''$$

(see Lemma 5.3 in [19]). Thus, x is considered as an $\mathcal{M}_{\varphi}$-valued function on $\mathbf{R}$ (after the Fourier transform). Also since $\mathcal{B}$ contains $a \otimes I \cong a \in \pi_{\sigma^{\varphi}}(\mathcal{A})$ and x must commute with a, we get

$$x \in (\mathcal{M}_{\varphi} \cap \mathcal{A}') \otimes \lambda(\mathbf{R})'' \subseteq (\mathcal{M} \cap \mathcal{A}') \otimes \lambda(\mathbf{R})'' = \mathcal{A} \otimes \lambda(\mathbf{R})'',$$

showing that $\mathcal{B}$ is a MASA in $\widetilde{\mathcal{M}}$. For the existence of a conditional expectation, notice that $\widehat{\varphi}|_{\mathcal{B}}$ is the dual weight of $\varphi|_{\mathcal{A}}$ on $\mathcal{A} \otimes \lambda(\mathbf{R})'' = \mathcal{A} \rtimes_{\sigma^{\varphi}} \mathbf{R}$ from the construction. Hence $\widehat{\varphi}$ is semi-finite on $\mathcal{B}$, and $\sigma_t^{\hat{\varphi}}(\mathcal{B}) = \mathcal{B}$. Then it follows from Takesaki's theorem that there exists a conditional expectation from $\widetilde{\mathcal{M}}$ onto $\mathcal{B}$.

It remains to show that $\mathcal{B}$ contains enough normalizers. But, for $g \in G$ we claim $Ad\left(\pi_{\sigma^{\varphi}}(L_{\chi_{\Gamma(g)}})\right)(\pi_{\sigma^{\varphi}}(\mathcal{B})) = \pi_{\sigma^{\varphi}}(\mathcal{B})$ (and we know that $\pi_{\sigma^{\varphi}}(L_{\chi_{\Gamma(g)}})$'s together with $\mathcal{B}$ generate the whole algebra). At first we note

$$Ad\left(\pi_{\sigma^{\varphi}}(L_{\chi_{\Gamma(g)}})\right)(\pi_{\sigma^{\varphi}}(\mathcal{A})) = \pi_{\sigma^{\varphi}}\left(L_{\chi_{\Gamma(g)}} \mathcal{A} L^*_{\chi_{\Gamma(g)}}\right) = \pi_{\sigma^{\varphi}}(\mathcal{A}).$$

We also note

$$\lambda(t)\pi_{\sigma^{\varphi}}\left(L_{\chi_{\Gamma(g^{-1})}}\right)\lambda(t)^* = \pi_{\sigma^{\varphi}}\left(\sigma_t^{\varphi}\left(L_{\chi_{\Gamma(g^{-1})}}\right)\right) = \pi_{\sigma^{\varphi}}\left(L_{\delta^{it}\chi_{\Gamma(g^{-1})}}\right).$$

This is equivalent to $\lambda(t)\pi_{\sigma^{\varphi}}\left(L_{\chi_{\Gamma(g^{-1})}}\right) = \pi_{\sigma^{\varphi}}\left(L_{\delta^{it}\chi_{\Gamma(g^{-1})}}\right)\lambda(t)$, and hence

$$\pi_{\sigma^{\varphi}}\left(L_{\chi_{\Gamma(g^{-1})}}\right)^* \lambda(t)\pi_{\sigma^{\varphi}}\left(L_{\delta^{it}\chi_{\Gamma(g^{-1})}}\right) = \pi_{\sigma^{\varphi}}\left(L_{(\chi^*_{\Gamma(g^{-1})}) * (\delta^{it}\chi_{\Gamma(g^{-1})})}\right)\lambda(t).$$

We compute

$$\begin{aligned} ((\chi^*_{\Gamma(g^{-1})}) * (\delta^{it}\chi_{\Gamma(g^{-1})}))(x,y) &= \sum_{z \sim x} \overline{\chi_{\Gamma(g^{-1})}(z,x)}(\delta^{it}\chi_{\Gamma(g^{-1})})(z,y) \\ = \delta^{it}(gx,y)\chi_{\Gamma(g^{-1})}(gx,y) &= \begin{cases} \delta^{it}(gx,x) & \text{for } x = y, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

so that we conclude

$$\pi_{\sigma^{\varphi}}(L_{\chi_{\Gamma(g)}})\lambda(t)\pi_{\sigma^{\varphi}}(L_{\chi_{\Gamma(g)}})^* = \pi_{\sigma^{\varphi}}(L_{a_g^{it}})\lambda(t) \tag{2.4}$$

with the function a_g supported on D and defined by $a_g(x,x) = \delta(gx,x)$. Therefore, we have shown $Ad(L_{\chi_{\Gamma(g)}})(\lambda(t)) \in \mathcal{B}$ as desired. $\square$

The lemma shows that $\widetilde{\mathcal{M}}$ can be written by using an equivalence relation, which can be easily read from the computations in the lemma. Note that the above Cartan subalgebra $\mathcal{B} \subseteq \widetilde{\mathcal{M}}$ is

$$\mathcal{B} \cong \mathcal{A} \otimes \lambda(\mathbf{R})'' \cong \mathcal{A} \otimes L^\infty(\mathbf{R}) \cong L^\infty(X \times \mathbf{R})$$

under the Fourier transform on the second variable. Via the Fourier transform, the generators $\pi_{\sigma^\varphi}(a) = a \otimes I$ ($a \in \mathcal{A}$) and $\lambda(t)$ (or more precisely $I \otimes \lambda(t)$) correspond to

$$a \otimes I \quad \text{and} \quad I \otimes m_{e^{it\cdot}}$$

respectively, where $m_{e^{it\cdot}}$ means the multiplication operator induced by the character $u \mapsto e^{itu}$. Let f be a function supported on D and $f(x,x) = F(x)$, and we compute

$$\begin{aligned}
&\pi_{\sigma^\varphi}\left(L_{\chi_{\Gamma(g)}}\right)\left(\pi_{\sigma^\varphi}(L_f)\lambda(t)\right)\pi_{\sigma^\varphi}\left(L_{\chi_{\Gamma(g)}}\right)^* \\
&\quad = \pi_{\sigma^\varphi}\left(L_{\chi_{\Gamma(g)}} L_f L^*_{\chi_{\Gamma(g)}}\right)\pi_{\sigma^\varphi}\left(L_{\chi_{\Gamma(g)}}\right)\lambda(t)\pi_{\sigma^\varphi}\left(L_{\chi_{\Gamma(g)}}\right)^* \\
&\quad = \pi_{\sigma^\varphi}\left(L_{\chi_{\Gamma(g)}} L_f L^*_{\chi_{\Gamma(g)}}\right)\pi_{\sigma^\varphi}\left(L_{a_g^{it}}\right)\lambda(t),
\end{aligned}$$

where the last equality comes from (2.4). At first note that $L_f\lambda(t)$ appearing in the far left side corresponds to the multiplication operator arising from the function $(x,u) \mapsto F(x)e^{itu}$. Secondly, since $\chi_{\Gamma(g)} * f * \chi^*_{\Gamma(g)}$ (supported on D) corresponds to the function $x \in D \mapsto F(gx)$, the above far right side corresponds to the multiplication operator arising from the function

$$(x,u) \mapsto F(gx)a_g^{it}e^{itu} = F(gx)\delta(gx,x)^{it}e^{itu} = F(gx)e^{it(u+\log\delta(gx,x))}.$$

Thus, we observe that the transformation induced by $Ad\left(\pi_{\sigma^\varphi}\left(L_{\chi_{\Gamma(g)}}\right)\right)$ on $X \times \mathbf{R}$ is

$$(x,u) \mapsto (gx, u + \log\delta(gx,x)).$$

This is exactly the skew transformation of g (induced by the "cocycle" $\delta(\cdot,\cdot)$, the Jacobian of the transformation).

Based on the computations so far we set $\widetilde{X} = X \times \mathbf{R}$ equipped with the measure $d\mu \otimes e^{-u}du$ (the meaning of the factor e^{-u} will be clarified shortly). Let $g \in G$ act on $\widetilde{X}$ by

$$\tilde{g}(x,u) = (gx, u + \log\delta(gx,x)) \tag{2.5}$$

(we use the symbol $\tilde{g}$ to indicate the fact that it is acting on $\widetilde{X}$). Let $\widetilde{\mathcal{R}}$ $(\subseteq \widetilde{X} \times \widetilde{X})$ be the equivalence relation generated by the above action. From the discussions so far one gets

$$\mathcal{W}^*(\widetilde{\mathcal{R}}) \cong \widetilde{\mathcal{M}}.$$

A few remarks are in order: First notice

$$d\mu(gx) \otimes e^{-(u+\log\delta(gx,x))}du = d\mu(x) \otimes e^{-u}du$$

since $\delta(gx,x) = \frac{d\mu(gx)}{d\mu(x)}$. The measure preserving property here implies that

$$tr(L_f) = \int_{X\times\mathbf{R}} f((x,u),(x,u))d\mu(x)e^{-u}du$$

gives rise to a trace (which corresponds to the general fact that $\widetilde{\mathcal{M}} = \mathcal{W}^*(\widetilde{\mathcal{R}})$ is semi-finite). Secondly recall that the dual action θ_s acts trivially on $\mathcal{M}$ and as the multiplication of characters on $\lambda(\mathbf{R})$ (i.e., as the translation after the Fourier transform). Therefore, we should consider

$$\begin{array}{cccc} T_s : & \widetilde{X} & \rightarrow & \widetilde{X} \\ & (x,u) & \mapsto & (x, u+s). \end{array} \tag{2.6}$$

Then, T_s commutes with each $\tilde{g}$ so that it normalizes $\widetilde{\mathcal{R}}$ (i.e., $(T_s \times T_s)(\widetilde{\mathcal{R}}) = \widetilde{\mathcal{R}}$). As pointed out above, the dual action is given by $\theta_s(L_f) = L_{f(T_{-s}\cdot, T_{-s}\cdot)}$. Notice

$$\begin{aligned} tr(\theta_s(L_f)) &= \int_{X\times\mathbf{R}} f((x,u-s),(x,u-s))d\mu(x)e^{-u}du \\ &= \int_{X\times\mathbf{R}} f((x,u),(x,u))d\mu(x)e^{-(u+s)}du \;=\; e^{-s}tr(L_f) \end{aligned}$$

as expected (see (2.1)).

Lemma 2.5 and the argument in the proof of Theorem 2.6 show that the center $\mathcal{Z}(\mathcal{W}^*(\widetilde{\mathcal{R}}))$ is the fixed point subalgebra $L^\infty(\widetilde{X})^G$. We write $L^\infty(\widetilde{X})^G = L^\infty(Y)$, i.e., the ergodic decomposition of the G-action on $\widetilde{X}$. Since T_s and $\tilde{g}$ commute, T_s induces $\dot{T}_s$ on Y.

The discussions so far show

Theorem 2.11. *Let Y be the space of the ergodic decomposition of the G-action (2.5), and $\dot{T}_s$ be the transformation on Y induced by T_s (see (2.6)). Then, $(Y, \{\dot{T}_s\}_{s\in\mathbf{R}})$ is the flow of weights of $\mathcal{M} = \mathcal{W}^*(\mathcal{R}_G)$.*

We will use the theorem in the following examples:

Example 2.12. Let $X = \mathbf{R}$ with the Lebesgue measure dt, and G be the "$at + b$"-group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} ;\ a, b \in \mathbf{Q}\ ,\ a > 0 \right\}$$

with the natural action $t \mapsto at + b$. Since the subgroup

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ;\ b \in \mathbf{Q} \right\}$$

already acts ergodically on X, we see that G acts ergodically on X. For $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, we have $\delta(gt, t) = a$. Set $\widetilde{X} = \mathbf{R} \times \mathbf{R}$ with $dt \otimes e^{-u}du$, and $\tilde{g}(t, u) = (at + b, u + \log a)$. We need to look at the ergodic decomposition of $\mathbf{X} \times \mathbf{X}$ under the G-action. If we choose $a = 1$, then the first space is killed by the density of $\mathbf{Q}$. Then, $\log a$ $(a > 0)$ kills the second space. Hence the G-action on $\widetilde{X}$ is also ergodic and we conclude $L^\infty(\widetilde{X})^G \cong \mathbf{C}I$. Therefore, $\mathcal{W}^*(\mathcal{R}_G)$ is a type III_1 factor.

Example 2.13. We keep the same notations as in Example 2.12, but we use $a \in \lambda^{\mathbf{Z}}$ $(0 < \lambda < 1)$ and b in the countable group generated by $\lambda^{\mathbf{Z}}$ and $\mathbf{Q}$. In this case, we get $L^\infty(\widetilde{X})^G \cong L^\infty([0, -\log\lambda))$ and $\dot{T}_t$ is the translation (of period $-\log\lambda$). Therefore, $\mathcal{W}^*(\mathcal{R}_G)$ is a type III_λ factor.

Example 2.14. Let $\{F_s\}_{s\in\mathbf{R}}$ be a non-singular ergodic flow on (Γ, m). Here we construct a type III_0 factor whose flow of weights is exactly the given one, and the construction below is due to T. Hamachi ([22]).

Let (S_0, Y_0, ν) be a type III_1 ergodic transformation. For example one considers the infinite product space $Y_0 = \Pi_{-\infty}^{\infty}\{1, 2, 3\}$ with the shift S_0. The measure ν is the product measure of

$$\left\{ \frac{1}{1 + \lambda + \mu}, \frac{\lambda}{1 + \lambda + \mu}, \frac{\mu}{1 + \lambda + \mu} \right\}$$

on the three-point space $\{1,2,3\}$ with $\log\lambda, \log\mu$ rationally independent. Set $X = \Gamma\times Y_0\times \mathbf{R}$ with $m\otimes\nu\otimes e^{-v}dv$, and let

$$\begin{aligned}\tilde{F}_t(\gamma,y,v) &= \left(F_t\gamma, y, v+t+\log\frac{dm\circ F_t}{dm}(\gamma)\right),\\ \tilde{S}(\gamma,y,v) &= \left(\gamma, S_0 y, v+\log\frac{d\nu\circ S_0}{d\nu}(y)\right).\end{aligned}$$

From the definition $\tilde{S}$ is measure preserving. On the other hand, due to the presence of t in the third variable, $\tilde{F}_t$ scales the measure by e^{-t}. Note they commute so that one gets the action of the product group $\mathbf{R}\times\mathbf{Z}$. We would like to have an action of a countable discrete group. Therefore, we take $G=\mathbf{Q}\times\mathbf{Z}$ and restrict the action to this subgroup. More precisely, for $g=(q,n)\in G$ the action is

$$g(\gamma,y,v) = \left(F_q\gamma, S_0^n y, v+q+\log\frac{dm\circ F_q}{dm}(\gamma)+\log\frac{d\nu\circ S_0^n}{d\nu}(y)\right). \tag{2.7}$$

At first we note that this action is ergodic. The crucial fact here is that the above transformations for $q=0$ act ergodically on the second and third spaces. In fact, when $q=0$ the action on the last two spaces is exactly the one defining the Poincaré suspension of S_0 (see (2.5)). But it is ergodic thanks to the III_1 assumption. Note that a function (on Γ) invariant under $\{F_q\}_{q\in\mathbf{Q}}$ is also invariant under $\{F_s\}_{s\in\mathbf{R}}$ since $\mathbf{Q}$ is dense in $\mathbf{R}$. Therefore, the desired ergodicity of the G-action comes from that of the given flow F_s.

For the computation of the flow of weights, as usual we set $\widetilde{X}=\Gamma\times Y_0\times\mathbf{R}\times\mathbf{R}$ with $m\otimes\nu\otimes e^{-v}dv\otimes e^{-u}du$, and let

$$\tilde{g}(\gamma,y,v,u) = \left(F_q\gamma, S_0^n y, v+q+log\frac{d\mu\circ F_q}{dm}(\gamma)+log\frac{d\nu\circ S_0^n}{d\nu}(y), u-q\right), \tag{2.8}$$

which is measure preserving. We need to look at the ergodic decomposition under $\tilde{g}$'s. At first by specializing q to 0, as above we immediately see that the second and third spaces disappear. Thus, it suffices to look at the ergodic decomposition of $\Gamma\times\mathbf{R}$ under the $\mathbf{Q}$-action

$$(\gamma,u)\mapsto(F_q\gamma, u-q).$$

As above the density of $\mathbf{Q}$ permits us to go back to the $\mathbf{R}$-action $(\gamma,u)\mapsto(F_t\gamma,u-t)$, and the dual action θ_s here corresponds to $(\gamma,u)\mapsto(\gamma,u+s)$. We observe that an invariant function under the above $\mathbf{R}$-action is of the form $f(\gamma, u) = \hat{f}(F_t\gamma)$ with a function $\hat{f}$ on Γ. The dual action θ_s behaves like

$$f(\gamma, u) \mapsto f(\gamma, u+s) = \hat{f}(F_{u+s}\gamma) = \hat{f}(F_s(F_u\gamma))$$

(see (2.6)). Thus, we have seen that the flow of weights is exactly the given flow $(\Gamma, \{F_s\}_{s\in\mathbf{R}})$.

3. Index Theory

We briefly recall the Jones index theory ([33]) for II_1 factors. Let $\mathcal{M} \supseteq \mathcal{N}$ be an inclusion of II_1 factors, and we regard that they are acting on the GNS Hilbert space $\mathcal{H}_{tr} = L^2(\mathcal{M})$ induced by the unique normalized trace tr. The unitary involution J of $\mathcal{H}_{tr}$ determined by $\Lambda_{tr}(x) \mapsto \Lambda_{tr}(x^*)$ (where $\Lambda_{tr} : \mathcal{M} \to \mathcal{H}_{tr}$ is the canonical injection) satisfies $\mathcal{M}' = J\mathcal{M}J$. It is well-known that the commutant $\mathcal{N}'$ is a factor of either type II_1 or II_∞. Let $e_{\mathcal{N}}$ be the orthogonal projection on the closed subspace $\overline{\Lambda_{tr}(\mathcal{N})}$ ($\subseteq \overline{\Lambda_{tr}(\mathcal{M})} = L^2(\mathcal{M})$). This projection is referred to as the Jones projection, and it is easy to see that $e_{\mathcal{N}}$ satisfies $e_{\mathcal{N}} \in \mathcal{N}'$ and $Je_{\mathcal{N}} = e_{\mathcal{N}}J$. In fact, these properties come from the fact that $\Lambda_{tr}(\mathcal{N})$ (and hence its closure) is invariant under the left multiplication of $\mathcal{N}$ and the adjoint operation (i.e, J). When $\mathcal{N}'$ is a type II_1 factor, the Jones index of $\mathcal{M} \supseteq \mathcal{N}$ is defined by making use of coupling constants. In the present case, $\mathcal{M}$ is acting standardly on $L^2(\mathcal{M})$, and the Jones index $[\mathcal{M} : \mathcal{N}]$ is (defined by)

$$[\mathcal{M} : \mathcal{N}] = \frac{1}{tr_{\mathcal{N}'}(e_{\mathcal{N}})} \quad \left(= \frac{\dim_{\mathcal{N}} L^2(\mathcal{M})}{\dim_{\mathcal{M}} L^2(\mathcal{M})} \right),$$

where $tr_{\mathcal{N}'}$ denotes the unique normalized trace on $\mathcal{N}'$. When $\mathcal{N}'$ is of type II_∞, we simply set $[\mathcal{M} : \mathcal{N}] = \infty$.

Theorem 3.1. (Jones) *The index $[\mathcal{M} : \mathcal{N}]$ for an inclusion $\mathcal{M} \supseteq \mathcal{N}$ of type II_1 factors belongs to the set*

$$\left\{4\cos^2\left(\frac{\pi}{n}\right);\ n = 3, 4, 5, \cdots\right\} \cup [4, \infty].$$

Moreover, all of the above values are actually realized for some subfactors in $\mathcal{M} = \mathcal{R}_0$, the hyperfinite II_1 factor.

The restriction of the projection $e_{\mathcal{N}}: \ L^2(\mathcal{M}) \longrightarrow L^2(\mathcal{N})$ to $\mathcal{M}$ ($\cong \Lambda_{tr}(M) \subseteq L^2(\mathcal{M})$) gives rise a normal conditional expectation $E_{\mathcal{N}}: \ \mathcal{M} \longrightarrow \mathcal{N}$. Basic properties are:

Lemma 3.2. *(i)* $E_{\mathcal{N}}(n_1 m n_2) = n_1 E_{\mathcal{N}}(m) n_2$ *for* $m \in \mathcal{M}$ *and* $n_i \in \mathcal{N}$,
(ii) $E_{\mathcal{N}}|_{\mathcal{N}} = Id_{\mathcal{N}}$,
(iii) $e_{\mathcal{N}} x e_{\mathcal{N}} = E_{\mathcal{N}}(x) e_{\mathcal{N}}$ *for* $x \in \mathcal{M}$,
(iv) $x \in \mathcal{M}$ *satisfies* $e_{\mathcal{N}} x = x e_{\mathcal{N}}$ *if and only if* $x \in \mathcal{N}$.

Proof. (iii) It suffices to check $e_{\mathcal{N}} x e_{\mathcal{N}} y \xi_0 = E_{\mathcal{N}}(x) e_{\mathcal{N}} y \xi_0$ ($y \in \mathcal{M}$), where $\xi_0 = \Lambda_{tr}(1)$ is the GNS (trace) vector. However, we compute

$$e_{\mathcal{N}} x e_{\mathcal{N}} y \xi_0 = e_{\mathcal{N}} x \Lambda_{tr}(E_{\mathcal{N}}(y)) = \Lambda_{tr}(E_{\mathcal{N}}(x E_{\mathcal{N}}(y))) = \Lambda_{tr}(E_{\mathcal{N}}(x) E_{\mathcal{N}}(y)),$$
$$E_{\mathcal{N}}(x) e_{\mathcal{N}} y \xi_0 = E_{\mathcal{N}}(x) \Lambda_{tr}(E_{\mathcal{N}}(y)) = \Lambda_{tr}(E_{\mathcal{N}}(x) E_{\mathcal{N}}(y)).$$

(iv) When $[e_{\mathcal{N}}, x] = 0$, we have $E_{\mathcal{N}}(x) e_{\mathcal{N}} = e_{\mathcal{N}} x e_{\mathcal{N}} = x e_{\mathcal{H}}$ by (iii). Hitting the both sides to ξ_0, we get $E_{\mathcal{N}}(x)\xi_0 = x\xi_0$ due to $e_{\mathcal{N}}\xi_0 = \xi_0$, showing $x = E_{\mathcal{N}}(x) \in \mathcal{N}$. □

The above (iv) means $\mathcal{N} = \mathcal{M} \cap \{e_{\mathcal{N}}\}'$ so that we have $\mathcal{N}' = \mathcal{M}' \vee \{e_{\mathcal{N}}\}$. The basic extension $\mathcal{M}_1$ of $\mathcal{M} \supseteq \mathcal{N}$ is defined as

$$\mathcal{M}_1 = J\mathcal{N}'J = J\mathcal{M}'J \vee \{J e_{\mathcal{N}} J\} = \mathcal{M} \vee \{e_{\mathcal{N}}\} \ (= \langle \mathcal{M}\,,\, e_{\mathcal{N}} \rangle'').$$

We obviously have $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{M}_1$, and (when $[\mathcal{M} : \mathcal{M}] < \infty$) the index is preserved $[\mathcal{M} : \mathcal{N}] = [\mathcal{M}_1 : \mathcal{M}]$. Iterating this procedure, one obtains a canonical tower of type II_1 factors (called the Jones tower):

$$\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{M}_1 = \langle \mathcal{M}, e_0 \rangle \subseteq \mathcal{M}_2 = \langle \mathcal{M}_1, e_0, e_1 \rangle \subseteq \cdots,$$

with $e_0 = e_{\mathcal{N}}, \ e_1 = e_{\mathcal{M}}, \ \cdots$. The successive Jones projections $e_0, \ e_1, \ e_2, \ \cdots$ satisfy

$$e_i e_j = e_j e_i \quad \text{when } |i - j| \geq 2,$$
$$e_i e_{i\pm 1} e_i = [\mathcal{M} : \mathcal{N}]^{-1} e_i.$$

This is essentially (3.5) (to be proved later) and restriction of index values comes from these relations (see [33]).

When $[\mathcal{M} : \mathcal{N}] < \infty$, one can let the factors act on a Hilbert space $\mathcal{K}$ in such a way that the action of $\mathcal{N}$ is standard. For example, one can choose a projection p' in the commutant of $\mathcal{M}$ ($\subseteq \mathcal{B}(L^2(\mathcal{M}))$) satisfying $tr_{\mathcal{M}'}(p') = [\mathcal{M} : \mathcal{N}]^{-1}$, and we set $\mathcal{K} = p'L^2(\mathcal{M})$. The amplification $x \in \mathcal{M} \mapsto xp' \in \mathcal{M}p'$ being isomorphic (and the same for $\mathcal{N}$), we can regard that $\mathcal{M} \supseteq \mathcal{N}$ are acting on $\mathcal{K}$, and easily see that the action of $\mathcal{N}$ is standard (by looking at the coupling constant $\cdots \dim_{\mathcal{N}p'} \mathcal{K} = tr_{\mathcal{N}'}(p') \dim_{\mathcal{N}} L^2(\mathcal{M})$ and $tr_{\mathcal{M}'} = tr_{\mathcal{N}'} |_{\mathcal{M}'}$, see [33] for details). Let $J_{\mathcal{N}}$ be the unitary involution on $\mathcal{K}$ as above arising from $\mathcal{N}$, and we set $\mathcal{P} = J_{\mathcal{N}}\mathcal{M}'J_{\mathcal{N}}$, a subfactor of $\mathcal{N}$. It is obvious that $\mathcal{M}$ is the basic extension of $\mathcal{N} \supseteq \mathcal{P}$, and because of this reason $\mathcal{P}$ is called a down-ward basic extension of $\mathcal{M} \supseteq \mathcal{N}$.

3.1. Index for General Factor-subfactor Pairs. Let $\mathcal{M} \supseteq \mathcal{N}$ be (not necessarily type II_1) factors. Here, we explain how to define

$$Ind\ E \in \left\{4\cos^2(\frac{\pi}{n});\ n = 3, 4, 5, \cdots\right\} \cup [4, \infty]$$

for a given normal conditional expectation $E : \mathcal{M} \longrightarrow \mathcal{N}$. The definition is based on Connes' spatial theory ([5]) and Haagerup's theory ([18]) on operator valued weights (see Appendix **4**), and details can be found in [38].

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{K}$ (the action is not necessarily standard). Let ψ' be a weight on the commutant $\mathcal{M}'$ with the GNS representation $(\mathcal{H}_{\psi'}, \Lambda_{\psi'}, \pi_{\psi'})$. For $\xi \in \mathcal{K}$ let $R^{\psi'}(\xi)$ be the operator from $\mathcal{H}_{\psi'}$ to $\mathcal{K}$ defined by

$$R^{\psi'}(\xi)\Lambda_{\psi'}(x') = x'\xi \quad \text{with } \mathcal{D}(R^{\psi'}) = \Lambda_{\psi'}(\mathfrak{N}_{\psi'})\ (\subseteq \mathcal{H}_{\psi'}),$$

where $\mathfrak{N}_{\psi'} = \{x' \in \mathcal{N}';\ \psi'(x'^*x') < \infty\}$ is the definition ideal. It is plain to see

$$y'R^{\psi'}(\xi) \subseteq R^{\psi'}(\xi)\pi_{\psi'}(y') \quad \text{for } y' \in \mathcal{M}',$$
$$R^{\psi'}(y\xi) = yR^{\psi'}(\xi) \quad \text{for } y \in \mathcal{M}.$$

When $R^{\psi'}(\xi)$ is bounded (the extension is still denoted by $R^{\psi'}(\xi)$), ξ is called a ψ'-bounded vector. The set of all such vectors (in $\mathcal{K}$) is denoted by $D(\mathcal{K}, \psi')$ (the density of which can be proved). For each $\xi \in D(\mathcal{K}, \psi')$ we set

$$\theta^{\psi'}(\xi, \xi) = R^{\psi'}(\xi)R^{\psi'}(\xi)^*.$$

The above first (intertwining property) shows $\theta^{\psi'}(\xi, \xi) \in \mathcal{M}_+$ while the second means the bimodule property $\theta^{\psi'}(y\xi, y\xi) = y\theta^{\psi'}(\xi, \xi)y^*$. We now take a weight $\phi \in \mathcal{M}_*^+$, and set

$$\xi \in D(\mathcal{K}, \psi') \longrightarrow q(\xi) = \phi(\theta^{\psi'}(\xi, \xi)) \in \mathbf{R}_+ \cup \{\infty\}.$$

It is a quadratic form, and actually lower semi-continuous. We set

$$\mathcal{D}(q) = \{\xi \in D(\mathcal{K}, \psi');\ q(\xi) < \infty\},\ \text{the domain of } q.$$

By the Friedrich theorem, there is a unique positive self-adjoint operator H such that $\mathcal{D}(q)$ is a core for $H^{\frac{1}{2}}$ and $\bar{q}(\xi) = ||H^{\frac{1}{2}}\xi||^2$ for $\xi \in \mathcal{D}(q)$. This H is referred to as the spatial derivative of ϕ relative to ψ', and denoted by $\frac{d\phi}{d\psi'}$.

Remark 3.3. We consider three typical cases. Details are left to the reader as an exercise.

(i) Assume that the action of $\mathcal{M}$ on $\mathcal{K}$ is standard. Let $\psi' = \omega_{\xi_0}^{\mathcal{N}'}$ with a cyclic and separating vector $\xi_0 \in \mathcal{K}$. Then, we may regard $\mathcal{H}_{\psi'} = \mathcal{K}$ and $\Lambda_{\psi'}(x') = x'\xi_0$. It is easy to see $\theta^{\psi'}(\xi, \xi) = xx^*$ for $\xi = x\xi_0 \in \mathcal{M}\xi_0$. Therefore, the above quadratic form is $x\xi_0 \in \mathcal{M}\xi_0 \longrightarrow \phi(xx^*)$, and the spatial derivative $\frac{d\phi}{d\psi'}$ is nothing but the relative modular operator $\Delta_{\phi,\tilde{\psi}'}$ with $\tilde{\psi}' = \omega_{\xi_0}^{\mathcal{M}} \in \mathcal{M}_*^+$.

(ii) Assume $\mathcal{M} = \mathcal{B}(\mathcal{K})$. Then $\mathcal{M}'$ is one-dimensional, and let $\psi' = 1$. Then, we have $\theta^{\psi'}(\xi, \xi) = \xi \otimes \xi^c$, the rank-one operator determined by $\xi \in \mathcal{K}$.

(iii) Assume $\mathcal{M} = \mathbf{C}I$. Then, we have $\mathcal{M}' = \mathcal{B}(\mathcal{K})$ and $\psi' = Tr(h_{\psi'}\cdot)$ with the density operator $h_{\psi'}$. In this case we get $\theta^{\psi'}(\xi, \xi) = \langle h_{\psi'}^{-1}\xi, \xi\rangle$.

Important properties of spatial derivatives are

$$\frac{d\psi'}{d\phi} = \left(\frac{d\phi}{d\psi'}\right)^{-1} \quad \text{and} \quad \left(\frac{d\phi}{d\psi'}\right)^{it} x \left(\frac{d\phi}{d\psi'}\right)^{-it} = \sigma_t^{\phi}(x) \text{ for } x \in \mathcal{M}.$$

We also have

$$\left(\frac{d\phi_1}{d\psi'}\right)^{it} \left(\frac{d\phi}{d\psi'}\right)^{-it} = (D\phi_1; D\phi)_t \quad \text{(Connes' Radon-Nikodym cocycle)}$$

for another ϕ_1 on $\mathcal{M}$.

At first we see that the spatial theory gives us a canonical (order-reversing) bijection between the weights on $\mathcal{M}$ and the operator valued weights $\mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{M}'$. Let ψ' be a weight on $\mathcal{M}'$ as before. For a weight ϕ on $\mathcal{M}$, we consider the weight $\chi = Tr(\frac{d\psi'}{d\phi}\cdot)$ on $\mathcal{B}(\mathcal{K})$. Then, for $x' \in \mathcal{M}'$ we compute

$$\sigma_t^{\chi}(x') = \left(\frac{d\psi'}{d\phi}\right)^{it} x' \left(\frac{d\psi'}{d\phi}\right)^{-it} = \sigma_t^{\psi'}(x').$$

Therefore, we have a unique operator valued weight $F : \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{M}'$ such that $\chi = Tr(\frac{d\psi'}{d\phi}\cdot) = \psi' \circ F$. We claim that this F does not depend on ψ' and it is uniquely determined by ϕ. In fact, let ψ'_1 be another weight on $\mathcal{M}'$ (giving rise to F_1). Then we have

$$(D(\psi' \circ F); d(\psi'_1 \circ F))_t = (D\psi'; d\psi'_1)_t = \left(\frac{d\psi'}{d\phi}\right)^{it} \left(\frac{d\psi'_1}{d\phi}\right)^{-it},$$

which in the Radon-Nikodym derivative between $Tr(\frac{d\psi'_1}{d\phi}\cdot)$ and $\chi = Tr(\frac{d\psi'}{d\phi}\cdot)$. These are $\psi' \circ F$ and $\psi'_1 \circ F_1$ respectively so that we conclude

$$(D(\psi' \circ F); d(\psi'_1 \circ F))_t = (D(\psi' \circ F); d(\psi'_1 \circ F_1))_t.$$

This means $\psi'_1 \circ F = \psi'_1 \circ F_1$ and hence $F = F_1$ as desired. The operator valued weight F to $\mathcal{M}'$ constructed so far is denoted by ϕ^{-1}.

Conversely, assume that an operator valued weight F to $\mathcal{M}'$ is given. Let ϕ_1, ψ be weights on $\mathcal{M}, \mathcal{M}'$ respectively. Then, the composition $\psi' \circ F$ is a weight on $\mathcal{B}(\mathcal{K})$ so that $\psi' \circ F = Tr(K\cdot)$ with a non-singular positive self-adjoint operator K. For $x' \in \mathcal{M}'$ we have

$$K^{it}x'K^{-it} = \sigma_t^{\psi' \circ F}(x') = \sigma_t^{\psi'}(x') = \left(\frac{d\psi'}{d\phi_1}\right)^{it} x' \left(\frac{d\psi'}{d\phi_1}\right)^{-it} = \left(\frac{d\phi_1}{d\psi'}\right)^{-it} x' \left(\frac{d\phi_1}{d\psi'}\right)^{it}.$$

Therefore, $D_t = K^{-it}\left(\frac{d\phi_1}{d\psi'}\right)^{-it}$ is a unitary in $\mathcal{M}$, and it is actually a σ^{ϕ_1}-cocycle because of

$$D_{t+s} = D_t \left(\frac{d\phi_1}{d\psi'}\right)^{it} D_s \left(\frac{d\phi_1}{d\psi'}\right)^{-it} = D_t\sigma_t^{\phi_1}(D_s) \qquad (\text{for } t, s \in \mathbf{R}).$$

Thus, one finds a unique weight ϕ on $\mathcal{M}$ with $D_t = (D\phi; D\phi_1)$ thanks to the converse of Connes' Radon-Nikodym theorem. We denote this weight ϕ on $\mathcal{M}$ by F^{-1}. Since $(D\phi; D\phi_1) = \left(\frac{d\phi}{d\psi'}\right)^{it}\left(\frac{d\phi_1}{d\psi'}\right)^{-it}$, from the definition of D_t above we get $K = \left(\frac{d\phi}{d\psi'}\right)^{-1} = \frac{d\psi'}{d\phi}$. Thus, we get $\psi' \circ F = Tr\left(\left(\frac{d\psi'}{d\phi}\right)\cdot\right) = \psi' \circ \phi^{-1}$ so that $F = \phi^{-1}$ $(= (F^{-1})^{-1})$.

The above discussions show that ϕ^{-1} is characterized by

$$\psi' \circ \phi^{-1} = Tr\left(\frac{d\psi'}{d\phi}\cdot\right). \tag{3.1}$$

Figure 1 (the inverse ϕ^{-1})

For a rank-one operator $\xi \otimes \xi^c$ $(\xi \in \mathcal{K})$ we have

$$\psi' \circ \phi^{-1}(\xi \otimes \xi^c) = Tr\left(\frac{d\psi'}{d\phi}(\xi \otimes \xi^c)\right) = \left\| \left(\frac{d\psi'}{d\phi}\right)^{\frac{1}{2}} \xi \right\|^2,$$

which is equal to $\psi'(\theta^\phi(\xi, \xi))$ from the definition of the spatial derivative $\frac{d\psi'}{d\phi}$. Therefore, we see

$$\phi^{-1}(\xi \otimes \xi^c) = \theta^\phi(\xi, \xi). \tag{3.2}$$

The computation

$$\begin{aligned}(D\phi_1^{-1}; D\phi_2^{-1})_t &= (D(\psi' \circ \phi_1^{-1}); D(\psi' \circ \phi_2^{-1}))_t = \left(\frac{d\psi'}{d\phi_1}\right)^{it} \left(\frac{d\psi'}{d\phi_2}\right)^{-it} \\ &= \left(\frac{d\phi_1}{d\psi'}\right)^{-it} \left(\frac{d\phi_2}{d\psi'}\right)^{it} = (D\phi_1; D\phi_2)_{-t}\end{aligned}$$

shows that the bijective correspondence $\phi \longleftrightarrow \phi^{-1}$ is order-reversing.

If $\mathcal{M} = \mathcal{B}(\mathcal{K})$, then ϕ is a weight on $\mathcal{B}(\mathcal{K})$ and $\phi = Tr(h\cdot)$ with the (non-singular positive self-adjoint) density operator h. The operator valued weight $\phi^{-1} : \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{M}' = \mathbf{C}I$ actually means a weight on $\mathcal{B}(\mathcal{K})$. Remark 3.3,(iii) and the above imply $\psi^{-1} = Tr(h^{-1}\cdot)$ in this case.

Let $\mathcal{M} \supseteq \mathcal{N}$ be a factor-subfactor pair with an operator valued weight E. We assume that they act on a Hilbert space $\mathcal{K}$ (not necessary standard). Let ψ' be a weight on $\mathcal{M}'$ respectively. Then, ψ'^{-1} is an operator valued weight $\mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{M}$ and we consider the composition $E \circ \psi'^{-1} : \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{N}$. Its converse $(E \circ \psi'^{-1})^{-1}$ is a weight on $\mathcal{N}$. Let φ be a weight on $\mathcal{N}$. From (3.1) we note

$$Tr\left(\frac{d(E \circ \psi'^{-1})^{-1}}{d\varphi} \cdot\right) = (E \circ \psi'^{-1})^{-1} \circ \varphi^{-1} = (\varphi \circ E \circ \psi'^{-1})^{-1}.$$

The composition $\varphi \circ E \circ \psi'^{-1}$ is a weight on $\mathcal{B}(\mathcal{K})$. Recall that the density of its inverse weight is just the inverse operator of the density of $\varphi \circ E \circ \psi'^{-1}$. This means

$$Tr\left(\left(\frac{d(E \circ \psi'^{-1})^{-1}}{d\varphi}\right)^{-1} \cdot\right) = \varphi \circ E \circ \psi'^{-1} = Tr\left(\left(\frac{d(\varphi \circ E)}{d\psi'}\right) \cdot\right)$$

and hence

$$\frac{d(E \circ \psi'^{-1})^{-1}}{d\varphi} = \frac{d\psi'}{d(\varphi \circ E)}.$$

This equality obviously implies that modular automorphism groups associated with $(E \circ \psi'^{-1})^{-1}$ and ψ' (on $\mathcal{M}'$) agree on $\mathcal{N}'$. Thus, there is a unique operator valued weight $F : \mathcal{N}' \longrightarrow \mathcal{M}'$ such that $(E \circ \psi'^{-1})^{-1} = \psi' \circ F$, that is, $E \circ \psi'^{-1} = (\psi' \circ F)^{-1}$. Let us write $F = E^{-1}$, and since $\phi \circ (\psi' \circ F)^{-1} = (\phi \circ E) \circ \psi'^{-1}$ we get

$$\frac{d(\psi' \circ F)}{d\varphi} = \frac{d\psi'}{d(\varphi \circ E)}.$$

The construction of F does not depend on ψ'. Indeed, let ψ'_1 be another weight on $\mathcal{M}'$ (giving rise to F_1). Then, we see

$$\left(\frac{d(\psi' \circ F)}{d\varphi}\right)^{it} \left(\frac{d(\psi'_1 \circ F)}{d\varphi}\right)^{-it} = (D\psi'; D\psi'_1)_t$$
$$= \left(\frac{d\psi'}{d(\varphi \circ E)}\right)^{it} \left(\frac{d\psi'_1}{d(\varphi \circ E)}\right)^{-it} = \left(\frac{d(\psi' \circ F)}{d\varphi}\right)^{it} \left(\frac{d(\psi'_1 \circ F_1)}{d\varphi}\right)^{-it}.$$

Hence, we get $\psi'_1 \circ F = \psi'_1 \circ F_1$, that is, $F = F_1$ as desired.

Let us write $F = E^{-1}$, and we note

(3.3) $$\frac{d(\psi' \circ E^{-1})}{d\varphi} = \frac{d\psi'}{d(\varphi \circ E)}.$$

Figure 2 ($E^{-1} : \mathcal{N}' \longrightarrow \mathcal{M}'$)

It is obvious that this relation uniquely determines E^{-1}. In particular, we get $(E^{-1})^{-1} = E$ and the chain rule

$$(E_0 \circ E)^{-1} = E^{-1} \circ E_0^{-1} \quad \text{for} \quad \mathcal{M} \xrightarrow{E} \mathcal{N} \xrightarrow{E_0} \mathcal{L}.$$

Since $E^{-1} \circ \phi^{-1} = (\phi \circ E)^{-1}$ (ϕ is a weight on $\mathcal{N}$), from (3.2) we have the following description of E^{-1}:

(3.4) $$E^{-1}(\theta^{\phi}(\xi, \xi)) = \theta^{\phi \circ E}(\xi, \xi).$$

From now on we assume that $E : \mathcal{M} \longrightarrow \mathcal{N}$ is a normal conditional expectation. For each unitary u' in $\mathcal{M}'$ we have

$$u' E^{-1}(I) u'^* = E^{-1}(u' I u'^*) = E^{-1}(I)$$

by the bimodule property. Since $\mathcal{M}$ is a factor, the above computation means that $E^{-1}(I)$ is a positive scalar (or $+\infty$).

Definition 3.4. *Ind* $E = E^{-1}(I)$.

It is possible to show that the value $E^{-1}(I)$ is independent on the choice of a Hilbert space on which $\mathcal{M}$ acts (by checking the effect under reduction and amplification). The above chain rule for E^{-1} readily says the multiplicativity of our index $Ind(E_0 \circ E) = (Ind\ E_0) \times (Ind\ E)$.

To get a basic extension and so on, we now assume that factors are acting on $\mathcal{H}_{\varphi\circ E} = L^2(\mathcal{M})$ (φ is a faithful state on $\mathcal{N}$). As in the type II_1 case we get the Jones projection $e_{\mathcal{N}} \in \mathcal{N}'$ defined by $e_{\mathcal{N}} x\xi_0 = E(x)\xi_0$ $(\varphi \circ E = \omega_{\xi_0}$ and $\xi_0 \in L^2(\mathcal{M})_+)$. This projection $e_{\mathcal{N}}$ does not depend on the choice of φ and satisfies the properties in Lemma 3.2. As in the II_1 case we set

$$\mathcal{M}_1 = J\mathcal{N}'J = \langle \mathcal{M}, e_{\mathcal{N}} \rangle, \text{ the basic extension of } \mathcal{M} \supseteq \mathcal{N}.$$

We assume $Ind\ E < \infty$. The map $(Ind\ E)^{-1}E^{-1} :\ \mathcal{N}' \longrightarrow \mathcal{M}'$ satisfies

$$\begin{aligned}(Ind\ E)^{-1}E^{-1}(x) &= (Ind\ E)^{-1}E^{-1}(\sqrt{x}I\sqrt{x}) \\ &= (Ind\ E)^{-1}\sqrt{x}E^{-1}(I)\sqrt{x} \ = \ x\end{aligned}$$

for $x \in \mathcal{N}'_+$, which means that $(Ind\ E)^{-1}E^{-1}$ is a conditional expectation. The basic extension E_1 of E is defined as

$$E_1 = (Ind\ E)^{-1}JE^{-1}(J \cdot J)J :\ \mathcal{M}_1 \longrightarrow \mathcal{M}.$$

We have $Ind\ E = Ind\ E_1$ (which easily follows from $(E^{-1})^{-1} = E$). Recall we saw $\theta^{\varphi\circ E}(\xi_0, \xi_0) = I$ (Remark 3.3,(i)). Since $e_{\mathcal{N}}\mathcal{H}_{\varphi\circ E}$ $(= L^2(\mathcal{N}))$ is a standard Hilbert space for $\mathcal{N}$, the operator $R^{\varphi}(\xi)$ used to define $\theta^{\varphi}(\xi, \xi)$ is given by

$$e_{\mathcal{N}} x\xi_0 \in e_{\mathcal{N}}\mathcal{H}_{\varphi\circ E} \longrightarrow x\xi \in \mathcal{H}_{\varphi\circ E}$$

in the present case. In particular, $\theta^{\varphi}(\xi_0, \xi_0) = e_{\mathcal{N}}$ and (3.4) say $E^{-1}(e_{\mathcal{N}}) = I$. Therefore, we conclude

$$E_1(e_{\mathcal{N}}) = (Ind\ E)^{-1}I. \tag{3.5}$$

As in the II_1 case, by iterating the above-explained procedure we get the tower

$$\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{M}_1 \subseteq \cdots$$

of factors and successive conditional expectations. When $\mathcal{M} \supseteq \mathcal{N}$ are type II_1 factors, we have the canonical conditional expectation $E = E_{\mathcal{N}}$ determined by the unique trace $tr_{\mathcal{M}}$. In this case we have $Ind\ E = [\mathcal{M} : \mathcal{N}]$, the Jones index. In fact, let

$\xi_0 \in L^2(\mathcal{M}; tr_{\mathcal{M}})$ be the unit trace vector for $\mathcal{M}$. Then, $\omega_{\xi_0}^{\mathcal{M}'}$ is a unique normalized trace $tr_{\mathcal{M}'}$. Since $tr_{\mathcal{M}} = tr_{\mathcal{N}} \circ E$, we observe

$$1 = \frac{dtr_{\mathcal{M}}}{dtr_{\mathcal{M}'}} = \frac{dtr_{\mathcal{N}}}{dtr_{\mathcal{M}'} \circ E^{-1}}$$

(see (3.3)). This shows that $tr_{\mathcal{M}'} \circ E^{-1}$ (on $\mathcal{N}'$) is tracial, and it is proportional to $tr_{\mathcal{N}'}$. Recall that the Jones index comes from the coupling constant

$$[\mathcal{M} : \mathcal{N}] = \dim_{\mathcal{N}}(L^2(\mathcal{M}, tr_{\mathcal{M}})) = tr_{\mathcal{N}}(e_{\mathcal{N}})^{-1}$$

while we get $tr_{\mathcal{M}'} \circ E^{-1}(e_{\mathcal{M}}) = 1$. Therefore, we must have

$$tr_{\mathcal{N}'} = [\mathcal{M} : \mathcal{N}]^{-1} tr_{\mathcal{M}'} \circ E^{-1}.$$

Comparing the values of the both sides against the identity operator I, we see $Ind\ E = [\mathcal{M} : \mathcal{N}]$.

3.2. Longo's Approach. Let $\mathcal{M} \supseteq \mathcal{N}$ be properly infinite von Neumann algebras acting on the Hilbert space $L^2(\mathcal{M})$ with a common cyclic and separating vector ξ_0. Therefore, we have the modular conjugations $J_{\mathcal{N},\xi_0}, J_{\mathcal{M},\xi_0}$ acting on the same space.

The endomorphism $\gamma\ (= \gamma_{\xi_0}) = Ad J_{\mathcal{N},\xi_0} J_{\mathcal{M},\xi_0} :\ \mathcal{M} \longrightarrow \mathcal{N}$ is called the canonical endomorphism of $\mathcal{M} \supseteq \mathcal{N}$ ([50]). In fact, for another cyclic and separating vector ξ_1 we have $\gamma_{\xi_1} = Ad\, u \circ \gamma_{\xi_0}$ for some unitary $u \in \mathcal{N}$, i.e., γ doesn't depend on the choice of ξ_0 (up to inner perturbation). Actually we need not use a common cyclic and separating vector in our further theory. Any modular conjugations $J_{\mathcal{N}}, J_{\mathcal{M}}$ (coming from a state or a weight) work fine for our purpose.

We begin with the semi-finite case. Let $A \supseteq B$ an inclusion of type II_1 factors. Then, $\mathcal{M} = A \otimes \mathcal{B}(\mathcal{H}) \supseteq \mathcal{N} = B \otimes \mathcal{B}(\mathcal{H})$ is that of type II_∞ factors, and we have the canonical endomorphism γ. Let $tr_{\mathcal{M}}$ be a trace on $\mathcal{M}$. Since a trace is unique up to a scalar multiple, $tr_{\mathcal{M}}$ and $tr_{\mathcal{M}} \circ \gamma$ are proportional. Actually we have $tr_{\mathcal{M}} \circ \gamma = [A : B]\ tr_{\mathcal{M}}$. Instead of showing this, we look at the following more general situation:

Lemma 3.5. *Let $\mathcal{M} \supseteq \mathcal{N}$ be an inclusion of type II_∞ factors. Furthermore, we assume that a trace $tr_{\mathcal{M}}$ is semi-finite on $\mathcal{N}$ so that $tr_{\mathcal{M}}$ gives us a conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$ satisfying $tr_{\mathcal{M}} = tr_{\mathcal{N}} \circ E$ with $tr_{\mathcal{N}} = tr_{\mathcal{M}}\,|_{\mathcal{N}}$. In this case we have $tr_{\mathcal{M}} \circ \gamma = (Ind\ E)\ tr_{\mathcal{M}}$.*

Proof. We set

$$tr_{\mathcal{M}'} = tr_{\mathcal{M}}(J_{\mathcal{M}} \cdot^* J_{\mathcal{M}}) \quad \text{and} \quad tr_{\mathcal{N}'} = tr_{\mathcal{N}}(J_{\mathcal{N}} \cdot^* J_{\mathcal{N}}),$$

where everything is acting on $\mathcal{H} = L^2(\mathcal{M}; tr_{\mathcal{M}})$. At first we show $\frac{dtr_{\mathcal{M}'}}{dtr_{\mathcal{M}}} = 1$. For $x \in \mathfrak{N}_{tr_{\mathcal{M}}}$ (the definition ideal of $tr_{\mathcal{M}}$), $R^{tr_{\mathcal{M}}}(\Lambda_{tr_{\mathcal{M}}}(x))$ is given by

$$\Lambda_{tr_{\mathcal{M}}}(y) \longrightarrow y\Lambda_{tr_{\mathcal{M}}}(x) = \Lambda_{tr_{\mathcal{M}}}(yx).$$

Therefore, $R^{tr_{\mathcal{M}}}(\Lambda_{tr_{\mathcal{M}}}(x))$ is the right multiplication by x, that is, $J_{\mathcal{M}}x^*J_{\mathcal{M}}$, and hence the quadratic form giving rise to the spatial derivative $\frac{dtr_{\mathcal{M}'}}{dtr_{\mathcal{M}}}$ is

$$\begin{aligned} \Lambda_{tr_{\mathcal{M}}}(x) \quad \longrightarrow \quad & tr_{\mathcal{M}'}(\theta^{tr_{\mathcal{M}}}(\Lambda_{tr_{\mathcal{M}}}(x), \Lambda_{tr_{\mathcal{M}}}(x))) \\ & = tr_{\mathcal{M}'}(J_{\mathcal{M}}x^*xJ_{\mathcal{M}}) = tr_{\mathcal{M}}(x^*x) = \langle \Lambda_{tr_{\mathcal{M}}}(x), \Lambda_{tr_{\mathcal{M}}}(x) \rangle. \end{aligned}$$

Therefore, we get $\frac{dtr_{\mathcal{M}'}}{dtr_{\mathcal{M}}} = 1$, and the identical argument also shows $\frac{dtr_{\mathcal{N}'}}{dtr_{\mathcal{N}}} = 1$. From (3.3) we compute

$$\frac{d(tr_{\mathcal{M}'} \circ E^{-1})}{dtr_{\mathcal{N}}} = \frac{dtr_{\mathcal{M}'}}{d(tr_{\mathcal{N}} \circ E)} = \frac{dtr_{\mathcal{M}'}}{dtr_{\mathcal{M}}} = 1 = \frac{dtr_{\mathcal{N}'}}{dtr_{\mathcal{N}}}$$

so that we see $tr_{\mathcal{M}'} \circ E^{-1} = tr_{\mathcal{N}'}$. For $x \in \mathcal{M}'_+$, we compute

$$tr_{\mathcal{N}'}(x) = tr_{\mathcal{M}'}(E^{-1}(x)) = (Ind\ E)\ tr_{\mathcal{M}'}(x),$$

showing $tr_{\mathcal{N}'} |_{\mathcal{M}'} = (Ind\ E) tr_{\mathcal{M}'}$. From this (for $x \in \mathcal{M}_+$) we compute

$$\begin{aligned} & tr_{\mathcal{M}}(\gamma(x)) = tr_{\mathcal{N}}(\gamma(x)) = tr_{\mathcal{N}}(J_{\mathcal{N}}(J_{\mathcal{M}}x^*J_{\mathcal{M}})^*J_{\mathcal{N}}) \\ & \quad = tr_{\mathcal{N}'}(J_{\mathcal{M}}x^*J_{\mathcal{M}}) = (Ind\ E)\ tr_{\mathcal{M}'}(J_{\mathcal{M}}x^*J_{\mathcal{M}}) = (Ind\ E)\ tr_{\mathcal{M}}(x). \end{aligned}$$

□

Note that the arguments in the above proof work as long as $E^{-1}(1)$ is a scalar (even if $\mathcal{M}, \mathcal{N}$ are not factors).

Next let $\mathcal{M} \supseteq \mathcal{N}$ be type III factors with a normal conditional expectation $E : \mathcal{M} \longrightarrow \mathcal{N}$. For a faithful $\varphi \in \mathcal{N}_*^+$, we consider the composition $\varphi \circ E \in \mathcal{M}_*^+$. Then the modular automorphism $\sigma_t^{\varphi \circ E}$ satisfies

$$(1) \quad \sigma_t^{\varphi \circ E}|_{\mathcal{N}} = \sigma_t^{\varphi}, \quad \text{and} \quad (2) \quad E \circ \sigma_t^{\varphi \circ E} = \sigma_t^{\varphi} \circ E.$$

From (1) we get the inclusions $\widetilde{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma^{\varphi \circ E}} \mathbf{R} \supseteq \widetilde{\mathcal{N}} = \mathcal{N} \rtimes_{\sigma^{\varphi}} \mathbf{R}$ of type II_∞ algebras, and (2) implies that E extends to $\widetilde{E} : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{N}}$ via $\widetilde{E}(\int_{-\infty}^{\infty} x(t)\lambda(t)dt) =$

$\int_{-\infty}^{\infty} E(x(t))\lambda(t)dt$. From the construction, we get the compatibility of all the relevant structures (such as the dual actions, the canonical traces, and so on). The canonical trace-scaling traces actually satisfy $tr_{\tilde{\mathcal{M}}}|_{\tilde{\mathcal{N}}} = tr_{\tilde{\mathcal{N}}}$, and $\tilde{E}$ indeed arises from $tr_{\widetilde{\mathcal{M}}}$ (details are at the beginning of §5.1).

Let us assume that $\mathcal{M} \supseteq \mathcal{N}$ are type III_1 factors at first, and let $\tilde{\gamma} : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{N}}$ be the canonical endomorphism. Since $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{N}}$ are type II_∞ factors by the assumption, as before we conclude $tr_{\widetilde{\mathcal{M}}} \circ \tilde{\gamma} = \lambda\, tr_{\widetilde{\mathcal{M}}}$. However, we point out that this phenomenon is completely general.

Lemma 3.6. *The same conclusion holds in general, i.e.,* $tr_{\widetilde{\mathcal{M}}} \circ \tilde{\gamma} = \lambda\, tr_{\widetilde{\mathcal{M}}}$.

Proof. At first note that $tr_{\tilde{\mathcal{M}}} \circ \tilde{\gamma}$ is also a trace and hence $tr_{\tilde{\mathcal{M}}} \circ \tilde{\gamma} = tr_{\tilde{\mathcal{M}}}(h\cdot)$ with $h = \frac{d\,tr_{\tilde{\mathcal{M}}}\circ\gamma}{d\,tr_{\tilde{\mathcal{M}}}}$ affiliated to the center $\mathcal{Z}(\widetilde{\mathcal{M}})$. Also the modular conjugations $J_{\widehat{\varphi\circ E}}$ and $J_{\hat{\varphi}}$ coming from the respective dual weights are given by

$$(J_{\widehat{\varphi\circ E}}\xi)(t) = \Delta_{\varphi\circ E}^{it} J_{\mathcal{M}}\xi(-t), \ \ (J_{\hat{\varphi}}\xi)(t) = \Delta_{\varphi}^{it} J_{\mathcal{N}}\xi(-t)$$

for a vector $\xi \in L^2(\mathcal{M}) \otimes L^2(\mathbf{R}) \cong L^2(\mathbf{R} : L^2(\mathcal{M}))$. The unitary $U(s)$ implementing the dual action (see §2.1) obviously commutes with the above J's so that the canonical endomorphism $\tilde{\gamma} = AdJ_{\widehat{\varphi\circ E}}J_{\hat{\varphi}}$ and the dual action θ_s commute. From (2.1) we notice

$$(tr_{\widetilde{\mathcal{M}}} \circ \tilde{\gamma})(\theta_{-s}(x)) = tr_{\widetilde{\mathcal{M}}} \circ \theta_{-s} \circ \tilde{\gamma}(x) = e^s\, tr_{\tilde{\mathcal{M}}} \circ \tilde{\gamma}(x),$$
$$tr_{\widetilde{\mathcal{M}}}(h\,\theta_{-s}(x)) = tr_{\tilde{\mathcal{M}}} \circ \theta_{-s}(\theta_s(h)x) = e^s\, tr_{\widetilde{\mathcal{M}}}(\theta_s(h)x).$$

Thus, we see $tr_{\widetilde{\mathcal{M}}} = tr_{\widetilde{\mathcal{M}}} \circ \tilde{\gamma}(\theta_s(h)\cdot)$ and $h = \theta_s(h)$ for each $s \in \mathbf{R}$ thanks to the uniqueness of a Radon-Nikodym derivative. Hence, we get the lemma from the central ergodicity of $\{\theta_s\}_{s\in\mathbf{R}}$. □

In [51] the constant λ in Lemma 3.6 was defined as the index of E. Although $\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ are not factors, we actually have $(\tilde{E})^{-1}(I) = (Ind\ E)I$ (as will be seen based on Theorem 3.12). Therefore, Lemma 3.5 (and the comment after the proof) shows that the index in this sense is the same as $Ind\ E$.

3.3. Minimal Expectation. If $\mathcal{M} \cap \mathcal{N}' = \mathbf{C}I$, then there exists a unique normal conditional expectation E (if it exists). But generally there are many $E's$, and let us minimize $Ind\ E$ over all possible E's. At first we consider the simple-minded example:

$$\mathbf{C}I = \mathcal{N} \subseteq M_n(\mathbf{C}) = \mathcal{M}$$

acting on $\mathbf{C}^n$. Note $\mathcal{N}' = M_n(\mathbf{C})$ and $\mathcal{M}' = \mathbf{C}I$. A normal conditional expectation (onto $\mathbf{C}$) means a state $E = Tr(h\cdot)$ while an operator valued weight $E^{-1} = Tr(k\cdot) : M_n(\mathbf{C}) = \mathcal{N}' \longrightarrow \mathbf{C} = \mathcal{M}'$ means a positive functional. We have $k = h^{-1}$ as was pointed out before, and the eigenvalues $\{\lambda_i\}_{i=1,2,\cdots,n}$ of h sum up to 1 due to $Tr(h) = 1$. We get $Ind\ E = E^{-1}(I) = \sum_i \lambda_i^{-1}$, and hence

$$\min \left\{ \sum_{i=1}^{n} \lambda_i^{-1};\ \lambda_i > 0,\ \sum \lambda_i = 1 \right\} = n^2.$$

Note that the minimum is attained exactly when $\lambda_i = \frac{1}{n}$, i.e., $E : M_n(\mathbf{C}) \longrightarrow \mathbf{C}$ is the normalized trace.

Let us go back to the general case, and we assume $Ind\ E < \infty$ (which does not depend on the choice of E). We will here see that $\min\{Ind\ E\}$ is attained for a unique conditional expectation ([26, 27, 51]).

At first we recall the notion of a local index. From a projection p in $\mathcal{M} \cap \mathcal{N}'$ we get the inclusion $p\mathcal{M}p \supseteq \mathcal{N}p$. Note $E(p) \in \mathcal{N} \cap \mathcal{N}' = \mathbf{C}I$, and the map E_p defined by

$$x \in p\mathcal{M}p \longrightarrow E(p)^{-1}E(x)p \in \mathcal{N}p$$

is a conditional expectation. As was seen in Propositions 4.2, 4.3, [38], the index of E_p (i.e., a local index) is given by

$$E(p)E^{-1}(p) \geq Ind\ E_p \quad \text{(and the equality holds if } \sigma_t^E(p) = p). \tag{3.6}$$

The conditional expectations are parameterized by the state space of the finite-dimensional algebra $\mathcal{M} \cap \mathcal{N}'$. More precisely, $E \longleftrightarrow E\ |_{\mathcal{M}\cap\mathcal{N}'}$ is known to give rise to a bijective correspondence to the state space of the finite-dimensional relative commutant $\mathcal{M} \cap \mathcal{N}'$ (see Appendix **3**). Let

$$\mathcal{M} \cap \mathcal{N}' = \sum_{i=1}^{k} {}^{\oplus} M_{n_i}(\mathbf{C}),$$

and we at first require that $E\ |_{\mathcal{M}\cap\mathcal{N}'}\ I$ be tracial. Let p_i be a minimal projection in the i-th component $M_{n_i}(\mathbf{C})$ (note that the values $E(p_i) \in \mathcal{N} \cap \mathcal{N}' = \mathbf{C}$ $(i = 1, 2, \cdots, k)$

determine the trace $E\mid_{\mathcal{M}\cap\mathcal{N}'}$). We secondly require $E^{-1} = \text{const.} \times E$ on $\mathcal{M}\cap\mathcal{N}'$ (the constant is of course $Ind\ E$). Look at the reduced system $p_i\mathcal{M}p_i \supseteq \mathcal{N}p_i$. This is an irreducible inclusion so that the index $(=\mu_i)$ is unambiguously determined. Our requirement and (3.6) force $\mu_i = \mu \times E(p_i)^2$, that is, $E(p_i) = \frac{\sqrt{\mu_i}}{\sqrt{\mu}}$ with $\mu = Ind\ E$ for simplicity. On the other hand, we must have the normalization property

$$I = E(I) = \sum_{j=1}^{k} n_j E(p_j) = \sum_{j=1}^{k} \frac{n_j\sqrt{\mu_j}}{\sqrt{\mu}}.$$

It means $\sqrt{\mu} = \sum_{i=1}^{k} n_i\sqrt{\mu_i}$, and hence we conclude

$$E(p_i) = \frac{\sqrt{\mu_i}}{\sum_{j=1}^{k} n_j\sqrt{\mu_j}} \quad \left(\text{and} \quad E^{-1}(p_i) = \sqrt{\mu_i}\left(\sum_{j=1}^{k} n_j\sqrt{\mu_j}\right)\right).$$

We have seen so far that the two conditions

(i) E is tracial on $\mathcal{M}\cap\mathcal{N}'$

(ii) $E^{-1} = \text{const.} \times E$ on $\mathcal{M}\cap\mathcal{N}'$

uniquely determine a state on $\mathcal{M}\cap\mathcal{N}'$ (and hence a normal conditional expectation). By E_0 we denote the unique conditional expectation specified in this way ([27]).

Theorem 3.7. *We have* $Ind\ E_0 = \min Ind\ E$, *furthermore* E_0 *is a unique minimizer.*

Thanks to this theorem, E_0 is referred to as the minimal conditional expectation of $\mathcal{M} \supseteq \mathcal{N}$. We often write

$$Ind\ E_0 = [\mathcal{M} : \mathcal{N}]_0.$$

We point out that in the II_1 case the conditional expectation $E = E_{\mathcal{N}}$ is not necessarily minimal. A typical example is the locally trivial inclusion

$$\mathcal{M} \supseteq \mathcal{N} = \{x + \theta(x);\ x \in p\mathcal{M}p\}$$

arising from a projection $p \in \mathcal{M}$ and an isomorphism $\theta : p\mathcal{M}p \longrightarrow (1-p)\mathcal{M}(1-p)$. In fact, the Jones index is $[\mathcal{M} : \mathcal{N}] = Ind\ E = \frac{1}{tr(p)} + \frac{1}{1-tr(p)}$ (Corollary 2.2.5, [33]) while the minimal index in $[\mathcal{M} : \mathcal{N}]_0 = 4$. An inclusion for which $E = E_{\mathcal{N}}$ is minimal is called extremal in the literature.

Proof. We start from an arbitrary conditional expectation F (i.e., a state on $\mathcal{M}\cap\mathcal{N}'$). For each $i \in \{1,2,\cdots,k\}$ we choose and fix a family $\{p_{ij}\}_{j=1,2,\cdots,n_i}$ of orthogonal minimal projections in $M_{n_i}(\mathbf{C})$ summing up to the identity in this matrix algebra. We set $\alpha_{ij} = F(p_{ij})$, and notice that the normalization condition $F(I) = I$ is

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha_{ij} = 1. \tag{3.7}$$

On the other hand, p_{ij} and p_i are unitarily equivalent via a unitary in $\mathcal{M}\cap\mathcal{N}'$ and hence $p_{ij}\mathcal{M}p_{ij} \supseteq \mathcal{N}p_{ij}$ is conjugate to the irreducible inclusion $p_i\mathcal{M}p_i \supseteq \mathcal{N}p_i$ so that (3.6) says $F^{-1}(p_{ij}) \geq \frac{\mu_i}{\alpha_{ij}}$ and

$$Ind\ F = \sum_{i=1}^{k}\sum_{j=1}^{n_i}E^{-1}(p_{ij}) \geq \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\mu_i}{\alpha_{ij}}.$$

The use of Lagrange multipliers tells that the minimum of $\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\mu_i}{\alpha_{ij}}$ subject to (3.7) is $Ind\ E_0 = (\sum_{i=1}^{k} n_i\sqrt{\mu_i})^2$. Therefore, we establish the minimality $Ind\ F \geq Ind\ E_0$. Moreover, it is straight-forward to see that this minimum is attained only when $\alpha_{ij} = \frac{\sqrt{\mu_i}}{\sum_{\ell=1}^{k} n_\ell\sqrt{\mu_\ell}}$ (independent of j for each i). If $Ind\ F = Ind\ E_0$, then this condition has to be satisfied for all p_{ij}'s. This means that F must be at first tracial on $\mathcal{M}\cap\mathcal{N}'$ and that the value of F against a minimal projection in each component $M_{n_i}(\mathbf{C})$ has to be $\frac{\sqrt{\mu_i}}{\sum_{\ell=1}^{k} n_\ell\sqrt{\mu_\ell}}$. Therefore, we must have $F = E_0$. □

If $E : \mathcal{M} \longrightarrow \mathcal{N}$ is minimal, then so is the basic extension $E_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$ (notice the general fact $\sigma_t^E = \sigma_{-t}^{E^{-1}}$, a consequence of (3.3)). Note that even if $\mathcal{M}\cap\mathcal{N}' = \mathbf{C}I$ we have $\mathcal{M}_1\cap\mathcal{N}' \neq \mathbf{C}I$ and the minimality of $E\circ E_1 : \mathcal{M}_1 \longrightarrow \mathcal{N}$ is not so clear. But it is indeed the case as was shown in [43], and more generally the composition of minimal expectations is always minimal ([52]). A very interesting proof (based on bases, see Theorem 2.9) of this fact can be found in [37].

Let $\{p_i\}_{i=1,2,\cdots,n}$ be a partition of I consisting of minimal projections in $\mathcal{M}\cap\mathcal{N}'$. The arguments before the above lemma tell the additivity of square roots of indices

$$[\mathcal{M}:\mathcal{N}]_0^{\frac{1}{2}} = \sum_{i=1}^{n}[p_i\mathcal{M}p_i : \mathcal{N}p_i]_0^{\frac{1}{2}} \tag{3.8}$$

($p_i\mathcal{M}p_i \supseteq \mathcal{N}p_i$ is irreducible). From the arguments in the above proof we observe that the validity of the additivity (3.8) for all partitions consisting of minimal projections actually characterize the minimal expectation E_0 ([26, 51]).

We claim that the reduced expectation $E_p : p\mathcal{M}p \longrightarrow \mathcal{N}p$ ($p \in \mathcal{M} \cap \mathcal{N}'$) is also minimal as long as E is. In fact, let q be a (minimal) projection in $p\mathcal{M}p \cap (\mathcal{N}p)' = p(\mathcal{M} \cap \mathcal{N}')p$. Then we note

$$(E_p)_q(x) = (E_p(q))^{-1}E_p(x)q = (E(p)^{-1}E(q)p)^{-1}E(p)^{-1}E(x)pq = E(q)^{-1}E(x)q.$$

Thus, we get $(E_p)_q = E_q$, and $\sqrt{Ind\ (E_p)_q} = \sqrt{Ind\ E_q} = \sqrt{Ind\ E}\ E(q)$ since E is minimal. On the other hand, we have $Ind\ E_p = (Ind\ E)E(p)^2$, and hence

$$\sqrt{Ind (E_p)_q} = \sqrt{Ind\ E_p}\ E(p)^{-1}E(q)\ \left(= \sqrt{Ind\ E_p}\ E_p(q)\right)$$

(p is the identity in $p\mathcal{M}p$). Therefore, the additivity (3.8) is valid for E_p and E_p is minimal as desired.

Note that this claim shows the additivity (3.8) is also valid for any partition even if p_i's in $\mathcal{M} \cap \mathcal{N}'$ are not necessarily minimal projections. In fact, in this case (3.6) becomes $[p_i\mathcal{M}p_i : \mathcal{N}p_i]_0 = E^{-1}(p_i)E(p_i) = [\mathcal{M} : \mathcal{N}]_0 E(p_i)^2$.

3.4. Pimsner-Popa Paper. There are various useful ways to compute $Ind\ E$. Let us briefly mention the work by Pimsner and Popa ([56]).

Theorem 3.8. (Pimsner-Popa inequality)

$$E(x) \geq (Ind\ E)^{-1}x \ \ \textit{for all} \ \ x \in \mathcal{M}_+.$$

The equality is attained for $x = e_{\mathcal{N}}$. Furthermore, (except type I case) if $\varepsilon \geq 0$ is the best constant satisfying $E(x) \geq \varepsilon x$ for $x \in \mathcal{M}_+$, then $Ind\ E$ is the reciprocal of ε.

Proof. We consider the standard action of $\mathcal{M}$ on $\mathcal{H} = L^2(\mathcal{M})$. Let $\chi' \in \mathcal{M}'^+_*$ (faithful) with the vector $\xi_{\chi'} = \Lambda_{\chi'}(I)$ satisfying $\chi' = \omega_{\xi_{\chi'}}$. A vector $\xi = y\xi_{\chi'} \in \mathcal{M}\xi_{\chi'}$ is a χ'-bounded vector and $R^{\chi'}(\xi) = y$. Hence, we see $\theta^{\chi'}(\xi, \xi) = yy^* \in \mathcal{M}_+$ (Remark 3.3, (i)). Since $E^{-1}(I) = Ind\ E < \infty$, we have $\chi' \circ E^{-1} \in \mathcal{N}'^+_*$ and we consider the operator

$$I :\ \Lambda_{\chi'}(x') \in \mathcal{K} \longrightarrow \Lambda_{\chi' \circ E^{-1}}(x') \in \mathcal{H}_{\chi' \circ E^{-1}}.$$

Since

$$||\Lambda_{\chi' \circ E^{-1}}(x')||^2 = \chi' \circ E^{-1}(x'^* x') = (Ind\ E) \times \chi'(x'^* x') = (Ind\ E) \times ||\Lambda_{\chi'}(x')||^2,$$

I gives rise to a bounded operator with norm less than $(Ind\ E)^{\frac{1}{2}}$. Notice

$$R^{\chi'}(\xi) \subseteq R^{\chi'\circ E^{-1}}(\xi)I \text{ and } I^* R^{\chi'\circ E^{-1}}(\xi)^* \subseteq R^{\chi'}(\xi)^*.$$

Therefore, we get

$$\begin{aligned}\theta^{\chi'}(\xi,\xi) = R^{\chi'}(\xi)R^{\chi'}(\xi)^* &\leq ||I||^2 \times R^{\chi'\circ E^{-1}}(\xi)R^{\chi'\circ E^{-1}}(\xi)^* \\ &\leq (Ind\ E) \times R^{\chi'\circ E^{-1}}(\xi)R^{\chi'\circ E^{-1}}(\xi)^* = (Ind\ E) \times \theta^{\chi'\circ E^{-1}}(\xi,\xi).\end{aligned}$$

(If $\xi = y\xi_{\chi'}$ is a $(\chi' \circ E^{-1})$-bounded vector, the above estimate is obvious. Otherwise, the both sides still make sense as elements in the extended positive part of $\mathcal{M}$, see [18].) Recall $E(\theta^{\chi'}(\xi,\xi)) = \theta^{\chi'\circ E^{-1}}(\xi,\xi)$ because of $(E^{-1})^{-1} = E$ (see §3.1 and especially (3.4)) while we have $\theta^{\chi'}(\xi,\xi) = yy^*$. Therefore, the above inequality means $yy^* \leq (Ind\ E)E(yy^*)$, and we are done. □

Note that the above proof works even if $\mathcal{M} \supseteq \mathcal{N}$ are not factors.

Theorem 3.9. (Pimsner-Popa basis) *If $Ind\ E < \infty$, then we can find elements $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subseteq \mathcal{M}$ such that*

$$\sum \lambda_i e_{\mathcal{N}} \lambda_i^* = I. \tag{3.9}$$

This $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is called a Pimsner-Popa basis for $\mathcal{N} \subseteq \mathcal{M}$ (it is really a basis as will be seen below), and the theorem can be proved based on the next lemma. (In fact, after performing a down-ward basic extension $\mathcal{M} \supseteq \mathcal{N} \supseteq \mathcal{P}$, one can repeat the argument in the first paragraph of the proof of Theorem 3.12 for the Jones projection $e_{\mathcal{P}} \in \mathcal{M}$.)

Lemma 3.10. (Push-down lemma) *When $Ind\ E < \infty$, for each $x \in \mathcal{M}_1$ there exists $\tilde{x} \in \mathcal{M}$ satisfying $xe_{\mathcal{N}} = \tilde{x}e_{\mathcal{N}}$.*

The lemma readily follows from $xe_{\mathcal{N}} = (Ind\ E)E_1(xe_{\mathcal{N}})e_{\mathcal{N}}$ $(x \in \mathcal{M}_1)$. This formula is trivial for $x \in \mathcal{M}$ and $x = ae_{\mathcal{N}}b$ $(a, b \in \mathcal{M})$, and hence it comes from the fact that an element generated (algebraically) by $\mathcal{M}$ and $e_{\mathcal{N}}$ can be rewritten as a sum of the preceding two kinds of x's (a consequence of basic properties of $e_{\mathcal{N}}$).

For $x \in \mathcal{M}$, thanks to (3.9) we compute

$$xe_{\mathcal{N}} = \sum \lambda_i e_{\mathcal{N}} \lambda_i^* x e_{\mathcal{N}} = \sum \lambda_i E_{\mathcal{N}}(\lambda_i^* x)e_{\mathcal{N}}.$$

Then, by hitting the both sides to $\xi_0 = \Lambda_\varphi(I)$, we get $x\xi_0 = \sum \lambda_i E_{\mathcal{N}}(\lambda_i^* x)\xi_0$, that is,

$$x = \sum \lambda_i E_{\mathcal{N}}(\lambda_i^* x).$$

This means that $\mathcal{M}$ is a finitely generated (actually projective) $\mathcal{N}$-module (when $Ind\ E < \infty$). Furthermore, by applying E_1 to (3.9), we get

$$I = E_1(I) = \sum E_1(\lambda_i e_{\mathcal{N}} \lambda_i^*) = \sum \lambda_i E_1(e_{\mathcal{N}})\lambda_i^* = (Ind\ E)^{-1} \sum \lambda_i \lambda_i^*,$$

and hence $Ind\ E = \sum \lambda_i \lambda_i^*$. Such bases also play important roles in Watatani's C^*-algebra index theory ([66]), and indeed the above is the definition of $Ind\ E$ in his theory.

Example 3.11. Let $\mathcal{N}$ be a factor equipped with an outer action $\alpha : G \longrightarrow Aut(\mathcal{N})$ of a finite group G. We set $\mathcal{M} = \mathcal{N} \rtimes_\alpha G$, which is a factor thanks to the outerness. Let

$$E : \sum_{g \in G} x_g \lambda_g \in \mathcal{M} \longrightarrow x_e \in \mathcal{N}$$

be the unique (due to $\mathcal{M} \cap \mathcal{N}' = \mathbf{C}I$) normal conditional expectation. Recall $L^2(\mathcal{M}) = L^2(\mathcal{N}) \otimes \ell^2(G)$ and an operator here can be expressed as $\mathcal{B}(L^2(\mathcal{N}))$-valued matrix with indices from G. In this matrix picture we have $\lambda_g = \sum_{h \in G} e_{gh,h}$ and $e_{\mathcal{N}} = e_{e,e}$ (where e denotes the unit in G). It is plain to see $\lambda_g e_{\mathcal{N}} \lambda_g^* = e_{g,g}$ and hence

$$\sum_{g \in G} \lambda_g e_{\mathcal{N}} \lambda_g^* = \sum_{g \in G} e_{g,g} = I.$$

Therefore, $\{\lambda_g\}_{g \in G}$ gives rise to a Pimsner-Popa basis, and $Ind\ E = \sum_{g \in G} \lambda_g \lambda_g^* = \#G$ as expected.

3.5. Characterization of the Basic Extension. The following characterization of the basic extension is useful:

Theorem 3.12. *Let $E : \mathcal{M} \longrightarrow \mathcal{N}$ be a normal conditional expectation. Assume that $\widetilde{\mathcal{L}}$ is a von Neumann algebra containing $\mathcal{M}$ and a normal conditional expectation $\widetilde{E} : \widetilde{\mathcal{L}} \longrightarrow \mathcal{M}$ is given. If $\widetilde{\mathcal{L}}$ possesses a projection $\tilde{e}$ with the central support I such* *that*

$$\widetilde{E}(\tilde{e}) = \lambda^{-1} I \ \ (\lambda > 0 \ \textit{scalar})$$
$$\lambda \widetilde{E}(\tilde{x}\tilde{e})\tilde{e} = \tilde{x}\tilde{e} \ \ \textit{for} \ \ \tilde{x} \in \widetilde{\mathcal{L}}$$
$$\tilde{e}x\tilde{e} = E(x)\tilde{e} \ \ \textit{for} \ \ x \in \mathcal{M},$$

then there exists an isomorphism π *from* $\widetilde{\mathcal{L}}$ *onto the basic extension* $\mathcal{M}_1$ *satisfying* $\pi(\tilde{e}) = e_1,\ \pi(x) = x\ (x \in \mathcal{M})$ *and* $\pi \circ \widetilde{E} \circ \pi^{-1} = \lambda^{-1} J E^{-1}(J \cdot J) J$. *In particular, we have* $E^{-1}(I) = \lambda I$.

The second condition is nothing but the content of the push-down lemma (see the paragraph after Lemma 3.10). Some characterization results are known for factors (see [57] and also [25]). But note that algebras are not necessarily factors in the theorem.

Proof. Choose partial isometries $\tilde{v}_1, \tilde{v}_2, \cdots$ in $\widetilde{\mathcal{L}}$ such that $\sum_{i=1}^{\infty} \tilde{v}_i \tilde{e} \tilde{v}_i^* = I$ (when $\mathcal{M} \supseteq \mathcal{N}$ are type II_1 factors the number of v_i's needed here is $[Ind\ E] + 1$). Hence, each $x \in \widetilde{\mathcal{L}}^+$ can be written as $x = \sum_{i=1}^{\infty} x^{\frac{1}{2}} \tilde{v}_i \tilde{e} \tilde{v}_i^* x^{\frac{1}{2}}$. Therefore, the second condition guarantees

$$x = \sum_{i=1}^{\infty} x_i \tilde{e} x_i^* \quad \text{with} \quad x_i = \lambda \widetilde{E}(x^{\frac{1}{2}} \tilde{v}_i \tilde{e}) \in \mathcal{M}.$$

At first we note $E \circ \widetilde{E}(x\tilde{e}) = E \circ \widetilde{E}(\tilde{e}x)$ for each $x \in \widetilde{\mathcal{L}}$. In fact, we may and do assume $x = a\tilde{e}a^*$ $(a \in \mathcal{M})$ from the above and then this follows from the first and third conditions. We choose and fix a faithful normal state φ on $\mathcal{N}$ and set $\psi = \varphi \circ E \in \mathcal{M}_*^+$. Let $\Lambda = \Lambda_\psi$ be the canonical injection from $\mathcal{M}$ into the GNS Hilbert space $\mathcal{H}_\psi$, and we set $\pi(x)\Lambda(m) = \lambda \Lambda(\widetilde{E}(xm\tilde{e}))$ for $x \in \mathcal{M}$ and $m \in \mathcal{M}$. We estimate

$$\begin{aligned} ||\lambda \Lambda(\widetilde{E}(xm\tilde{e}))||^2 &= \lambda^2 \psi\left(\widetilde{E}(xm\tilde{e})^* \widetilde{E}(xm\tilde{e})\right) \le \lambda^2 \psi\left(\widetilde{E}(\tilde{e}m^*x^*xm\tilde{e})\right) \\ &\le \lambda^2 ||x||^2 \psi\left(\widetilde{E}(\tilde{e}m^*m\tilde{e})\right) = \lambda^2 ||x||^2 \varphi \circ E\left(\widetilde{E}(E(m^*m)\tilde{e})\right) \\ &= \lambda ||x||^2 \varphi \circ E\left(m^*m\right) = \lambda ||x||^2 ||\Lambda(m)||^2. \end{aligned}$$

Therefore, $\pi(x)$ extends to a bounded operator on $\mathcal{H}_\psi$ (still denoted by $\pi(x)$). It is routine to see that π is a normal $*$-homomorphism. For example we compute

$$\begin{aligned}\langle\Lambda(m_1),\pi(x^*)\Lambda(m_2)\rangle &= \lambda\langle\Lambda(m_1),\Lambda(\widetilde{E}(x^*m_2\tilde{e})\rangle = \lambda\psi\left(\widetilde{E}(\tilde{e}m_2^*x)m_1\right)\\ &= \lambda\psi\circ\widetilde{E}(\tilde{e}m_2^*xm_1) = \lambda\psi\circ\widetilde{E}(m_2^*xm_1\tilde{e}) = \lambda\psi\left(m_2^*\widetilde{E}(xm_1\tilde{e})\right)\\ &= \langle\pi(x)\Lambda(m_1),\Lambda(m_2)\rangle,\end{aligned}$$

where the fourth equation follows from the fact that $\tilde{e}$ is in the centralizer of $E\circ\widetilde{E}$.

The image $\pi(\widetilde{\mathcal{L}})$ is a von Neumann algebra acting on $\mathcal{H}_\psi$. We claim that π is injective. To see this, let us assume $\pi(x)=0$. As was seen in the first paragraph we have $\sum_{i=1}^\infty v_ie_{\mathcal{M}}v_i^*=1$ with $v_i=\lambda\widetilde{E}(\tilde{v}_i\tilde{e})\in\mathcal{M}$. Since Λ is injective, by hitting $\pi(x)$ to the vector $\Lambda(v_i)$ we see $\widetilde{E}(xv_i\tilde{e})=0$ and hence $xv_i\tilde{e}v_i^*=\lambda\widetilde{E}(xv_i\tilde{e})\tilde{e}v_i=0$. Thus, we get $x=0$ by summing up over i.

For $x\in\mathcal{M}$ and $e_{\mathcal{N}}$ we compute

$$\begin{aligned}\pi(x)\Lambda(m) &= \lambda\Lambda(\widetilde{E}(xm\tilde{e})) = \lambda\Lambda(xm\widetilde{E}(\tilde{e})) = \Lambda(xm) = \pi_\psi(x)\Lambda(m),\\ \pi(\tilde{e})\Lambda(m) &= \lambda\Lambda(\widetilde{E}(\tilde{e}m\tilde{e})) = \lambda\Lambda(\widetilde{E}(E(m)\tilde{e})) = \Lambda(E(m)) = e_{\mathcal{N}}\Lambda(m),\end{aligned}$$

where $\pi_\psi(x)\cong x$ denotes the GNS representation. Since $\mathcal{M}$ and $\tilde{e}$ generate $\widetilde{\mathcal{L}}$, we have $\pi(\tilde{\mathcal{L}})=\langle\mathcal{M},e_{\mathcal{N}}\rangle''=\mathcal{M}_1$.

We claim $\widetilde{E}(\cdot)=\lambda^{-1}\pi^{-1}(JE^{-1}(J\pi(\cdot)J)J)$. By the discussion in the first paragraph, it suffices to check the both sides against $x=a\tilde{e}a^*$ $(a\in\mathcal{M})$. But, the both sides obviously give us $\lambda^{-1}aa^*$. Thus, we conclude $\pi\circ\widetilde{E}\circ\pi^{-1}=\lambda^{-1}JE^{-1}(J\cdot J)J$. $\square$

As a typical application of the above characterization, we consider the compatibility of taking crossed products and the basic construction. Let $\mathcal{M}\supseteq\mathcal{N}$ be a factor-subfactor pair with a normal conditional expectation $E:\mathcal{M}\longrightarrow\mathcal{N}$ $(Ind\ E<\infty)$. Assume that $\alpha: G\longrightarrow Aut(\mathcal{M})$ is an action with $\alpha_g(\mathcal{N})=\mathcal{N}$ (i.e., $\alpha_g\in Aut(\mathcal{M};\mathcal{N})$) and $\alpha_g\circ E=E\circ\alpha_g$. Note that the second requirement is automatic if $\mathcal{M}\cap\mathcal{N}'=\mathbf{C}I$ or more generally if E is minimal since $\alpha_g\circ E\circ(\alpha_g)^{-1}$ is minimal. This condition is also automatic when $\mathcal{M}\supseteq\mathcal{N}$ are type II_1 and E comes from the unique trace. We set $\mathcal{M}\rtimes_\alpha G\supseteq\mathcal{N}\rtimes_\alpha G$. Note that E extends to $\tilde{E}$ from $\mathcal{M}\rtimes_\alpha G$ onto $\mathcal{N}\rtimes_\alpha G$.

Let $e_{\mathcal{N}}$ be the Jones projection, and u_g be the canonical implementation of α_g. We compute

$$u_g e_{\mathcal{N}} u_g^* x\xi_\varphi = u_g e_{\mathcal{N}} \alpha_g^{-1}(x)\xi_\varphi = u_g E(\alpha_g^{-1}(x))\xi_\varphi$$
$$= u_g \alpha_g^{-1}(E(x))\xi_\varphi = E(x)\xi_\varphi = e_{\mathcal{N}} x\xi_\varphi.$$

($\varphi \circ \alpha_g = \varphi = \omega_{\xi_\varphi}$) so that we have $u_g e_{\mathcal{N}} u_g^* = e_{\mathcal{N}}$. Therefore, Adu_g gives rise to an action of $\mathcal{M}_1 = \langle \mathcal{M}, e_{\mathcal{N}} \rangle''$, which is denoted by α_g again. Note that this is an extension of the original action α_g (on $\mathcal{M}$) and $\alpha_g(e_{\mathcal{N}}) = e_{\mathcal{N}}$. We have $e_{\mathcal{N}} = \pi_\alpha(e_{\mathcal{N}}) \in \mathcal{M}_1 \rtimes G$. Let $E_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be the basic extension. Since $E_1(e_{\mathcal{N}}) = (Ind\ E)^{-1} I$ (see (3.5)) and $\alpha_g(e_{\mathcal{N}}) = e_{\mathcal{N}}$, we see $E_1 \circ \alpha_g = \alpha_g \circ E_1$ and we get the extension $\tilde{E}_1 :\ \mathcal{M}_1 \rtimes G \longrightarrow \mathcal{M} \rtimes G$.

Proposition 3.13. *The above $\mathcal{M}_1 \rtimes_\alpha G$ is the basic extension of $\mathcal{M} \rtimes_\alpha G \supseteq \mathcal{N} \rtimes_\alpha G$. Moreover, $\tilde{E}_1$ is the basic extension of $\tilde{E} :\ \mathcal{M} \rtimes_\alpha G \longrightarrow \mathcal{N} \rtimes_\alpha G$.*

Proof. The central support of $e_{\mathcal{N}}$ in $\mathcal{M}_1$ is I, and hence that of $e_{\mathcal{N}} = \pi_\alpha(e_{\mathcal{N}})$ as an element in $\mathcal{M}_1 \rtimes_\alpha G$ is also I. We easily check all the conditions in Theorem 3.12 for the triple $\langle \mathcal{M}_1 \rtimes_\alpha G, e_{\mathcal{N}}, \tilde{E}_1 \rangle$ based on the fact the same is true for $\mathcal{M}_1, e_{\mathcal{N}}$ and E_1. For example we check the second condition $(Ind\ E)\tilde{E}_1(xe_{\mathcal{N}})e_{\mathcal{N}} = xe_{\mathcal{N}}$ for $x \in \mathcal{M}_1 \rtimes_\alpha$G. By the continuity, we may and do assume that x is of the form $\sum_g x_g \lambda_g$ (finite sum with $x_g \in \mathcal{M}_1$). Since $\lambda_g e_{\mathcal{N}} \lambda_g^* = \alpha_g(e_{\mathcal{N}}) = e_{\mathcal{N}}$, we get

$$xe_{\mathcal{N}} = \sum_g x_g \lambda_g e_{\mathcal{N}} = \sum_g x_g \lambda_g e_{\mathcal{N}} \lambda_g^* \lambda_g = \sum_g x_g e_{\mathcal{N}} \lambda_g.$$

Note $\tilde{E}_1$ acts on the above "coefficient-wise". Therefore, by applying $(Ind\ E)\tilde{E}_1$, we get

$$(Ind\ E)\tilde{E}_1(xe_{\mathcal{N}}) = (Ind\ E)\tilde{E}_1(\sum_g x_g e_{\mathcal{N}} \lambda_g) = \sum_g (Ind\ E)E_1(x_g e_{\mathcal{N}})\lambda_g = \sum_g x_g e_{\mathcal{N}} \lambda_g,$$

which is exactly $xe_{\mathcal{N}}$. □

The original inclusion $\mathcal{M} \supseteq \mathcal{N}$ and $\mathcal{M} \rtimes_\alpha G \supseteq \mathcal{N} \rtimes_\alpha G$ are sometimes quite different. In fact, they may have completely different invariants (for inclusions), and the simultaneous crossed product is sometimes used to get new inclusions (called the orbifold construction). Very thorough investigation on this construction was made in [10] (see also [17] for the non AFD case).

4. Factor-subfactor pairs arising from group-subgroup pairs

Here we consider factor-subfactor pairs arising from group-subgroup pairs to see many concrete examples of principal and dual principal graphs. The discussions here are common for all (non-type I) factors.

Let $\mathcal{P}$ be a factor equipped with an outer action α of a finite group G. For a subgroup H we get the factor-subfactor pair

$$\mathcal{M} = \mathcal{P} \rtimes_\alpha G \supseteq \mathcal{N} = \mathcal{P} \rtimes_\alpha H$$

by looking at the crossed products. We have $\mathcal{M} \cap \mathcal{N}' \subseteq \mathcal{M} \cap \mathcal{P}' = \mathbf{C}I$ and $Ind\ E = \sharp(G)/\sharp(H)$, and furthermore the unique conditional expectation is given by

$$E(\sum_{g \in G} x_g \lambda_g) = \sum_{h \in H} x_h \lambda_h \quad (x_g \in \mathcal{P}).$$

Set $H_0 = \{h \in H;\ ghg^{-1} \in H \quad \text{for all } g \in G\}$, which is the maximal normal subgroup of H. If $H_0 \neq \{e\}$, then H_0 is normal in G and it is easy to see that the pair $\mathcal{M} \supseteq \mathcal{N}$ is isomorphic to the pair

$$(\mathcal{P} \rtimes H_0) \rtimes (H/H_0) \supseteq (\mathcal{P} \rtimes H_0) \rtimes (G/H_0).$$

By considering H/H_0 instead of H, we may assume that $H_0 = \{e\}$.

We will compute the principal and dual principal graphs of $\mathcal{M} \supseteq \mathcal{N}$ in this section. Our argument here are very direct, and the (better in some sense) proof based on bimodules can be found in [45].

4.1. Basic Extensions. For $\xi \in \ell^2(G/H)$ we set

$$(\rho(g)\xi)(g'H) = \xi(g^{-1}g'H).$$

Then, $g \mapsto \rho(g)$ is a unitary representation, and $\alpha_g \otimes Ad\rho(g)$ gives rise to a G-action on $\mathcal{P} \otimes \mathcal{B}(\ell^2(G/H))$. We imbed $\ell^\infty(G/H)$ into $\mathcal{B}(\ell^2(G/H))$ as diagonal matrices, i.e., $f \in \ell^\infty(G/H)$ is identified with the corresponding multiplication operator $m_f \in \mathcal{B}(\ell^2(G/N))$. For $m_f \in \ell^\infty(G/H)$ (or more precisely $f \in \ell^\infty(G/H)$), we compute

$$\begin{aligned}(\rho(g)m_f\rho(g)^*\xi)(g'H) &= (m_f\rho(g)^*\xi)(g^{-1}g'H)\\ &= f(g^{-1}g'H)(\rho(g)^*\xi)(g^{-1}g'H)\\ &= f(g^{-1}g'H)\xi(g'H) = (m_{f(g^{-1}\cdot)}\xi)(g'H),\end{aligned}$$

and hence $Ad\rho(g)$ leaves $\ell^\infty(G/H)$ invariant.

We set

$$\mathcal{M}_1 = (\mathcal{P} \otimes \ell^\infty(G/H)) \rtimes_{\alpha\otimes Ad\rho} G,$$

the crossed product. Then $\mathcal{M}_1$ is a factor since the action $\alpha_g \otimes Ad\rho(g)$ is outer on $\mathcal{P} \otimes \ell^\infty(G/H)$ and free on the center $\mathcal{Z}(\mathcal{P} \otimes \ell^\infty(G/H)) = \ell^\infty(G/H)$. The map $E_1(\sum_g x_g\lambda_g) = \sum_g y_g\lambda_g$ determines a normal conditional expectation $E_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$, where $y_g = \frac{1}{\#(G/H)}\sum_{g'H\in G/H} x_g(g'H)$ is the average of $x_g \in \mathcal{P} \otimes \ell^\infty(G/H)$. We set $e = \pi(1 \otimes \delta_H) \in \mathcal{M}_1$, and note that δ_H ($\cong m_{\delta_H}$) corresponds to the matrix $e_{H,H}$.

Lemma 4.1. *$\mathcal{M}_1$ is the basic extension of $\mathcal{M} \supseteq \mathcal{N}$ and e is the Jones projection.*

Proof. We check the conditions in Theorem 3.12. At first we note

$$E_1(e) = \frac{1}{\#(G/H)}(0 + \cdots + 0 + 1 + 0 + \cdots + 0) = \frac{1}{\#(G/H)}I.$$

Suppose $x = \sum x_{g\in G}x_g\lambda_g$ belongs to $\mathcal{M}$. Then, x_g is an element in $\mathcal{P} \cong \mathcal{P} \otimes I \hookrightarrow \mathcal{P} \otimes \ell^\infty(G/H)$ and we have $E(x) = \sum_{h\in H} x_h\lambda_h$. Since $(Ad\rho(h))(\delta_H) = \delta_H$, we have $[\lambda_h, e] = 0$ and

$$E(x)e = \sum_h x_h e\lambda_h = \sum_h (x_h \otimes I)(I \otimes \delta_H)\lambda_h = \sum_h (x_h \otimes \delta_H)\lambda_h.$$

On the other hand, we have

$$exe = \sum_g ex_g\lambda_g e\lambda_g^*\lambda_g = \sum_g (x_g \otimes \delta_H)\lambda_g e\lambda_g^*\lambda_g.$$

Since $\lambda_g e\lambda_g^* = I \otimes \delta_{gH}$, we get

$$exe = \sum_g (x_g \otimes \delta_H\delta_{gH})\lambda_g = \sum_h (x_h \otimes \delta_H)\lambda_h,$$

and hence $exe = E(x)e$ for $x \in \mathcal{M}$.

Let $x = \sum_g x_g\lambda_g \in \mathcal{M}_1$. Then, each x_g is an element in $\mathcal{P}\otimes\ell^\infty(G/H))$ so that it is a $\mathcal{P}$-valued function on G/H or the diagonal matrix $\sum_{g'H}(x_g(g'H) \otimes \delta_{g'H})$. We note

$$xe = \sum_g x_g\lambda_g e\lambda_g\lambda_g^* = \sum_g x_g(I \otimes \delta_{gH})\lambda_g = \sum_g (x_g(gH) \otimes \delta_{gH})\lambda_g.$$

Hence, we have

$$\#(G/H)E_1(xe)e = \sum_g (x_g(gH) \otimes I)\lambda_g e = \sum_g (x_g(gH) \otimes \delta_{gH})\lambda_g,$$

which means $\#(G/H)E_1(xe)e = xe$ for $x \in \mathcal{M}_1$. $\square$

Set

$$\mathcal{M}_2 = (\mathcal{P} \otimes \mathcal{B}(\ell^2(G/H))) \rtimes_{\alpha\otimes Ad\rho} G.$$

Then $\mathcal{M}_2$ is a factor since $\alpha_g \otimes Ad\rho(g)$ is an outer action on $\mathcal{P} \otimes \mathcal{B}(\ell^2(G/H))$. Let E_0 be the canonical conditional expectation from $\mathcal{B}(\ell^2(G/H))$ onto $\ell^\infty(G/H)$, i.e., the one kills all the off-diagonal components. Then, the formula $E_2(\sum_g x_g\lambda_g) = \sum_g(id_{\mathcal{P}} \otimes E_0)(x_g)\lambda_g$ determines a normal conditional expectation $E_2 : \mathcal{M}_2 \longrightarrow \mathcal{M}_1$. Note that $\mathcal{B}(\ell^2(G/H))$ is regarded as matrices with indices coming from G/H. In the lemma below the matrix

$$\sum_{gH,g'H} e_{gH,g'H} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

plays a crucial role. Namely, what is so special about this matrix is the following rules:

$$\begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \sum_k d_{1k} & \sum_k d_{1k} & \cdots & \sum_k d_{1k} \\ \sum_k d_{2k} & \sum_k d_{2k} & \cdots & \sum_k d_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_k d_{nk} & \sum_k d_{nk} & \cdots & \sum_k d_{nk} \end{pmatrix}$$
$$= \begin{pmatrix} \sum_k d_{1k} & 0 & \cdots & 0 \\ 0 & \sum_k d_{2k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_k d_{nk} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \left(\sum_{i=1}^n d_i\right) \times \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Lemma 4.2. *The factor $\mathcal{M}_2$ is the basic extension of $\mathcal{M}_1 \supseteq \mathcal{M}$ and the Jones projection is given by* $e_1 = \dfrac{1}{\#(G/H)}\left(I \otimes \sum_{gH,g'H} e_{gH,g'H}\right)$.

Proof. We use Theorem 3.12 again. At first $E_2(e_1) = \frac{1}{\sharp(G/H)}I$ is obvious. Assume $x = \sum x_g\lambda_g \in \mathcal{M}_1$. Then, $x_g \in \mathcal{P} \otimes \ell^\infty(G/H)$, and we have $E_1(x) = \sum_g \left(\left(\frac{1}{\sharp(G/H)}\sum_{g'H} x_g(g'H)\right) \otimes I\right)\lambda_g$. Since $\sum_{gH,g'H} e_{gH,g'H}$ and λ_g commute, we get

$$E_1(x)e_1 = \frac{1}{(\#(G/H))^2}\sum_g \left(\left(\sum_{g'H} x_g(g'H)\right) \otimes \left(\sum_{lH,kH} e_{lH,kH}\right)\right)\lambda_g.$$

From the second rule before the lemma it is easy to see that e_1xe_1 gives rise to the same result.

Also one has to check $\#(G/H)E_2(xe_1)e_1 = xe_1$ for $x \in \mathcal{M}_2$. Note that $x = \sum x_g\lambda_g \in \mathcal{M}_1$ with $x_g \in \mathcal{P} \otimes \mathcal{B}(\ell^2(G/H))$, i.e., a full matrix with entries from $\mathcal{P}$. Based on the first rule before the lemma one easily checks the identity, and details are left to the reader. □

Repeated use of the two lemmas obviously gives us the following description of the Jones tower:

$$\begin{aligned}
&\mathcal{N} = \mathcal{P} \rtimes_\alpha H,\\
&\mathcal{M} = \mathcal{P} \rtimes_\alpha G,\\
&\mathcal{M}_1 = (\mathcal{P} \otimes \ell^\infty(G/H)) \rtimes_{\alpha\otimes Ad\rho} G,\\
&\mathcal{M}_2 = (\mathcal{P} \otimes \mathcal{B}(\ell^2(G/H))) \rtimes_{\alpha\otimes Ad\rho} G,\\
&\mathcal{M}_3 = (\mathcal{P} \otimes \mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H)) \rtimes_{\alpha\otimes Ad\rho\otimes Ad\rho} G,\\
&\mathcal{M}_4 = (\mathcal{P} \otimes \mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}(\ell^2(G/H))) \rtimes_{\alpha\otimes Ad\rho\otimes Ad\rho} G,\\
&\cdots
\end{aligned}$$

with product actions $\alpha \otimes Ad\rho \otimes \cdots \otimes Ad\rho$ of G. Also note that imbeddings are specified by

$$\begin{aligned}
&\underbrace{\mathcal{B}(\ell^2(G/H)) \otimes \cdots \otimes \mathcal{B}(\ell^2(G/H))}_{n\text{-times}}\\
&\qquad \hookrightarrow \underbrace{\mathcal{B}(\ell^2(G/H)) \otimes \cdots \otimes \mathcal{B}(\ell^2(G/H))}_{n\text{-times}} \otimes \ell^\infty(G/H)
\end{aligned}$$

with $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \otimes \cdots \otimes x_n \otimes I$, and

$$\underbrace{\mathcal{B}(\ell^2(G/H)) \otimes \cdots \otimes \mathcal{B}(\ell^2(G/H))}_{n\text{-times}} \otimes \ell^\infty(G/H) \hookrightarrow \underbrace{\mathcal{B}(\ell^2(G/H))) \otimes \cdots \otimes \mathcal{B}(\ell^2(G/H))}_{(n+1)\text{-times}}$$

as diagonal matrices. It is of course possible to write down successive Jones projections.

Let us compute the relative commutants, and we begin with

$$\mathcal{M}_1 \cap \mathcal{N}' = \{(\mathcal{P} \otimes \ell^\infty(G/H)) \rtimes_{\alpha \otimes Ad\rho} G\} \cap \{\mathcal{P} \rtimes_\alpha H\}'.$$

An element $x = \sum x_g \lambda_g \in \mathcal{M}_1$ ($x_g \in \mathcal{P} \otimes \ell^\infty(G/H)$) belongs to $\in \mathcal{M}_1 \cap \mathcal{P}'$ if and only if

$$yx_g = x_g(\alpha_g \otimes Ad\rho(g))(y) \quad \text{for all } y \in \mathcal{P} \text{ and } g \in G. \tag{4.1}$$

Note that everything here is a ($\mathcal{P}$-valued) diagonal matrix and moreover $y \in \mathcal{P} \cong \mathcal{P} \otimes I$ is constant along the diagonal. For $g \neq e$, comparing $(g'H, g'H)$-components of the both sides, we get $yx_g(g'H) = x_g(g'H)\alpha_g(y)$. Since α_g is outer, we must have $x_g(g'H) = 0$. This is true for each $g'H$ so that we get $x_g = 0$. On the other hand, for $g = e$ we must have $x_e(g'H) \in \mathcal{P} \cap \mathcal{P}' = \mathbf{C}I$. Therefore, we conclude $\mathcal{M}_1 \cap \mathcal{P}' = \ell^\infty(G/H)$ and hence $\mathcal{M}_1 \cap \mathcal{N}' = \ell^\infty(G/H)^H$, the fixed-point algebra under the H-action $Ad\rho(h)$.

Next we compute

$$\mathcal{M}_2 \cap \mathcal{N}' = \{(\mathcal{P} \otimes \mathcal{B}(\ell^2(G/H))) \rtimes_{\alpha \otimes Ad\rho} G\} \cap \{\mathcal{P} \rtimes_\alpha H\}'.$$

Let $x = \sum x_g \lambda_g \in \mathcal{M}_2$, and as before we begin with the computation of $\mathcal{M}_2 \cap \mathcal{P}'$. The requirement is (4.1), but notice $x_g \in \mathcal{P} \otimes \mathcal{B}(\ell^2(G/H))$ this time, i.e., x_g is a full matrix. Based on the outerness as before we observe that the (g_1H, g_2H)-component $x_g(g_1H, g_2H)$ can be non-zero only when $g = e$ and $x_e(g_1H, g_2H)$ must be a scalar. Therefore, we get $\mathcal{M}_2 \cap \mathcal{P}' = \mathcal{B}(\ell^2(G/H))$ and hence $\mathcal{M}_2 \cap \mathcal{N}' = \mathcal{B}(\ell^2(G/H))^H$.

Inductively, the algebras $\mathcal{M}_3 \cap \mathcal{N}'$, $\mathcal{M}_4 \cap \mathcal{N}'$, $\mathcal{M}_5 \cap \mathcal{N}'$, $\cdots$ can be computed. The similar arguments also give us description of the algebras $\{\mathcal{M}_k \cap \mathcal{M}'\}_k$.

Theorem 4.3. *With the imbeddings described before, we have*

$$\begin{aligned}
&\mathcal{M}_1 \cap \mathcal{N}' = (\ell^\infty(G/H))^H, \\
&\mathcal{M}_2 \cap \mathcal{N}' = (\mathcal{B}(\ell^2(G/H)))^H, \\
&\mathcal{M}_3 \cap \mathcal{N}' = (\mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H))^H, \\
&\mathcal{M}_4 \cap \mathcal{N}' = (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}(\ell^2(G/H)))^H, \\
&\mathcal{M}_5 \cap \mathcal{N}' = (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}((\ell^2(G/H) \otimes \ell^\infty(G/H))^H, \\
&\cdots,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{M}_2 \cap \mathcal{M}' = (\mathcal{B}(\ell^2(G/H)))^G, \\
&\mathcal{M}_3 \cap \mathcal{M}' = (\mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H))^G, \\
&\mathcal{M}_4 \cap \mathcal{M}' = (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}(\ell^2(G/H)))^G, \\
&\mathcal{M}_5 \cap \mathcal{M}' = (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}((\ell^2(G/H)) \otimes \ell^\infty(G/H))^G, \\
&\cdots.
\end{aligned}$$

Here, the action (of H or G) is given by $Ad(\rho(g) \otimes \rho(g) \otimes \cdots \otimes \rho(g))$.

4.2. Principal and Dual Principal Graphs. We begin with the computation of the dual principal graph, which is much easier. The tower $\mathcal{M}_1 \cap \mathcal{M}' \subseteq \mathcal{M}_2 \cap \mathcal{M}' \subseteq \mathcal{M}_3 \cap \mathcal{M}' \subseteq \cdots$ is determined by the inclusions

$$\begin{aligned}
(\mathcal{B}(\ell^2(G/H)))^G &\subseteq (\mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H))^G \\
&\subseteq (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}(\ell^2(G/H)))^G \subseteq \cdots .
\end{aligned} \tag{4.2}$$

Since the G-action here is transitive, we have the following:

Lemma 4.4.

$$(\mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H))^G \cong \mathcal{B}(\ell^2(G/H))^H.$$

Proof. Let

$$\Phi \ : \ (\mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H))^G \longrightarrow \mathcal{B}(\ell^2(G/H))^H$$

be the map defined by $\sum x_{gH} \otimes \delta_{gH} \mapsto x_H$. (Note that since the H-action does not move δ_H we observe $x_H \in \mathcal{B}(\ell^2(G/H))^H$.) Then this gives the required isomorphism. In fact, when $x \in \mathcal{B}(\ell^2(G/H))^H$, we have $(Ad\rho(gh))(x) = (Ad\rho(g))(x)$. Therefore, the map Φ^{-1} defined by $\Phi^{-1}(x) = \sum_{gH}(Ad\rho(g))(x) \otimes \delta_{gH}$ is well-defined. It is easy to see that Φ^{-1} is indeed the inverse of Φ. $\square$

It follows from Lemma 4.4 that the first two steps of the inclusions (4.2) become

$$\mathcal{B}(\ell^2(G/H))^G \subseteq \mathcal{B}(\ell^2(G/H))^H \subseteq (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}(\ell^2(G/H)))^G.$$

The first imbedding is the natural one as a subset. In fact, we have

$$\begin{array}{ccccc} \mathcal{B}(\ell^2(G/H))^G & \subseteq & (\mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H))^G & \cong & \mathcal{B}(\ell^2(G/H))^H, \\ x & \hookrightarrow & x \otimes I = \sum_{gH} x \otimes \delta_{gH} & \stackrel{\Phi}{\mapsto} & x. \end{array}$$

The above second imbedding is given by

$$\begin{array}{ccccc} \mathcal{B}(\ell^2(G/H))^H & \cong & (\mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H))^G & \subseteq & (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}(\ell^2(G/H)))^G \\ x & \stackrel{\Phi^{-1}}{\mapsto} & \sum_{gH}(Ad\rho(g))(x) \otimes \delta_{gH} & \hookrightarrow & \sum_{gH}(Ad\rho(g))(x) \otimes e_{gH,gH}. \end{array}$$

Note that the fixed-point algebra $\mathcal{B}(\ell^2(G/H))^G$ is the algebra of self-intertwiners for the representation ρ (on $\ell^2(G/H)$) of G. Hence, its algebra structure determines the irreducible decomposition. Note that $\mathcal{B}(\ell^2(G/H))^H$ is the algebra of self-intertwiners of $\rho|_H$, and the above first imbedding (i.e., the natural one) corresponds to $\pi \in \hat{G} \to Rest_H^G\pi$ at the level of representations. On the other hand, the second imbedding corresponds to $\pi \in \hat{H} \to Ind_H^G\pi$ (see standard textbooks for details on induced representations). Hence the dual principal graph is determined by

$$\begin{cases} \hat{G} & \stackrel{ind}{\longleftarrow} & \hat{H}, \\ \hat{G} & \stackrel{rest}{\longrightarrow} & \hat{H}. \end{cases}$$

(The starting point (corresponding to $\mathcal{M} \cap \mathcal{M}' = \mathbf{C}I$) is the identity representation of G.) The assumption $H_0 = \{e\}$ at the beginning of the section implies that all the irreducible representations appear in the n-fold tensor product $\rho \otimes \rho \otimes \cdots \otimes \rho$ (for n large enough). Therefore, the above induction and restriction procedures exhaust all the irreducible representations.

We next go to the principal graph. From the relative commutants $\mathcal{M} \cap \mathcal{N}' \subseteq \mathcal{M}_1 \cap \mathcal{N}' \subseteq \mathcal{M}_2 \cap \mathcal{N}' \subseteq \cdots$ we get

$$\begin{aligned} \mathcal{B}(\ell^2(G/H))^H &\subseteq (\mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H))^H \\ &\subseteq (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}(\ell^2(G/H)))^H \subseteq \cdots . \end{aligned} \tag{4.3}$$

The H-action on $\ell^\infty(G/H)$ is not transitive here, and let

$$\mathcal{O}_1 = \{H\}, \mathcal{O}_2 = \{gH, g'H, \cdots\}, \cdots, \mathcal{O}_n$$

be the orbits of the H-action on G/H (i.e., we consider the double cosets). Choose and fix a representative $g_iH \in \mathcal{O}_i$ for each i ($g_1 = e$). The isotropy subgroups of g_iH is

$$H_i = \{h \in H;\ hg_iH = g_iH\} = g_iHg_i^{-1} \cap H.$$

(Note $H_1 = H$ and $H/H_i = \mathcal{O}_i$.)

Lemma 4.5. *The map*

$$\Phi :\ (\mathcal{B}(\ell^2(G/H)) \otimes \ell^\infty(G/H))^H \longrightarrow \sum_{i=1}^{n}{}^{\oplus}\mathcal{B}(\ell^2(G/H))^{H_i}$$

defined by $\Phi\left(\sum_{gH} x_{gH} \otimes \delta_{gH}\right) = \sum_{i=1}^{n}{}^{\oplus} x_{g_iH}$ *gives rise to an isomorphism.*

Proof. At first we note $x_{g_iH} \in \mathcal{B}(\ell^2(G/H))^{H_i}$ so that the above map makes sense. Let $x = \sum_{i=1}^{n}{}^{\oplus} x_i$ with $x_i \in \mathcal{B}(\ell^2(G/H))^{H_i}$. When $gH \in \mathcal{O}_i$ we can choose $h \in H$ satisfying $gH = hg_iH$ and we set $x_{gH} = Ad\rho(h)(x_i)$. Note that h is not unique but ambiguity comes from the isotropy subgroup H_i. Hence, the H_i-invariance of x_i shows that the above x_{gH} is well-defined. We easily observe that $\sum_{gH}{}^{\oplus} x_{gH} \otimes \delta_{gH}$ is invariant under the H-action and that the map assigning this to x is obviously the inverse of Φ. $\square$

Via the above isomorphism the first two steps of the inclusions (4.3) become

$$\mathcal{B}(\ell^2(G/H))^H \subseteq \sum_{i=1}^{n}{}^{\oplus}\mathcal{B}(\ell^2(G/H))^{H_i} \subseteq (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}(\ell^2(G/H)))^H.$$

The first imbedding $(\mathcal{B}(\ell^2(G/H))^H \subseteq \sum_{i=1}^{n}{}^{\oplus}\ \mathcal{B}(\ell^2(G/H))^{H_i}$ is given by $x \hookrightarrow \sum_{i=1}^{n}{}^{\oplus}\ x$ while the second imbedding $\sum_{i=1}^{n}{}^{\oplus}\ \mathcal{B}(\ell^2(G/H))^{H_i} \subseteq (\mathcal{B}(\ell^2(G/H)) \otimes \mathcal{B}(\ell^2(G/H))^H$ is given by $\sum_{i=1}^{n}{}^{\oplus}\ x_i \hookrightarrow \sum_{gH} x_{gH} \otimes e_{gH,gH}$. Here, in the second imbedding x_{gH}'s are determined as in the proof of Lemma 4.5.

Summing up the discussions so far, we see that the principal graph is described by

$$\begin{cases} \hat{H} & \stackrel{ind}{\longleftarrow} & \amalg_{i=1}^{n} \hat{H}_i, \\ \hat{H} & \stackrel{rest}{\longrightarrow} & \amalg_{i=1}^{n} \hat{H}_i. \end{cases}$$

(The starting point (corresponding to $\mathcal{N} \cap \mathcal{N}' = \mathbf{C}I$) is the identity representation of $\hat{H}_1 = \hat{H}$)

Example 4.6. Let $\mathcal{M} = \mathcal{P} \rtimes \mathfrak{S}_n \supseteq \mathcal{N} = \mathcal{P} \rtimes \mathfrak{S}_{n-1}$ with the symmetric group $\mathfrak{S}_n$ ($n \geq 3$) and the subgroup

$$\mathfrak{S}_{n-1} = \{\sigma \in \mathfrak{S}_n;\ \sigma(n) = n\}.$$

Notice $\mathfrak{S}_n/\mathfrak{S}_{n-1} = \{1, 2, \cdots, n\}$ on which $\mathfrak{S}_{n-1}$ acts. Since $\mathfrak{S}_{n-1}$ shuffles the first $n-1$ points and fixes the last n, we have two orbits: the trivial one $\mathcal{O}_1 = \mathfrak{S}_{n-1}$ and the other $\mathcal{O}_2$. Take $g_1 = e$ and $g_2 = (n, n-1)$ (transposition) as representatives, and we observe

$$H_1 = \mathfrak{S}_{n-1} \quad \text{and} \quad H_2 = g_2 \mathfrak{S}_{n-1} g_2^{-1} \cap \mathfrak{S}_{n-1} = \mathfrak{S}_{n-2}$$

with $\mathfrak{S}_{n-2} = \{\sigma \in \mathfrak{S}_n;\ \sigma(n) = n \text{ and } \sigma(n-1) = n-1\}$. Therefore, the vertices of the principal graph are parameterized by $\hat{S}_{n-1} \amalg \hat{S}_{n-2}$ (even levels) and $\hat{S}_{n-1}$ (odd levels) while those of the dual principal graph are parameterized by $\hat{S}_n$ (even levels) and $\hat{S}_{n-1}$ (odd levels). For the pair $\mathcal{P} \rtimes \mathfrak{A}_n \supseteq \mathcal{P} \rtimes \mathfrak{A}_{n-1}$ arising from the alternating groups $\mathfrak{A}_n$, we can do the same thing. For $n \geq 4$ we have two orbits (when $n = 3$ we have $\mathfrak{A}_2 = \{e\}$), and the two isotropy subgroups are $\mathfrak{A}_{n-1}$ and $\mathfrak{A}_{n-2}$ again. For $\mathfrak{A}_n$ and $\mathfrak{S}_n$ irreducible representations are parameterized by Young tables, and the irreducible decomposition of their induction and restriction is described by the branching rule (for Young tables). For $n = 3, 4$ the principal and dual principal graphs coincide. However, for $n = 5$ they are different (for both of $\mathfrak{S}_n$ and $\mathfrak{A}_n$).

Figure 3 (the graphs for $\mathfrak{S}_4/\mathfrak{S}_3$)

Figure 4 (the graphs for $\mathfrak{A}_4/\mathfrak{A}_3$)

Figure 5 (the graphs for $\mathfrak{S}_5/\mathfrak{S}_4$)

Figure 6 (the graphs for $\mathfrak{A}_5/\mathfrak{A}_4$)

5. Inclusions of type III factors

Here we analyze structure of inclusions of type III factors (of finite indices). When a pair $\mathcal{M} \supseteq \mathcal{N}$ of type II_1 factors has finite index, $\mathcal{M}$ and $\mathcal{N}$ possess many properties in common (see [56]). For example $\mathcal{N}$ is AFD if and only if $\mathcal{M}$ is, and hence in this case $\mathcal{M}$ and $\mathcal{N}$ are isomorphic as factors due to the uniqueness of the hyperfinite II_1 factor. On the other hand, in the (AFD) type III setting the assumption of finite index does not guarantee that involved factors are isomorphic (see Example 5.5 for most typical examples). Therefore, when analyzing a pair of type III factors, we should begin by investigating how close the III_λ-types ($0 \leq \lambda \leq 1$) of involved factors are. This can be done by comparing flows of weights.

5.1. Stability of III_λ Types. Assume that $\mathcal{M} \supseteq \mathcal{N}$ are type III factors and $E : \mathcal{M} \longrightarrow \mathcal{N}$ is a normal conditional expectation with $Ind\ E < \infty$. Let $E_1 : \mathcal{M}_1 \to \mathcal{M}$ be the basic extension of E. Since $\sigma_t^{\varphi\circ E} \mid_{\mathcal{N}} = \sigma_t^{\varphi}$ and $\sigma_t^{\varphi\circ E\circ E_1} \mid_{\mathcal{M}} = \sigma_t^{\varphi\circ E}$, we can consider the inclusions

$$\widetilde{\mathcal{M}_1} = \mathcal{M}_1 \rtimes_{\sigma^{\varphi\circ E\circ E_1}} \mathbf{R} \supseteq \widetilde{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma^{\varphi\circ E}} \mathbf{R} \supseteq \widetilde{\mathcal{N}} = \mathcal{N} \rtimes_{\sigma^{\varphi}} \mathbf{R}$$

of crossed products (which are algebras of type II_∞). Recall that dual actions and dual weights, etc. are around for these algebras. From their definitions we easily see the compatibility:

$$\begin{aligned} &\theta_t^{\tilde{\mathcal{M}}_1} \mid_{\tilde{\mathcal{M}}} = \theta_t^{\tilde{\mathcal{M}}} \quad \text{and} \quad \widehat{\varphi \circ E \circ E_1} \mid_{\tilde{\mathcal{M}}} = \widehat{\varphi \circ E}, \\ &\theta_t^{\tilde{\mathcal{M}}} \mid_{\tilde{\mathcal{N}}} = \theta_t^{\tilde{\mathcal{N}}} \quad \text{and} \quad \widehat{\varphi \circ E} \mid_{\tilde{\mathcal{N}}} = \hat{\varphi}. \end{aligned}$$

In particular, the dual weights $\widehat{\varphi \circ E \circ E_1}, \widehat{\varphi \circ E}$ give rise to the natural conditional expectations $\widetilde{E_1}$ and $\widetilde{E}$ respectively. It is easy to see

$$\widetilde{E}\left(\int_{-\infty}^{\infty} x(t)\lambda(t)dt\right) = \int_{-\infty}^{\infty} E(x(t))\lambda(t)dt \quad \text{for} \quad \int_{-\infty}^{\infty} x(t)\lambda(t)dt \in \widetilde{\mathcal{M}}$$

and the similar expression for $\widetilde{E}_1$ (think about the projection from $L^2(\widetilde{\mathcal{M}})$ onto $L^2(\widetilde{\mathcal{N}})$). It is important to notice (from Proposition 3.13) that $\widetilde{\mathcal{M}}_1$ and $\widetilde{E}_1$ are the basic extensions of $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{N}}$ and $\widetilde{E} : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{N}}$. Note that Theorem 3.12 says $\widetilde{E}^{-1}(I) = (Ind\ E)I$. Moreover, the compatibility of the canonical traces is also valid

$$tr_{\widetilde{\mathcal{M}}} \mid_{\widetilde{\mathcal{N}}} = tr_{\widetilde{\mathcal{N}}},$$

since the dual weights are compatible and the Radon-Nikodym derivative H between the dual weight and the trace (recall (2.2)) is affiliated with $\mathcal{N}$. Also we may consider $\widetilde{E}, \widetilde{E}_1$ as the ones arising from the canonical traces because of

$$tr_{\widetilde{\mathcal{M}}} \circ \widetilde{E} = \widehat{\varphi \circ E}(H^{-1}\widetilde{E}(\cdot)) = \widehat{\varphi}(H^{-1}\widetilde{E}(\cdot)) = tr_{\widetilde{\mathcal{N}}} \circ E.$$

From $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{N}}$, we have

$$\widetilde{E} \mid_{\tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}'} \colon \ \widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}' \longrightarrow \mathcal{Z}(\widetilde{\mathcal{N}})$$

by passing to the relative commutant, and moreover the Pimsner-Popa inequality (Theorem 3.8)

$$\widetilde{E}(x) \geq (Ind\ E)^{-1}x \quad \text{for} \quad x \in (\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')_+$$

is valid. Notice

$$\mathcal{Z}(\widetilde{\mathcal{N}}) = \widetilde{\mathcal{N}} \cap \widetilde{\mathcal{N}}' \subseteq \widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}' \cap (\widetilde{\mathcal{M}}' \vee \widetilde{\mathcal{N}}) = \mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}').$$

Thus, we can consider the further restriction

$$\widetilde{E} \mid_{\mathcal{Z}(\tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}')} \colon \ \mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}') \longrightarrow \mathcal{Z}(\widetilde{\mathcal{N}}),$$

and of course the Pimsner-Popa inequality remains valid. Let $\widetilde{\mathcal{N}} = \int_{X_{\mathcal{N}}}^{\oplus} \widetilde{\mathcal{N}}(\omega) d\mu(\omega)$ be the central decomposition so that we have

$$\mathcal{Z}(\widetilde{\mathcal{N}}) = \int_{X_{\mathcal{N}}}^{\oplus} \mathbf{C} I_\omega d\mu(\omega) \cong L^\infty(X_{\mathcal{N}}).$$

Notice

$$\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}') \subseteq \widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}' \subseteq \widetilde{\mathcal{N}} \vee \widetilde{\mathcal{N}}' = \mathcal{Z}(\mathcal{N})'$$

so that the center $\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')$ of the relative commutant is decomposable over $\mathcal{Z}(\mathcal{N})$ and we have

$$\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}') = \int_{X_{\mathcal{N}}}^{\oplus} \mathcal{A}_\omega d\mu(\omega)$$

with abelian algebras $\mathcal{A}_\omega$.

Let

$$\widetilde{E}|_{\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')} = \int_{X_{\mathcal{N}}}^{\oplus} E_\omega d\mu(\omega)$$

be the disintegration of $\widetilde{E}|_{\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')}$, and notice that $\{E_\omega : \mathcal{A}_\omega \to \mathbf{C}\}_\omega$ is a field of states in the present case. We obviously have $E_\omega(x) \geq (Ind\ E)^{-1}x$ for a.e. $x \in (\mathcal{A}_\omega)_+$. At first we claim $\dim \mathcal{A}_\omega \leq Ind\ E$. In fact, let $\{p_i\}_{i=1,2,\cdots,n}$ be a partition of I consisting of projections in $\mathcal{A}_\omega$. Let $\lambda_i = E_\omega(p_i) > 0$ so that $1 = E_\omega(I) = \sum_{i=1}^n E_\omega(p_i) = \sum_{i=1}^n \lambda_i$. On the other hand, the Pimsner-Popa inequality shows $\lambda_i \geq (Ind\ E)^{-1}$ so that we we get

$$1 = \sum_{i=1}^n \lambda_i \geq n \times (Ind\ E)^{-1},$$

that is, $n \leq Ind\ E$. Secondly, notice that the measurable function $\omega \mapsto \dim \mathcal{A}_\omega$ is invariant under the dual action $\{\theta_s\}_{s\in\mathbf{R}}$ since it commutes with $\widetilde{E}$. Therefore, the central ergodicity implies that $\dim \mathcal{A}_\omega$ is constant, and we conclude

$$\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}') = L^\infty(X_{\mathcal{N}} \times \{1, 2, \cdots, n\})$$

with some $n \leq Ind\ E$. Hence, we have seen that the restriction of the dual action to $\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')$ gives us an n to 1 extension of the flow of weights of $\mathcal{N}$.

Starting from $\widetilde{E_1} : \widetilde{\mathcal{M}_1} \longrightarrow \widetilde{\mathcal{M}}$, we can of course repeat the identical arguments. In this way we get

$$\mathcal{Z}(\widetilde{\mathcal{M}_1} \cap \widetilde{\mathcal{M}}') = L^\infty(X_{\mathcal{M}} \times \{1, 2, \cdots, m\})$$

so that the restriction of the dual action to $\mathcal{Z}(\widetilde{\mathcal{M}_1} \cap \widetilde{\mathcal{M}}')$ gives us an m to 1 extension of the flow of weights of $\mathcal{M}$. On the other hand, $\widetilde{\mathcal{M}_1}$ is the basic extension of $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{N}}$, i.e., $\widetilde{\mathcal{M}_1} = J\widetilde{\mathcal{N}}'J$ with the modular conjugation J associated with $\widetilde{\mathcal{M}}$. The map j defined by $j(x) = Jx^*J$ gives rise to an anti-isomorphism from $\widetilde{\mathcal{M}_1} \cap \widetilde{\mathcal{M}}'$ onto $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}'$, and hence we have the isomorphism between $\mathcal{Z}(\widetilde{\mathcal{M}_1} \cap \widetilde{\mathcal{M}}')$ and $\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')$. Note that J commutes with the unitary defining the dual action (see the description of J in the proof of Lemma 3.6) so that j intertwines the respective dual actions. Therefore, we can identify $\mathcal{Z}(\widetilde{\mathcal{M}_1} \cap \widetilde{\mathcal{M}}')$ and $\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')$ equipped with the respective dual actions, and we have shown

Theorem 5.1. (Hamachi-Kosaki, [23]) *Let $\mathcal{M} \supseteq \mathcal{N}$ be type III factors and we assume that there is a normal conditional expectation $E : \mathcal{M} \longrightarrow \mathcal{N}$ with $Ind\ E < \infty$. Then, the flow $(X_{\mathcal{N}}, F_t^{\mathcal{N}})$ of weights of $\mathcal{N}$ and the flow $(X_{\mathcal{M}}, F_t^{\mathcal{M}})$ of weights of $\mathcal{M}$ admit a common finite-to-one extension (X, F_t):*

$$(X_{\mathcal{N}}, F_t^{\mathcal{N}}) \underset{n\ to\ 1}{\longleftarrow} (X, F_t) \underset{m\ to\ 1}{\longrightarrow} (X_{\mathcal{M}}, F_t^{\mathcal{M}}).$$

Moreover, integers n, m are equal or less than $Ind\ E$.

For example assume that $\mathcal{M}$ is of type III_1, i.e., $X_{\mathcal{M}}$ is a singleton. Then, the theorem says that X and $X_{\mathcal{N}}$ are finite sets. Therefore, the ergodic flow $(X_{\mathcal{N}}, F_s)$ must be the trivial one, i.e., $\mathcal{N}$ is of type III_1. In this way we get the following corollary ([23, 47]):

Corollary 5.2. *Assume $\mathcal{M}, \mathcal{N}$ are of type III_λ, III_μ $(0 \leq \lambda, \mu \leq 1)$ respectively and $Ind\ E < \infty$. Then, we have*
(i) $\lambda = 1$ if and only if $\mu = 1$,
(ii) $\lambda = 0$ if and only if $\mu = 0$,
(iii) if $0 < \lambda, \mu < 1$ then $\mu = \lambda^{\frac{m}{n}}$ with some integers $n, m \leq Ind\ E$.

Note that the only measure <u>class</u> on the flow space $X_{\mathcal{M}}$ is specified (from $\mathcal{Z}(\widetilde{\mathcal{M}}) = L^\infty(X_{\mathcal{M}})$). Fine measure-theoretic structure on this matter and related quantities (such as an entropy) seems to have direct relevance to study on finer structure of the corresponding type III_0 factor. One typical and (supporting) result is: the flow of weights of an AFD type III_0 factor admits a finite invariant measure if and only if the factor contains an irreducible type III_1 subfactor which is the range of a normal conditional expectation (of typically infinite index). This area probably deserves further investigation, and here analysis on subfactors of infinite index might be important. Note that the preceding theorem says (for example) that the flow of weights of $\mathcal{M}$ admits a finite or σ-finite invariant measure if and only if so does the flow of weights of $\mathcal{N}$.

One interesting open problem on stability is the following: Assume that (say) $\mathcal{M}$ is ITPFI. Even in the case of index 2 it is not known if $\mathcal{N}$ is ITPFI.

5.2. Decomposition of Conditional Expectation. In the above Theorem 5.1 the flow (X, F_t) naturally arises as a common finite-to-one extensions of the flows of weights of $\mathcal{M}$ and $\mathcal{N}$. Here, we will construct (two) intermediate algebras whose flow of weights is exactly (X, F_t).

Let $\mathcal{M} \supseteq \mathcal{N}$ be a pair of type III_λ $(\lambda \neq 1)$ factors and $E : \mathcal{M} \longrightarrow \mathcal{N}$ a normal conditional expectation with finite index. Then a normal conditional expectation $\widetilde{E} : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{N}}$ comes from the dual weight (or equivalently from the canonical trace) and satisfies $\widetilde{E} \circ \theta_t = \theta_t \circ \widetilde{E}$. Thus, $\widetilde{E}$ extends to

$$\tilde{\widetilde{E}} : \widetilde{\mathcal{M}} \rtimes_\theta \mathbf{R} \longrightarrow \widetilde{\mathcal{N}} \rtimes_\theta \mathbf{R}.$$

More precisely, $\tilde{\tilde{E}}$ comes from the bidual weight $\hat{\hat{\psi}}$ (with $\psi = \varphi \circ E$). By the Takesaki duality, we have the conjugacy

$$\widetilde{\mathcal{M}} \rtimes_\theta \mathbf{R} \supseteq \widetilde{\mathcal{N}} \rtimes_\theta \mathbf{R} \quad \cong \quad \mathcal{M} \otimes \mathcal{B}(L^2(\mathbf{R})) \supseteq \mathcal{N} \otimes \mathcal{B}(L^2(\mathbf{R})).$$

Via this conjugacy the bidual weight $\hat{\hat{\psi}}$ is known to be transformed to $\psi \otimes Tr_{\mathcal{B}(L^2(\mathbf{R}))}$. The conjugacy sends $\hat{\hat{\varphi}}$ (which is a restriction of $\hat{\hat{\psi}}$) to $\varphi \otimes Tr_{\mathcal{B}(L^2(\mathbf{R}))}$, and hence we see that $\tilde{\tilde{E}}$ is transformed to $E \otimes Id_{\mathcal{B}(L^2(\mathbf{R}))}$.

On the other hand, we claim

$$(\mathcal{M} \otimes \mathcal{B}(L^2(\mathbf{R})), \mathcal{N} \otimes \mathcal{B}(L^2(\mathbf{R})), E \otimes Id_{\mathcal{B}(L^2(\mathbf{R}))}) \quad \cong \quad (\mathcal{M}, \mathcal{N}, E).$$

In fact, from $\mathcal{N}$ one chooses matrix units $\{e_{ij}\}$ and an isometry v with $vv^* = e_{11}$ so that

$$\mathcal{M} \otimes \mathcal{B}(L^2(\mathbf{R})) \cong e_{11}\mathcal{M}e_{11} \otimes \mathcal{B}(L^2(\mathbf{R})) \cong \mathcal{M},$$
$$\mathcal{N} \otimes \mathcal{B}(L^2(\mathbf{R})) \cong e_{11}\mathcal{N}e_{11} \otimes \mathcal{B}(L^2(\mathbf{R})) \cong \mathcal{N}.$$

It is easy to see that these conjugacies send $E \otimes Id_{\mathcal{B}(L^2(\mathbf{R}))}$ to E because e_{ij} and v are taken from $\mathcal{N}$.

Therefore, we may identify $(\mathcal{M} \supseteq \mathcal{N}, E)$ with $(\widetilde{\mathcal{M}} \rtimes_\theta \mathbf{R} \supseteq \widetilde{\mathcal{N}} \rtimes_\theta \mathbf{R}, \tilde{\tilde{E}})$. We set

$$\widetilde{\mathcal{A}} = \widetilde{\mathcal{M}} \cap \mathcal{Z}', \ \widetilde{\mathcal{B}} = \widetilde{\mathcal{N}} \vee \mathcal{Z} \text{ with } \mathcal{Z} = \mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}').$$

We have $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{A}} \supseteq \widetilde{\mathcal{B}} \supseteq \widetilde{\mathcal{N}}$, and it is easy to see $\mathcal{Z}(\widetilde{\mathcal{A}}) = \mathcal{Z}(\widetilde{\mathcal{B}}) = \mathcal{Z}$.

Lemma 5.3. $\widetilde{E} : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{N}}$ *is decomposed into the following three conditional expectations:*

$$\widetilde{\mathcal{M}} \underset{\widetilde{G}}{\rightarrow} \widetilde{\mathcal{A}} \underset{\widetilde{H}}{\rightarrow} \widetilde{\mathcal{B}} \underset{\widetilde{F}}{\rightarrow} \widetilde{\mathcal{N}}.$$

Moreover, each of $\widetilde{G}, \widetilde{H}, \widetilde{F}$ *commutes with the dual action* θ_s.

Proof. Let χ be a faithful normal state on $\widetilde{\mathcal{N}}$. Notice

$$\sigma_t^{\chi\circ\widetilde{E}}|_{\widetilde{\mathcal{N}}} = \sigma_t^{\chi} \quad \text{and} \quad \sigma_t^{\chi\circ\widetilde{E}}|_{\widetilde{\mathcal{M}}\cap\widetilde{\mathcal{N}}'} = \sigma_t^{tr\circ\widetilde{E}}|_{\widetilde{\mathcal{M}}\cap\widetilde{\mathcal{N}}'} = id.$$

Therefore, we get the invariance

$$\sigma_t^{\chi\circ\widetilde{E}}(\widetilde{\mathcal{A}}) = \widetilde{\mathcal{A}} \quad \text{and} \quad \sigma_t^{\chi\circ\widetilde{E}}(\widetilde{\mathcal{B}}) = \widetilde{\mathcal{B}}.$$

The first invariance guarantees the existence of a conditional expectation $\widetilde{G} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{A}}$ (uniquely) determined by $\chi \circ \widetilde{E} = (\chi \circ \widetilde{E})|_{\tilde{\mathcal{A}}} \circ \widetilde{G}$ (and $\sigma_t^{\chi \circ \widetilde{E}|_{\tilde{\mathcal{A}}}} = \sigma_t^{\chi \circ \widetilde{E}}|_{\widetilde{\mathcal{A}}}$). From the second invariance, there is a conditional expectation $\widetilde{H} : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{B}}$ determined by $(\chi \circ \widetilde{E})|_{\tilde{\mathcal{A}}} = (\chi \circ \widetilde{E})|_{\tilde{\mathcal{B}}} \circ \widetilde{H}$. We finally set $\widetilde{F} = \widetilde{E}|_{\tilde{\mathcal{B}}} : \widetilde{\mathcal{B}} \longrightarrow \widetilde{\mathcal{N}}$, which is obviously a conditional expectation. Notice that the composition $\widetilde{F} \circ \widetilde{H} \circ \widetilde{G}$ is a normal conditional expectation from $\widetilde{\mathcal{M}}$ onto $\widetilde{\mathcal{N}}$. Since we have $\chi \circ \widetilde{F} \circ \widetilde{H} \circ \widetilde{G} = \chi \circ \widetilde{E}$ from the construction, we conclude $\widetilde{E} = \widetilde{F} \circ \widetilde{H} \circ \widetilde{G}$ (since $\tilde{E}$ is unique subject to the condition $\chi \circ \widetilde{E} \circ \widetilde{E} = \chi \circ \widetilde{E}$).

Let φ be a normal semi-finite faithful weight $\mathcal{N}$, and we set $\psi = \varphi \circ E$. Notice that the dual weights $\hat{\psi}, \hat{\varphi}$ are related by

$$\hat{\psi} = \hat{\varphi} \circ \widetilde{E} = \hat{\varphi} \circ \widetilde{F} \circ \widetilde{H} \circ \widetilde{G}.$$

This shows that $\hat{\psi}$ is semi-finite on $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ and that $\widetilde{F}, \widetilde{H}, \widetilde{G}$ come from $\hat{\psi}|_{\widetilde{\mathcal{B}}}, \hat{\psi}|_{\widetilde{\mathcal{A}}}, \hat{\psi}$ respectively. We claim $\theta_s \circ \widetilde{F} \circ \theta_{-s} = \widetilde{F}$. Since $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}'$ and its center $\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')$ are invariant under θ_s, so is $\widetilde{\mathcal{A}}$. Therefore, $\theta_s \circ \widetilde{F} \circ \theta_{-s}$ is a conditional expectation from $\widetilde{\mathcal{M}}$ onto $\widetilde{\mathcal{A}}$. We observe

$$\hat{\psi}|_{\widetilde{\mathcal{A}}} \circ \theta_s \circ \widetilde{F} \circ \theta_{-s} = \hat{\psi}|_{\widetilde{\mathcal{A}}} \circ \widetilde{F} \circ \theta_{-s} = \hat{\psi}|_{\tilde{\mathcal{A}}} \circ \theta_{-s} = \hat{\psi}|_{\widetilde{\mathcal{A}}}$$

due to the fact that the dual weight is invariant under the dual action. Therefore, by the uniqueness again, we get $\theta_s \circ \widetilde{F} \circ \theta_{-s} = \widetilde{F}$, that is, $\widetilde{F} \circ \theta_s = \theta_s \circ \widetilde{F}$. The commutativity with $\widetilde{H}$ and $\widetilde{G}$ can be proved in the same way. □

From the lemma we get

$$\widetilde{\mathcal{M}} \rtimes_\theta \mathbf{R} \underset{\tilde{\tilde{G}}}{\longrightarrow} \widetilde{\mathcal{A}} \rtimes_\theta \mathbf{R} \underset{\tilde{\tilde{H}}}{\longrightarrow} \widetilde{\mathcal{B}} \rtimes_\theta \mathbf{R} \underset{\tilde{\tilde{F}}}{\longrightarrow} \widetilde{\mathcal{N}} \rtimes_\theta \mathbf{R}.$$

Note that the composition of the three conditional expectation is $\tilde{\tilde{E}}$. Therefore, from the discussion in the paragraph before the lemma we get the following theorem ([39]):

Theorem 5.4. *Let $\mathcal{M} \supseteq \mathcal{N}$ be type III factors, and $E : \mathcal{M} \longrightarrow \mathcal{N}$ be a conditional expectation with Ind $E < \infty$. Then we can find subalgebras $\mathcal{A}, \mathcal{B}$ such that $\mathcal{M} \supseteq \mathcal{A} \supseteq \mathcal{B} \supseteq \mathcal{N}$. Also $E : \mathcal{M} \longrightarrow \mathcal{N}$ splits into the three conditional expectations*

$$\mathcal{M} \underset{G}{\longrightarrow} \mathcal{A} \underset{H}{\longrightarrow} \mathcal{B} \underset{F}{\longrightarrow} \mathcal{N}.$$

Let us investigate the covariant systems (crossed products by the relevant modular actions together with the dual actions) of the three-step inclusions in Theorem 5.4. appearing in Lemma 5.3. To do this, one should use the weight $\widehat{tr_{\widetilde{\mathcal{N}}}}$ (on $\mathcal{N} = \widetilde{\mathcal{N}} \rtimes_\theta \mathbf{R}$) dual to the canonical trace $tr_{\widetilde{\mathcal{N}}}$ on $\widetilde{\mathcal{N}}$. The proof is once again based on the Takesaki duality (in a general form). It is important to notice that the associated modular action σ_t $(= \sigma_t^{\widehat{tr_{\widetilde{\mathcal{N}}}}})$ is the action dual to θ_s (on $\widetilde{\mathcal{N}}$). Therefore, we can use the duality, which shows that the crossed product $\mathcal{N} \rtimes_\sigma \mathbf{R}$ is conjugate to $\widetilde{\mathcal{N}} \otimes \mathcal{B}(L^2(\mathbf{R}))$ and the dual action on the former corresponds to $\theta_s \otimes Adv(s)^*$ on the latter with the regular representation $(v(s)f)(t) = f(t-s)$. Since $(\widetilde{\mathcal{N}}, \theta_s)$ was the covariant system (of the original $\mathcal{N}$), we can find a unitary w in $\widetilde{\mathcal{N}} \otimes \mathcal{B}(L^2(\mathbf{R}))$ satisfying $I \otimes v(s)^* = w(\theta_s \otimes id)(w^*)$. For $x \in \widetilde{\mathcal{N}} \otimes \mathcal{B}(L^2(\mathbf{R}))$ we compute

$$\begin{aligned} w(\theta_s \otimes id)(w^* x w)w^* &= w(\theta_s \otimes id)(w^*)(\theta_s \otimes id)(x)(\theta_s \otimes id)(w)w^* \\ &= (I \otimes v(s)^*)(\theta_s \otimes id)(x)(I \otimes v(s)) = (\theta_s \otimes Adv(s)^*)(x). \end{aligned}$$

This means that $\theta_s \otimes Adv(s)^*$ is conjugate to $\theta_s \otimes id$ via Adw. One then chooses matrix units $\{e_{ij}\}$ and an isometry v with $vv^* = e_{11}$ from the fixed-point subalgebra $\left(\widetilde{\mathcal{N}}\right)^\theta = \mathcal{N}$. The discussion before Lemma 5.3 shows that $\left(\widetilde{\mathcal{N}} \otimes \mathcal{B}(L^2(\mathbf{R})), \theta_s \otimes id\right)$ and $\left(\widetilde{\mathcal{N}}, \theta_s\right)$ are conjugate.

We can consider the conjugations in the preceding paragraph simultaneously for all of $\mathcal{M}, \mathcal{A}, \mathcal{B}$ and $\mathcal{N}$ (and everything is compatible with inclusions) so that we conclude

Proposition 5.5. *The covariant systems associated with the three-step inclusions*

$$\mathcal{M} \underset{G}{\longrightarrow} \mathcal{A} \underset{H}{\longrightarrow} \mathcal{B} \underset{F}{\longrightarrow} \mathcal{N}$$

in Theorem 5.4 are exactly

$$\widetilde{\mathcal{M}} \underset{\widetilde{G}}{\longrightarrow} \widetilde{\mathcal{A}} \underset{\widetilde{H}}{\longrightarrow} \widetilde{\mathcal{B}} \underset{\widetilde{F}}{\longrightarrow} \widetilde{\mathcal{N}}$$

in Lemma 5.3.

We can show that the two finite-to-one extensions in Theorem 5.1 come from $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}} \supseteq \widetilde{\mathcal{N}}$ respectively, and full details are left to the reader.

5.3. Structure Results for Inclusions of Type III Factors I. Let $\mathcal{M} \supseteq \mathcal{N}$ be an inclusion of type III_λ ($\lambda \neq 1$) factors with finite index. For simplicity let us assume $\mathcal{M} \cap \mathcal{N}' = \mathbf{C}I$. Theorem 5.4 says that we can split study of such inclusions into the following three cases (by looking at $\mathcal{A} \supseteq \mathcal{B}, \mathcal{B} \supseteq \mathcal{N}$ and $\mathcal{M} \supseteq \mathcal{A}$ respectively):

1. $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{N}}$ admit the common central decomposition

$$\widetilde{\mathcal{M}} = \int_X^{\oplus} \widetilde{\mathcal{M}}(\omega) d\omega, \quad \widetilde{\mathcal{N}} = \int_X^{\oplus} \widetilde{\mathcal{N}}(\omega) d\omega,$$

with the field $\{\widetilde{\mathcal{M}}(\omega) \supseteq \widetilde{\mathcal{N}}(\omega)\}_{\omega \in X}$ of type II_∞ factors. In particular, in the type III_λ ($0 < \lambda < 1$) case we get the common discrete decomposition

$$\mathcal{M} = \mathcal{A} \rtimes_\theta \mathbf{Z} \supseteq \mathcal{N} = \mathcal{B} \rtimes_\theta \mathbf{Z}$$

with $\mathcal{A} \supseteq \mathcal{B}$ of type II_∞ factors and $\theta \in Aut(\mathcal{A}, \mathcal{B})$ satisfying $tr_{\mathcal{A}} \circ E = \lambda\, tr_{\mathcal{A}}$.

2. $\mathcal{Z}(\widetilde{\mathcal{M}}) = L^\infty(X_{\mathcal{N}} \times \{1, 2, \cdots, n\})$ and $\mathcal{Z}(\widetilde{\mathcal{N}}) = L^\infty(X_{\mathcal{N}})$.

3. $\mathcal{Z}(\widetilde{\mathcal{M}}) = L^\infty(X_{\mathcal{M}})$ and $\mathcal{Z}(\widetilde{\mathcal{N}}) = L^\infty(X_{\mathcal{M}} \times \{1, 2, \cdots, m\})$.

Example 5.6. Let $\mathcal{N}$ be the Powers factor of type III_λ with the Powers state φ. Let T_0 be the period of σ_t^φ ($\sigma_{T_0}^\varphi = Id$), and we set $\alpha = \sigma_{T_0/2}^\varphi$ (a $\mathbf{Z}_2$-action on $\mathcal{N}$). The crossed product $\mathcal{M} = \mathcal{N} \rtimes_\alpha \mathbf{Z}_2$ is of type III_{λ^2}, and the inclusion $\mathcal{M} \supseteq \mathcal{N}$ is of the above second type. On the other hand, the fixed-point algebra $\mathcal{N}^\alpha$ is also of type III_{λ^2}, and $\mathcal{N} \supseteq \mathcal{N}^\alpha$ is of the above third type.

The above second and third types are dual to the each other in the sense that one goes to the opposite type by passing to the basic extension. In fact, with the modular conjugation J of $\widetilde{\mathcal{M}}$ we compute the basic extension of $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{A}} = \widetilde{\mathcal{M}} \cap \mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')'$ as follows:

$$\begin{aligned} J\left(\widetilde{\mathcal{M}} \cap \mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')'\right)' J &= J\left(\widetilde{\mathcal{M}}' \vee \mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')\right) J \\ &= \widetilde{\mathcal{M}} \vee J\mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')J = \widetilde{\mathcal{M}} \vee \mathcal{Z}(\widetilde{\mathcal{M}}_1 \cap \widetilde{\mathcal{M}}'). \end{aligned}$$

Therefore, we here explain inclusions of just the second type. Those of the above first type will be explained in §5.5.

Theorem 5.7. (Loi, [48]) *Assume that* $E: \mathcal{M} \longrightarrow \mathcal{N}$ *satisfies:* (*i*) $\mathcal{M}$ *is the Powers factor of type* III_{λ^n} $(0<\lambda<1)$, (*ii*) $\mathcal{N}$ *is of type* III_λ, *and* (*iii*) $Ind\ E = n$, *then we have* $\mathcal{M}\cap\mathcal{N}' = \mathbf{C}I$. *Furthermore, such a subfactor is unique up to conjugacy. More precisely,* $\mathcal{M}\supseteq\mathcal{N}$ *is conjugate to* $\mathcal{R}_\lambda \rtimes_\alpha \mathbf{Z}_n \supseteq \mathcal{R}_\lambda$ *with the Powers factor* $\mathcal{R}_\lambda$ (*of type* III_λ) *and* $\alpha = \sigma^\varphi_{\frac{T_0}{n}}$, *where* φ *is the Powers state on* $\mathcal{R}_\lambda$.

In the type III_0 case, we have

Theorem 5.8. (Hamachi-Kosaki) *Let* $\mathcal{M}\supseteq\mathcal{N}$ *be AFD factors of type* III_0 *of the form* $\mathcal{M}=\mathcal{A}=\mathcal{B}$. *Assume that* $E:\mathcal{M}\longrightarrow\mathcal{N}$ *induces an* n *to* 1 *map between the flows of weights*

$$\mathcal{Z}(\widetilde{\mathcal{M}}) = L^\infty(X_\mathcal{N}\times\{1,2,\cdots,n\}),\quad \mathcal{Z}(\widetilde{\mathcal{N}}) = L^\infty(X_\mathcal{N}).$$

Then we have the following:
(*i*) $Ind\ E = n$ *and* $\mathcal{M}\cap\mathcal{N}'=\mathbf{C}I$,
(*ii*) *Classification* (*up to conjugacy*) *of pairs* $\mathcal{M}\supseteq\mathcal{N}$ (*described here*) *is equivalent to that* (*up to strong conjugacy*) *of* n *to* 1 *extensions between the flows of weights,*
(*iii*) $\mathcal{M}\supseteq\mathcal{N}$ *can be always written as* $\mathcal{M}=\mathcal{P}^H\supseteq\mathcal{N}=\mathcal{P}^G$ *with a pair* $G\supseteq H$ *of a finite group and its subgroup.*

The AFD assumption is of course irrelevant for the first assertion. Note that in the type III_λ case (i.e., in Theorem 5.7) the subfactor $\mathcal{R}_\lambda$ can be expressed as the fixed-point subalgebra of $\mathcal{R}_\lambda\rtimes_\alpha\mathbf{Z}_n$ under the dual $\mathbf{Z}_n$-action. However, in the type III_0 case the subgroup H in the theorem cannot be avoided. Typical examples can be found in §7 of [25] (where the symmetric groups $\mathfrak{S}_3\supseteq\mathfrak{S}_2$ appear).

5.4. Factor-subfactor Pairs Described by Equivalence Relations. We here look at two typical inclusions described by the Krieger construction. The first one comes from a factor map while the second comes from a subequivalence relation. They are dual to the each other, and the constructions explained here are common for all types. Typical examples of inclusions described in Theorem 5.8 (i.e., ones of type **2** at the beginning of §5.3) are ones arising from factors maps (and in fact always of this form in the AFD case). On the other hand, dual inclusions, i.e., ones of type **3** arise from subrelations. Detailed study on inclusions considered here and closely related (more general) ones can be found in [25] (see also [14, 64]).

(A) We begin with an inclusion arising from a factor map (i.e., an extension). Let $\mathcal{S}_X \subseteq X^2$ and $\mathcal{R}_Y \subseteq Y^2$ be two ergodic measured equivalence relations (with countable orbits), and we use the subscripts X, Y to indicate underlying spaces. Furthermore, we assume that the former is an n to 1 extension of the latter. This means at first $X = Y \times \{1, 2, \cdots, n\}$ and a measure $d\mu_X$ on X is the product of $d\mu_Y$ on Y and the equally distributed probability measure on $\{1, 2, \cdots, n\}$. Secondly we require that one can find an $\mathfrak{S}_n$-valued cocycle α on $\mathcal{R}_Y$ (i.e., $\alpha(u,v)\alpha(v,w) = \alpha(u,w)$ for $(u,v), (v,w) \in \mathcal{R}_Y$) such that the relation $\mathcal{S}_X$ is described by

$$x = (u,i) \sim y = (v,j) \quad \text{(i.e., } (x,y) \in \mathcal{S}_X\text{)}$$
$$\text{if and only if} \quad (u,v) \in \mathcal{R}_Y \text{ and } i = \alpha(u,v)(j).$$

Figure 7 (picture of extension)

Let π be the projection from X onto Y. When $(u,v) \in \mathcal{R}_Y$, each of points in the fiber $\pi^{-1}(u)$ is $\mathcal{S}_X$-equivalent to a unique point in $\pi^{-1}(v)$ (determined by $\alpha(v,u)$). We also point out that two points in the same fiber $\pi^{-1}(u)$ are equivalent (in the sense of $\mathcal{S}_X$) only when they are the same point.

We set $\mathcal{M} = \mathcal{W}^*(\mathcal{S}_X)$, the Krieger construction, and

$$\mathcal{N} = \{L_f \in \mathcal{M};\ f(x,y) = \hat{f}(\pi(x), \pi(y))\ ((x,y) \in \mathcal{S}_X) \text{ with a function } \hat{f} \text{ on } \mathcal{R}_Y\},$$

which is a subfactor isomorphic to $\mathcal{W}^*(\mathcal{R}_Y)$ via $L_f \longleftrightarrow L_{\hat{f}}$.

Lemma 5.9. *Let $\mathcal{A}$ ($\cong L^\infty(X)$), $\mathcal{B}$ ($\cong L^\infty(Y)$) be the Cartan subalgebras of $\mathcal{M}$ and $\mathcal{N}$ respectively. Then, we have $\mathcal{M} \cap \mathcal{B}' = \mathcal{A}$ and $\mathcal{M} \cap \mathcal{N}' = \mathbf{C}I$.*

Proof. Let us assume $L_g \in \mathcal{M} \cap \mathcal{B}'$. The commutativity against arbitrary elements in $\mathcal{B}$ means $F(\pi(x))g(x,y) = g(x,y)F(\pi(y))$ $((x,y) \in \mathcal{S}_X)$ for each function F on Y, that is,

$$g(x,y) = 0 \quad \text{as long as} \quad \pi(x) \neq \pi(y).$$

Note that g is defined on $\mathcal{S}_X$, and as noted above $(x,y) \in \mathcal{S}_X$ satisfies $\pi(x) = \pi(y)$ if and only if $x = y$. Therefore, g is supported on the diagonal $D \subseteq X^2$, i.e., $L_g \in \mathcal{A}$ and the first assertion has been shown.

We now assume $L_g \in \mathcal{M} \cap \mathcal{N}'$. Since $L_g \in \mathcal{M} \cap \mathcal{N}' \subseteq \mathcal{M} \cap \mathcal{B}' = \mathcal{A}$ by the first half of the proof, g is supported on the diagonal. We set $G(x) = g(x,x)$, and note that from the commutativity of L_g against L_f in the above definition of $\mathcal{N}$ we have

$$G(x)\hat{f}(\pi(x),\pi(y)) = \hat{f}(\pi(x),\pi(y))G(y) \ ((x,y) \in \mathcal{S}_X).$$

This means that the function G is $\mathcal{S}_X$-invariant, and hence due to the ergodicity of $\mathcal{S}_X$ it is constant. $\square$

Notice $\frac{dm(x)}{dm(y)} = \frac{dm'(\pi(x))}{dm'(\pi(y))}$ so that the value of the module $\delta(x,y)$ on $\mathcal{S}_X$ depends only on $(\pi(x),\pi(y)) \in \mathcal{R}_Y$. This means that $\mathcal{N}$ is invariant under the modular action arising from ω_{ξ_0} with $\xi_0 = \chi_D$ (thanks to $\sigma_t^{\omega_{\xi_0}}(L_f) = L_{\delta^{it}f}$). It is easy to see that the orthogonal projection from $L^2(\mathcal{M})$ onto $L^2(\mathcal{N}) = \overline{\mathcal{N}\xi_0}$ is given by the following averaging procedure:

$$e_{\mathcal{N}} : \xi \mapsto \tilde{\xi} \quad \text{with} \quad \tilde{\xi}(x,y) = \frac{1}{n} \sum_{\substack{(x',y') \in \mathcal{S}_X \\ x' \in \pi^{-1}(\pi(x)) \\ y' \in \pi^{-1}(\pi(y))}} \xi(x',y'), \tag{5.1}$$

which implies that the corresponding (unique) normal conditional expectation E is given by $E(L_f) = L_{\tilde{f}}$.

We introduce a larger equivalence relation $\mathcal{R}'_X$ on X by

$$(x,y) \in \mathcal{R}'_X \ (\subseteq X^2) \quad \text{if } (\pi(x),\pi(y)) \in \mathcal{R}_Y.$$

Note $\mathcal{S}_X \subseteq \mathcal{R}'_X$ $(\subseteq X^2)$, and all points in the same fiber $\pi^{-1}(u)$ are equivalent in the sense of $\mathcal{R}'_X$. Since $\mathcal{S}_X$ is already ergodic, so is the larger $\mathcal{R}'_X$. The factor $\mathcal{W}^*(\mathcal{R}'_X)$, the Krieger construction, contains the subfactor

$$\{L_f \in \mathcal{W}^*(\mathcal{R}'_X);\ f \text{ is supported on } \mathcal{S}_X\}, \tag{5.2}$$

which is of course isomorphic to $\mathcal{W}^*(\mathcal{S}_X) = \mathcal{M}$. We note that $E_1(L_f) = L_{\chi_{\mathcal{S}_X}f}$ (killing "components outside of $\mathcal{S}_X$") gives rise to a normal conditional expectation from $\mathcal{W}^*(\mathcal{R}'_X)$ onto the subfactor (5.2).

Let f_0 be the function on $\mathcal{R}'_X$ defined by

$$f_0(x,y) = \begin{cases} \frac{1}{n} & \text{if } \pi(x) = \pi(y), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\tilde{e}_0 = L_{f_0} \in \mathcal{W}^*(\mathcal{R}'_X)$ is a projection and $E_1(\tilde{e}_0) = \frac{1}{n}\, I$ (since $(x,y) \in \mathcal{S}_X$ satisfies $\pi(x) = \pi(y)$ if and only if $x = y$).

Proposition 5.10. *The triple $(\mathcal{W}^*(\mathcal{R}'_X), E_1, \tilde{e}_0)$ satisfies the conditions in Theorem 3.12, and hence $\mathcal{M}_1 = \mathcal{W}^*(\mathcal{R}'_X)$ is the basic extension of $\mathcal{M} \supseteq \mathcal{N}$ with $Ind\ E = n$. Here, $\mathcal{M}$ means the factor given by* (5.2).

Proof. Let $L_f \in \mathcal{W}^*(\mathcal{R}'_X)$. Then, $L_f\tilde{e}_0 = L_{f*f_0}$, and for $(x,y) \in \mathcal{R}'_X$ we compute

$$(f*f_0)(x,y) = \sum_{(z,y)\in\mathcal{R}'_X} f(x,z)f_0(z,y) = \frac{1}{n}\sum_{z\in\pi^{-1}(\pi(y))} f(x,z) \tag{5.3}$$

from the definition of f_0. We note $E_1(L_f\tilde{e}_0) = L_{\chi_{\mathcal{S}_X}(f*f_0)}$ and compute

$$\begin{aligned}((\chi_{\mathcal{S}_X}(f*f_0))*f_0)(x,y) &= \sum_{(z,y)\in\mathcal{R}'_X} (\chi_{\mathcal{S}_X}(f*f_0))(x,z)f_0(z,y)\\ &= \frac{1}{n}\sum_{z\in\pi^{-1}(\pi(y))} (\chi_{\mathcal{S}'_X}(f*f_0))(x,z).\end{aligned}$$

Notice that in the last sum only one z in $\pi^{-1}(\pi(y))$ is $\mathcal{S}_X$-equivalent to x (recall what an extension means), and this unique point is denoted by $p(x,\pi(y))$. Then, we have

$$((\chi_{\mathcal{S}_X}(f*f_0))*f_0)(x,y) = \frac{1}{n}(f*f_0)(x,p(x,\pi(y))).$$

Since $\pi(p(x,\pi(y))) = \pi(y)$, it is equal to $\frac{1}{n^2}\sum_{z\in\pi^{-1}(\pi(y))} f(x,z)$ by (5.3), which is the quantity (5.3) divided by n. Therefore, we have shown $nE_1(L_f\tilde{e}_0)\tilde{e}_0 = L_f\tilde{e}_0$ for $L_f \in \mathcal{W}^*(\mathcal{R}'_X)$.

We then take L_f from the subfactor (5.2). We have

$$(f_0*f*f_0)(x,y) = \frac{1}{n^2}\sum_{\substack{z_1\in\pi^{-1}(\pi(x))\\ z_2\in\pi^{-1}(\pi(y))}} f(z_1,z_2).$$

Since f is supported on $\mathcal{S}_X$, only n of n^2 summands could be non-zero and we have

$$(f_0*f*f_0)(x,y) = \frac{1}{n^2}\sum_{z\in\pi^{-1}(\pi(x))} f(z,p(z,\pi(y))) \quad ((x,y)\in\mathcal{R}'_X). \tag{5.4}$$

From (5.1) we get $E(L_f) = L_{\tilde{f}}$, where the function $\tilde{f}$ (on $\mathcal{R}'_X$) supported on $\mathcal{S}_X$ is defined by

$$\tilde{f}(x,y) = \frac{1}{n}\sum_{z\in\pi^{-1}(\pi(x))} f(z,p(z,\pi(y))) \quad ((x,y)\in\mathcal{S}_X).$$

Thus, by noting $\pi(p(z,\pi(y)))=\pi(y)$, from (5.3) we get

$$(\tilde f * f_0)(x,y)=\frac{1}{n^2}\sum_{\substack{z\in\pi^{-1}(\pi(x))\\ w\in\pi^{-1}(\pi(y))}} f(z,w). \tag{5.5}$$

Since f is supported on $\mathcal{S}_X$, (5.4) and (5.5) are actually the same and we have shown $\tilde e_0 L_f \tilde e_0 = E(L_f)\tilde e_0$ for L_f in (5.2). $\square$

(B) This time we start from two ergodic equivalence relations $\mathcal{S}_X \subseteq \mathcal{R}_X$ ($\subseteq X^2$) on the same space X. We set $\mathcal{M}=\mathcal{W}^*(\mathcal{R}_X)$, the Krieger construction and

$$\mathcal{N}=\{L_f\in\mathcal{W}^*(\mathcal{R}_X);\ f \text{ is supported on } \mathcal{S}_X\}$$

which is isomorphic to $\mathcal{W}^*(\mathcal{S}_X)$. Note that $\mathcal{M}\supseteq\mathcal{N}$ contain the common Cartan subalgebra $\mathcal{A}\cong L^\infty(X)$ so that

$$\mathcal{M}\cap\mathcal{N}'\subseteq\mathcal{M}\cap\mathcal{A}'=\mathcal{A}.$$

Thus, the ergodicity of $\mathcal{S}_X$ (corresponding to $\mathcal{N}$) shows $\mathcal{M}\cap\mathcal{N}'=\mathbf{C}I$. Note that $E(L_f)=L_{\chi_{\mathcal{S}_X}f}$ as above gives us a (unique) normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$, and the corresponding orthogonal projection $e_{\mathcal{N}}$ from $L^2(\mathcal{M})$ onto $L^2(\mathcal{N})=\overline{\mathcal{N}\xi_0}$ is of course given by $e_{\mathcal{N}}\xi=\chi_{\mathcal{S}_X}\xi$.

Since $\mathcal{S}_X\subseteq\mathcal{R}_X$, each $(\mathcal{R}_X)$-orbit $(\mathcal{R}_X)_x$ is decomposed into several $(\mathcal{S}_X)$-orbits. Let $\#(x)$ be the number of $(\mathcal{S}_X)$-orbits. The integer-valued function $\#(x)$ being invariant under $\mathcal{R}_X$, from the the ergodicity $\#(x)$ is constant ($=n$). This constant n turns out to be $Ind\ E$.

Note that $\mathcal{S}_X$ acts on $\mathcal{R}_X$ from the left in the obvious way, i.e., $(x,y)\in\mathcal{S}_X$ acts like $(z,x)\in\mathcal{R}_X\mapsto(z,x)(x,y)=(z,y)\in\mathcal{R}_X$. Let X_1 be the space of the ergodic decomposition of this action, that is,

$$X_1=\{(x,U);\ x\in X \text{ and } U \text{ is an } (\mathcal{S}_X)\text{-orbit in the } (\mathcal{R}_X)\text{-orbit } (\mathcal{R}_X)_x\}.$$

Note that $(x,U)\in X_1\mapsto x\in X$ is an n to 1 map. From the measure $d\mu_X$ on X we define the measure $d\mu_{X_1}$ by $d\mu_{X_1}(x,U)=\frac{1}{n}d\mu_X(x)$. Assume $(x,U),(y,V)\in X_1$ satisfy $(\pi(x,U),\pi(y,V))\in\mathcal{R}_X$, i.e., x and y are $(\mathcal{R}_X)$-equivalent. Therefore, U,V are $(\mathcal{S}_X)$-orbits in the same $(\mathcal{R}_X)$-orbit $(\mathcal{R}_X)_x=(\mathcal{R}_X)_y$. We now introduce an equivalence relation $\mathcal{S}'_{X_1}$ on X_1:

$$((x,U),(y,V)) \in \mathcal{S}'_{X_1} \ (\subseteq X_1^2) \quad \text{if} \quad (x,y) \in \mathcal{R}_X \text{ and } U = V.$$

We note that $\mathcal{S}'_{X_1}$ is an extension of $\mathcal{R}_X$. In fact, when x and y are $\mathcal{R}_X$-equivalent, we look at the fibers

$$\pi^{-1}(x) = \{(x,U_1),(x,U_2),\cdots,(x,U_n)\},$$
$$\pi^{-1}(y) = \{(y,V_1),(y,V_2),\cdots,(y,V_n)\}.$$

Note that both of the U_i's and the V_i's exhaust the n $(\mathcal{S}_X)$-orbits in the $(\mathcal{R}_X)$-orbit $(\mathcal{R}_X)_x = (\mathcal{R}_X)_y$. Thus, each of U_i's is the same as exactly one of V_i's and this determines the $\mathfrak{S}_n$-valued cocycle α at the beginning of **(A)**.

Let $\mathcal{M}_1 = \mathcal{W}^*(\mathcal{S}'_{X_1})$, the Krieger construction. We get the subfactor

$$\begin{aligned}\{L_f \in \mathcal{W}^*(\mathcal{S}_{X_1});\ f((x,U)(y,U)) = \hat{f}(x,y)\ ((x,y) \in \mathcal{R}_X)\\ \text{with a function } \hat{f} \text{ on } \mathcal{R}_X\}\end{aligned} \tag{5.6}$$

isomorphic to $\mathcal{M} = \mathcal{W}^*(\mathcal{R}_X)$ as in **(A)**, $\mathcal{S}'_{X_1}$ being an extension of $\mathcal{R}_X$. Moreover, we have the normal conditional expectation E_1 given by (5.1). More precisely, let $L_f \in \mathcal{W}^*(\mathcal{S}'_{X_1})$, and we notice that if $E_1(L_f) = L_g$ then the function $g((x,U),(y,V))$ depends only on $(\pi(x,U),\pi(y,U)) = (x,y) \in \mathcal{R}_X$. From the definition of the relation $\mathcal{S}'_{X_1}$ and (5.1) we easily see

$$g((x,U),(y,U)) = \hat{g}(x,y) \quad \text{with} \quad \hat{g}(x,y) = \frac{1}{n}\sum_{W\in(\mathcal{R}_X)_x} f((x,W),(y,W)), \tag{5.7}$$

i.e., the average over $(\mathcal{S}_X)$-orbits W in the $(\mathcal{R}_X)$-orbit $(\mathcal{R}_X)_x = (\mathcal{R}_X)_y$. (Remark that f is defined on $\mathcal{S}'_{X_1}$ so that $U = V$ always.)

Let f_1 be the function on $\mathcal{S}'_{X_1}$ defined by

$$f_1((x,U),(y,U)) = \begin{cases} 1 & \text{if } x = y \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Note the above first condition means $x = y$ and U must be the $(\mathcal{S}_X)$-orbit $(\mathcal{S}_X)_x$. We set $\tilde{e}_1 = L_{f_1} \in \mathcal{M}_1$, which is a projection. We easily see $E_1(\tilde{e}_1) = \frac{1}{n}\ I$.

Proposition 5.11. *The triple $(\mathcal{W}^*(\mathcal{S}'_{X_1}), E_1, \tilde{e}_1)$ satisfies the conditions in Theorem 3.12, and hence $\mathcal{M}_1 = \mathcal{W}^*(\mathcal{S}'_{X_1})$ is the basic extension of $\mathcal{M} \supseteq \mathcal{N}$ with $Ind\ E = n$. Here, the factor $\mathcal{M}$ is the one given by* (5.6)

Proof. Let $L_f \in \mathcal{W}^*(\mathcal{S}'_{X_1})$. We compute

$$(5.8) \qquad (f * f_1)((x,U)(y,U)) = \sum_{((z,U)(y,U))\in \mathcal{S}'_{X_1}} f((x,U),(z,U))f_1((z,U),(y,U))$$
$$= \begin{cases} f((x,U),(y,U)) & \text{if } y \in U, \\ 0 & \text{otherwise,} \end{cases}$$

from the definition of f_1. Thus, we see $E_1(L_f\tilde{e}_1) = L_g$ with $g((x,U),(y,U)) = \hat{g}(x,y)$ and

$$\hat{g}(x,y) = \frac{1}{n}\sum_{W\in(\mathcal{R}_X)_x} (f * f_1)((x,W)(y,W)) = \frac{1}{n} f((x,(\mathcal{S}_X)_y)(y,(\mathcal{S}_X)_y))$$

from (5.7). Now (5.8) says $(g * f_1)((x,U),(y,U))$ can be non-zero only when $y \in U$ and in this case we get

$$(g * f_1)((x,U),(y,U)) = g((x,U),(y,U)) = \hat{g}(x,y) = \frac{1}{n} f((x,(\mathcal{S}_X)_y)(y,(\mathcal{S}_X)_y)).$$

Thus, we have shown $nE_1(L_f\tilde{e}_1)\tilde{e}_1 = L_f\tilde{e}_1$ for $L_f \in \mathcal{W}^*(\mathcal{S}'_{X_1})$.

We take L_f described in (5.6) (together with $\hat{f}$). Almost the same computation as (5.8) shows

$$(f_0 * f * f_0)((x,U),(y,U)) = \begin{cases} f((x,U),(y,U)) = \hat{f}(x,y) & \text{if } x,y \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Let $E(L_f) = L_g$ with $g((x,U),(y,U)) = \hat{g}(x,y)$. By (5.8) we get

$$(g * f_1)((x,U)(y,U)) = \begin{cases} g((x,U),(y,U)) = \hat{g}(x,y) & \text{if } y \in U, \\ 0 & \text{otherwise.} \end{cases}$$

However, from the definition of E we have $\hat{g} = \chi_{\mathcal{S}_X}\hat{f}$. Also note if $(x,y) \in \mathcal{S}_X$ and $y \in U$ then one gets $x \in U$. Therefore, we observe that the preceding two convolution products are the same, i.e., $\tilde{e}_1 L_f \tilde{e}_1 = E(L_f)\tilde{e}_1$ for L_f in (5.6). □

Example 5.12. ([24]) Recall the construction in Example 2.14 from a non-singular ergodic flow $\{F_s\}_{s\in\mathbf{R}}$ on (Γ, m). We assume that $\{F_s\}_{s\in\mathbf{R}}$ is an n to 1 extension of another (automatically ergodic) flow $\{F'_s\}_{s\in\mathbf{R}}$ lives on (Γ', m'). In that example we introduced the action of $G = \mathbf{Q}\times\mathbf{Z}$ on the product space $X = \Gamma\times Y_0\times\mathbf{R}$ (see (2.7)). Note that one can also introduce the G-action on $Y = \Gamma'\times Y_0\times\mathbf{R}$ in the same way (but by using F'_s instead in (2.7)). Of course the former action is an n to 1 extension of the latter. Therefore, from this we get $\mathcal{S}_X$ $(= (\mathcal{S}_X)_G)$ and $\mathcal{R}_Y$ $(= (\mathcal{R}_Y)_G)$, and of course the former is an n to 1 extension of the latter in the sense specified at the beginning of **(A)**. Therefore, as in **(A)** we get the pair $\mathcal{M} = \mathcal{W}^*(\mathcal{S}_X) \supseteq \mathcal{N}$ of factors (where $\mathcal{N}$ is the one defined as in the paragraph before Lemma 5.9 and it is isomorphic to $\mathcal{W}^*(\mathcal{R}_Y)$). The resulting inclusion is irreducible with index n and of the type described in Theorem 5.8, i.e., $\mathcal{M} = \mathcal{A} = \mathcal{B}$.

To see this, we look at th inclusion $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{N}}$ of von Neumann algebras of type II_∞ (i.e., the crossed products with respect to the relevant modular actions). As explained in §2.3 they appear as the Poincaré suspension. Namely, let $\widetilde{X} = X \times \mathbf{R}$ and $\widetilde{X} = X \times \mathbf{R}$, and we consider the equivalence relations

$$\begin{cases} \widetilde{\mathcal{S}_X} \ (\subseteq (\widetilde{X})^2) & = \ \text{the equivalence relation generated by (2.8)}, \\ \widetilde{\mathcal{R}_Y} \ (\subseteq (\widetilde{Y})^2) & = \ \text{the equivalence relation generated by (2.8) with } F'_q \text{ instead.} \end{cases}$$

From the construction it is plain to see that $\widetilde{\mathcal{S}_X}$ is an n to 1 extension of $\widetilde{\mathcal{R}_Y}$ and that the inclusion of algebras arising from this extension as in **(A)** is exactly $\widetilde{\mathcal{M}} \supseteq \widetilde{\mathcal{N}}$ (although they are no longer factors due to the lack of ergodicity). Note that the arguments in the proof of Lemma 5.9 show

$$\mathcal{Z}(\widetilde{\mathcal{M}}) = \widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}' \ (= \mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')).$$

Therefore, the relevant expectation induces a map from $\mathcal{Z}(\widetilde{\mathcal{M}}) = \mathcal{Z}(\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{N}}')$ onto $\mathcal{Z}(\widetilde{\mathcal{N}})$, and it is plain to see that this is exactly the n to 1 extension $F_s \longrightarrow F'_s$ we started from. Hence, the index between $\mathcal{A}$ and $\mathcal{N}$ is n, and we conclude $\mathcal{M} = \mathcal{A} = \mathcal{B}$.

Let $\mathcal{M} \supseteq \mathcal{N}$ be the inclusion in the above example, and $\mathcal{M}_1$ be the basic extension. By Proposition 5.10, $\mathcal{M}_1 \supseteq \mathcal{M}$ is an inclusion described in **(B)**, i.e., the one arising from $\mathcal{R}'_X \supseteq \mathcal{S}_X$, a relation-subrelation pair. Let $\widetilde{X} = X \rtimes \mathbf{R}$. By passing to the Poincaré suspension, we get the relation-subrelation pair $\widetilde{\mathcal{R}'_X} \supseteq \widetilde{\mathcal{S}_X}$ in $(\widetilde{X})^2$ (which are not necessarily ergodic unless in the III_1 case), and the inclusion $\widetilde{\mathcal{M}_1} \supseteq \widetilde{\mathcal{M}}$ of algebras of type II_∞ is

$$\mathcal{W}^*(\widetilde{\mathcal{R}'_X}) \supseteq \{L_f \in \mathcal{W}^*(\widetilde{\mathcal{R}'_X});\ f \text{ is supported on } \widetilde{\mathcal{S}_X}\}$$

described in **(B)**. Since they contain the common Cartan subalgebra (isomorphic to $L^\infty(\widetilde{X})$), as before the relative commutant $\widetilde{\mathcal{M}_1} \cap \widetilde{\mathcal{M}}'$ falls into this Cartan subalgebra and

$$\widetilde{\mathcal{M}_1} \cap \widetilde{\mathcal{M}}' = \mathcal{Z}(\widetilde{\mathcal{M}}).$$

The flow of weights of $\mathcal{M}_1$ (= the one for $\mathcal{N}$) appears as a factor flow of that of $\mathcal{M}$, which is exactly $F_s \longrightarrow F'_s$ in Example 5.12. This happens because $\widetilde{\mathcal{R}'_X}$ has more relations than $\widetilde{\mathcal{S}_X}$ and hence the space of the ergodic decomposition of the former is smaller than that of the latter.

5.5. Structure Results for Inclusions of Type III Factors II. Here we consider inclusions of the first type described at the beginning of §5.3. Assume that $\mathcal{A} \supseteq \mathcal{B}$ is an inclusion of type II_1 factors and $\mathcal{N}$ is a type III factor. Of course the very simple-minded inclusion $\mathcal{N} \otimes \mathcal{A} \supseteq \mathcal{N} \otimes \mathcal{B}$ of type III factors is of the above first type. An important problem here is: For an inclusion $\mathcal{M} \supseteq \mathcal{N}$ of the first type how do we decide if it splits into the tensor product as above ? In the type III_λ $(0 < \lambda < 1)$ case, quite a satisfactory answer can be found in [49, 60] (see also [34]).

Before going further, let us recall graphs canonically attached to an inclusion in question ([16]). Let $\mathcal{M} \supseteq \mathcal{N}$ be (type III) factors with finite index, and

$$\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots$$

be the Jones tower as usual. Since $Ind(E \circ E_1 \circ \cdots \circ E_n) = (IndE)^{n+1} < \infty$, each $\mathcal{M}_n \cap \mathcal{N}'$ is finite dimensional and we get the increasing sequence

$$\mathcal{N} \cap \mathcal{N}' \subseteq \mathcal{M} \cap \mathcal{N}' \subseteq \mathcal{M}_1 \cap \mathcal{N}' \subseteq \cdots$$

of finite dimensional higher relative commutants. Hence it is described by a Bratteli diagram, which yields the principal graph. Similarly from

$$\mathcal{M} \cap \mathcal{M}' \subseteq \mathcal{M}_1 \cap \mathcal{M}' \subseteq \mathcal{M}_2 \cap \mathcal{M}' \subseteq \cdots ,$$

we get another graph, the dual principal graph. When $IndE < 4$, the two graphs are known to coincide. Furthermore, in this case the following Dynkin diagrams appear from inclusions:

$$A_n \ (n \geq 3), \quad D_{2n} \ (n \geq 2), \quad E_6, \quad E_8$$

Figure 8 (Dynkin diagrams)

We assume that $\mathcal{M} \supseteq \mathcal{N}$ is an inclusion of type III_λ $(0 < \lambda < 1)$ factors (of the first type). As noted before we get the common discrete decomposition

$$\mathcal{M} = \mathcal{A} \rtimes_\theta \mathbf{Z} \supseteq \mathcal{N} = \mathcal{B} \rtimes_\theta \mathbf{Z}.$$

Note that $\mathcal{A} \supseteq \mathcal{B}$ is an inclusion of type II_∞ factors with the index equal to $IndE$, and this gives rise to the principal and dual principal graphs again. For convenience let us call them type II graphs. On the other hand, the graphs coming from the original inclusion $\mathcal{M} \supseteq \mathcal{N}$ is referred to as type III graphs in what follows. Note that the trace-scaling automorphism $\theta \in Aut(\mathcal{A}, \mathcal{B})$ can be canonically extended to the basic extensions $\mathcal{A}_n$ (of $\mathcal{A} \supseteq \mathcal{B}$). Indeed, the extensions are specified by the requirement that they fix the successive Jones projections. Note that the (extended) θ gives us an automorphism on $\mathcal{A}_n \cap \mathcal{B}'$.

When E is a minimal conditional expectation (automatic when $Ind\ E < 4$ or more generally when $\mathcal{M} \cap \mathcal{N}' = \mathbf{C}I$), we have

$$\mathcal{M}_n \cap \mathcal{N}' = (\mathcal{A}_n \cap \mathcal{B}')^\theta .$$

In fact, this follows from $\mathcal{M}_n \cap \mathcal{B}' = \mathcal{A}_n \cap \mathcal{B}'$ (and this happens because a trace-scaling automorphism is strongly outer $\cdots$ see §6.4, **3**). The action of θ on the tower of higher relative commutants is called the Loi invariant or the standard invariant (of θ). For simplicity let us assume that $Ind\ E < 4$. For example when the type II graph is A_n, all of $\mathcal{A}_n \cap \mathcal{B}'$ are generated by the Jones projections so that θ acts trivially on $\mathcal{A}_n \cap \mathcal{B}'$. Similarly (by looking at relevant trace values) we also observe that no graph change occurs for E_6 and E_8. The only (type II) graph admitting a non-trivial Loi invariant is D_{2n}, and the invariant is described by

Figure 9 (non-trivial Loi-invariant for D_{2n})

In this case the type III graph "shrinks to" A_{4n-3}. The meaning of graph changes was completely clarified in [32] by making use of the sector theory.

The discussions so far imply:

(i) if the type III graph is one of A_m $(m \neq 4n-3)$, D_{2n}, E_6, E_8 then the type II graph is the same and no Loi invariant,

and (ii) if the type III graph is A_{4n-3}, then the type II graph is either D_{2n} or A_{4n-3}. Of course the Loi invariant is as above in the former case while it is trivial in the latter. The classification result of Loi and Popa (when $Ind\ E < 4$) states

Theorem 5.13. *Let $\mathcal{M} \supseteq \mathcal{N}$ be an inclusion of Powers factors of type III_λ $(0 < \lambda < 1)$ (of the first type) with $Ind\ E < 4$.*

(i) *If the type III graph is one of A_m $(m \neq 4n-3)$, D_{2n}, E_6, E_8 then this inclusion splits into the tensor product* $($*as explained above*$)$.
(ii) *If the type III and type II graph are both A_{4n-3}, the inclusion splits again,*
(iii) *If the type III and type II graphs are A_{4n-3}, D_{2n} respectively, then a subfactor is unique* $($*up to conjugacy*$)$ *subject to this condition.*

Of course a model inclusion for the case (iii) can be constructed in an explicit fashion. Description is particularly easy when $n = 2$ (i.e., $Ind\ E = 3$ and the graphs are A_5, D_4). Let $\alpha : \ \mathfrak{S}_3 \longrightarrow Aut(\mathcal{R}_0)$ be a (unique) outer action of the symmetric group $\mathfrak{S}_3$ on the hyperfinite II_1 factor $\mathcal{R}_0$. In the crossed product $\mathcal{R}_0 \rtimes_\alpha \mathfrak{S}_3 = \langle \mathcal{R}_0, \lambda_g \rangle''$ we consider the two subfactors

$$\mathcal{A}_0 = \mathcal{R}_0 \rtimes_\alpha \mathfrak{A}_3 \supseteq \mathcal{B}_0 = \mathcal{R}_0,$$

where $\mathfrak{A}_3 \cong \mathbf{Z}_3$ is the subgroup generated by the 3-cycle $(1, 2, 3)$. We also consider the automorphism $\beta = Ad\ \lambda_{(1,2)} \in Aut(\mathcal{A}_0, \mathcal{B}_0)$ of period 2. Let $\mathcal{A}_1$ be the basic extension of $\mathcal{A}_0 \supseteq \mathcal{B}_0$. Then,

$$\mathcal{A}_1 \cap \mathcal{B}_0' = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} ;\ a, b, c \in \mathbf{C} \right\} \quad (\cong \ell^\infty(\mathfrak{A}_3))$$

is generated by the three projections $e_{\mathcal{B}_0}$, $\lambda_{(1,2,3)} e_{\mathcal{B}_0} \lambda^*_{(1,2,3)}$, $\lambda_{(1,2,3)^2} e_{\mathcal{B}_0} \lambda^*_{(1,2,3)^2}$, where $e_{\mathcal{B}_0} = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is the Jones projection. Remark that the (extended) β switches the last two projections because of $(1, 2)(1, 2, 3)(1, 2) = (1, 2, 3)^2$, which means that β acts on $\mathcal{A}_1 \cap \mathcal{B}_0'$ as in the above figure (with $n = 2$). Let $\mathcal{R}_\lambda = \mathcal{R}_{0,1} \rtimes_\theta \mathbf{Z}$ be the discrete decomposition of the Powers factor. Then, the discussions so far imply that the inclusion

$$(\mathcal{R}_{0,1} \otimes \mathcal{A}_0) \rtimes_{\theta \otimes \beta} \mathbf{Z} \supseteq (\mathcal{R}_{0,1} \otimes \mathcal{B}_0) \rtimes_{\theta \otimes \beta} \mathbf{Z},$$

serves as a model inclusion.

Another description for $(\mathcal{A}_0, \mathcal{B}_0, \beta)$ is also possible. At first we consider

$$\mathcal{C}_0 = \mathcal{R}_0 \rtimes_\alpha \mathfrak{S}_3 \supseteq \mathcal{D}_0 = \mathcal{R}_0 \rtimes_\alpha \mathfrak{S}_2.$$

Then the (one-dimensional) "signature representation" ϵ of $\mathfrak{S}_3$ gives us the automorphism π_ϵ of period 2 for the above pair

$$\pi_\epsilon(x) = \sum_{g \in \mathfrak{S}_3} \epsilon(g) x_g \lambda_g \quad \text{for} \quad x = \sum_{g \in \mathfrak{S}_3} x_g \lambda_g \in \mathcal{C}_0 = \mathcal{R}_0 \rtimes_\alpha \mathfrak{S}_3.$$

We set $\mathcal{A}_0 = \mathcal{C}_0 \rtimes_{\pi_\epsilon} \mathbf{Z}_2$ and $\mathcal{B}_0 = \mathcal{D}_0 \rtimes_{\pi_\epsilon} \mathbf{Z}_2$, and let β be the dual action of $\pi_\epsilon \in Aut(\mathcal{A}_0, \mathcal{B}_0)$. The triple consisting of these also does the job.

As was explained so far subfactors (with index say less than 4) in the Powers factor are quite rigid although they are slightly less rigid than those in the hyperfinite II_1 factors. On the other hand, situation is completely different in the (AFD) type III_0 case, where one can usually distinguish uncountably many non-conjugate subfactors (see [41, 44] for typical examples).

6. Sector Theory

The theory of super selection sectors originally occurred in QFT ([9], see also [61]). This was further polished by R. Longo, and it played an fundamental role in his approach on index theory ([51]).

At first we recall that an $\mathcal{M}$-$\mathcal{M}$ bimodule $\mathcal{X}$ means a Hilbert space ($= \mathcal{X} = {}_\mathcal{M}\mathcal{X}_\mathcal{M}$) equipped with commuting representations of $\mathcal{M}$ and its opposite algebra $\mathcal{M}^\circ$ (it is called an $\mathcal{M}$-$\mathcal{M}$ correspondence by A. Connes, see Chap. V, Appendix B, [6]). The representations of $\mathcal{M}, \mathcal{M}^\circ$ give rise to left and right $\mathcal{M}$-actions, and we use the notation $m_1 \cdot \xi \cdot m_2$ to indicate them. Two $\mathcal{M}$-$\mathcal{M}$ bimodules are said to be unitarily equivalent when there exists a surjective isometry (between the two underlying Hilbert spaces) which intertwines both of the left and right $\mathcal{M}$-actions, and unitarily equivalent $\mathcal{M}$-$\mathcal{M}$ bimodules are sometimes identified.

Assume that $\mathcal{M}$ is a type III factor throughout with the standard Hilbert space $\mathcal{H} = L^2(\mathcal{M})$. Let $End(\mathcal{M})$ be the unital (normal) $*$-endomorphisms of $\mathcal{M}$ into itself. For each $\rho \in End(\mathcal{M})$ we consider the commuting normal representations of $\mathcal{M}$ and $\mathcal{M}^\circ$ on the standard Hilbert space $\mathcal{H}$ defined by

$$m_1 \cdot \xi \cdot m_2 = \rho(m_1) J m_2^* J \xi \quad (\text{for } m_i \in \mathcal{M} \text{ and } \xi \in \mathcal{H}),$$

where J ($= J_\mathcal{M}$) denotes the modular conjugation. This $\mathcal{M}$-$\mathcal{M}$ bimodule is denoted by $\mathcal{H}_\rho$. Assume that two $\mathcal{M}$-$\mathcal{M}$ bimodules $\mathcal{H}_{\rho_1}, \mathcal{H}_{\rho_2}$ ($\rho_1, \rho_2 \in End(\mathcal{M})$) are unitarily equivalent, i.e., there exists a surjective isometry $u : \mathcal{H}_{\rho_1} \longrightarrow \mathcal{H}_{\rho_2}$ satisfying

$$u(m_1 \cdot \xi \cdot m_2) = m_1 \cdot (u\xi) \cdot m_2.$$

Note $\mathcal{H}_{\rho_1} = \mathcal{H}_{\rho_2} = \mathcal{H}$ as Hilbert spaces, and the compatibility between the right $\mathcal{M}$-actions forces $u \in (J\mathcal{M}J)' = \mathcal{M}$. Next, the compatibility between the left M-actions means $u\rho_1(m) = \rho_2(m)u$ for each $m \in \mathcal{M}$, i.e., inner conjugate $\rho_1 = u\rho_2(\cdot)u^*$ with a unitary $u \in \mathcal{M}$. Therefore, we have seen that $\mathcal{H}_{\rho_1}, \mathcal{H}_{\rho_2}$ are unitarily equivalent (as $\mathcal{M}$-$\mathcal{M}$ bimodules) if and only if ρ_1, ρ_2 are inner conjugate. In [51] the author mentions that the following important observation is due to A. Connes.

Lemma 6.1. *Any $\mathcal{M}$-$\mathcal{M}$ bimodule $\mathcal{X}$ is unitarily equivalent to $\mathcal{H}_\rho$ for some $\rho \in End(\mathcal{M})$.*

Proof. At first note that any representation is faithful since $\mathcal{M}$ is a factor. We denote the left and right $\mathcal{M}$-actions on $\mathcal{X}$ (i.e., the representation of $\mathcal{M}$ and that of $\mathcal{M}^\circ$) by $\pi_l^{\mathcal{X}}, \pi_r^{\mathcal{X}}$ respectively. Let π_l, π_r be the usual representation and anti-representation of $\mathcal{M}$ on the standard Hilbert space $L^2(M) = \mathcal{H}$ (i.e., $\pi_l(m) \cong m$ is the GNS-representation and $\pi_r(m) = J\pi_l(m)^*J = Jm^*J$). Ignoring the left $\mathcal{M}$-action on $\mathcal{X}$ at first, we regard that $\mathcal{M}^\circ$ is (faithfully) represented on $\mathcal{X}$ by $\pi_r^{\mathcal{X}}$. On the other hand, $\mathcal{M}^\circ$ is also represented on the standard Hilbert space $\mathcal{H}$ by π_r. Since $\mathcal{M}^\circ$ is of type III, the two representations of $\mathcal{M}^\circ$ are spatially implemented. Thus, there exists a surjective isometry $U : \mathcal{H} \longrightarrow \mathcal{X}$ satisfying $\pi_r^{\mathcal{X}}(m) = U\pi_r(m)U^*$ $(m \in \mathcal{M})$. We note

$$\pi_l^{\mathcal{X}}(\mathcal{M}) \subseteq \pi_r^{\mathcal{X}}(\mathcal{M})' = (U\pi_r(\mathcal{M})U^*)' = (UJ\mathcal{M}JU^*)' = UJ\mathcal{M}'JU^* = U\mathcal{M}U^*,$$

where the first inclusion comes from the fact that the left and right actions commute. From the above we have $U^*\pi_l^{\mathcal{X}}(\mathcal{M})U \subseteq \mathcal{M}$, and hence $\rho(m) = U^*\pi_l^{\mathcal{X}}(m)U$ gives rise to an endomorphism of $\mathcal{M}$. The $\mathcal{M}$-$\mathcal{M}$ bimodule $\mathcal{H}_\rho$ is unitarily equivalent to $\mathcal{X}$ via U. In fact, we compute

$$\begin{aligned} U(m_1 \cdot \xi \cdot m_2) &= U\rho(m_1)Jm_2^*J\xi = UU^*\pi_l^{\mathcal{X}}(m_1)UJm_2^*J\xi \\ &= \pi_l^{\mathcal{X}}(m_1)U\pi_r(m_2)\xi = \pi_l^{\mathcal{X}}(m_1)\pi_r^{\mathcal{X}}(m_2)U\xi = m_1 \cdot (U\xi) \cdot m_2, \end{aligned}$$

(where $m_1 \cdot \xi \cdot m_2$ and $m_1 \cdot (U\xi) \cdot m_2$ of course indicate the actions on $\mathcal{H}_\rho$ and $\mathcal{X}$ respectively). □

We define $\rho_1 \sim \rho_2$ when they are inner conjugate, and we set

$$Sect(\mathcal{M}) = End(\mathcal{M})/\sim, \text{ the sectors.}$$

A class $[\rho]$ is called a sector (or $\mathcal{M}$-sector). When no confusion is possible, we simply write ρ (instead of $[\rho]$) to ease the notation. The discussions so far show that (for a type III factor) study of $\mathcal{M}$-$\mathcal{M}$ bimodules up to unitary equivalence is completely the same as that on sectors.

From $\rho \in End(\mathcal{M})$, we get the inclusion $M \supseteq \rho(M)$ (both considered to be acting on $\mathcal{M}$). Since $\rho(M)$ is also a type III factor, it acts on $\mathcal{M}$ standardly. This means that we have the modular conjugation $J_{\rho(\mathcal{M})}$ of $\rho(\mathcal{M})$ on the same Hilbert space $\mathcal{H}$. Let γ be the canonical endomorphism defined by

$$\gamma = AdJ_{\rho(\mathcal{M})}J : \mathcal{M} \longrightarrow \rho(\mathcal{M})$$

(see [50]). Since $\gamma(\mathcal{M}) \subseteq \rho(\mathcal{M})$, we can set

$$\bar{\rho}(m) = \rho^{-1} \circ \gamma(m) \quad (m \in \mathcal{M}),$$

which is an endomorphism of $\mathcal{M}$. Its class $[\bar{\rho}]$ depends only on that of ρ, and we set $\overline{[\rho]} = [\bar{\rho}]$, the conjugate sector of $[\rho]$.

Recall that the contragredient $\mathcal{M}$-$\mathcal{M}$ bimodule $\overline{\mathcal{X}}$ of an $\mathcal{M}$-$\mathcal{M}$ bimodule $\mathcal{X}$ means the conjugate Hilbert space $\overline{\mathcal{X}}$ ($= \{\bar{\xi};\ \xi \in \mathcal{H}\}$ with $\overline{\alpha\xi_1 + \beta\xi_2} = \alpha\bar{\xi_1} + \beta\bar{\xi_2}$ and $\langle \bar{\xi_1}, \bar{\xi_2} \rangle = \langle \xi_2, \xi_1 \rangle$, etc.) equipped with the following $\mathcal{M}$-$\mathcal{M}$ action:

$$m_1 \cdot \bar{\xi} \cdot m_2 = \overline{m_2^* \cdot \xi \cdot m_1^*} \quad (\text{for } m_i \in \mathcal{M} \text{ and } \bar{\xi} \in \overline{\mathcal{X}}).$$

In the bimodule picture, passing from ρ to the conjugate sector $\bar{\rho}$ corresponds to considering a contragredient bimodule.

Lemma 6.2. *The bimodule $\mathcal{H}_{\bar{\rho}}$ arising from the conjugate sector $\bar{\rho}$ is unitarily equivalent to the contragredient bimodule $\overline{\mathcal{H}_\rho}$ of $\mathcal{H}_\rho$.*

Before going to a proof, we point out that an endomorphism ρ always admits an implementation as another consequence of being type III. In fact, we choose and fix a faithful normal state ϕ on $\mathcal{M}$, and set $\psi = \phi \circ \rho^{-1} \in \rho(\mathcal{M})_*^+$. Let $\psi = \omega_{\xi_1}$ (resp. $\phi = \omega_{\xi_2}$) with a cyclic and separating vector $\xi_1 \in L^2(\mathcal{M})$ for $\rho(\mathcal{M})$ (resp. $\phi = \omega_{\xi_2}$ with a cyclic and separating vector $\xi_2 \in L^2(\mathcal{M})$ for $\mathcal{M}$). Then, the map

$m\xi_2 \in \mathcal{M}\xi_2 \to \rho(m)\xi_1 \in \rho(\mathcal{M})\xi_1$ gives rise to the unitary U on $L^2(\mathcal{M})$ (due to $\psi \circ \rho = \phi$). Note that

$$UxU^*\rho(y)\xi_1 = Uxy\xi_2 = \rho(xy)\xi_1 = \rho(x)\rho(y)\xi_1$$

for each $x, y \in \mathcal{M}$. The subspace $\rho(\mathcal{M})\xi_1$ being dense, we get $\rho(x) = UxU^*$ ($x \in \mathcal{M}$). Let $S_{\mathcal{M}}, S_{\rho(\mathcal{M})}$ be the S-operators (in the modular theory) associated with $(\mathcal{M}, \xi_2), (\rho(\mathcal{M}), \xi_1)$ respectively. Note

$$US_{\mathcal{M}}U^*\rho(m)\xi_1 = US_{\mathcal{M}}m\xi_2 = Um^*\xi_2 = \rho(m)^*\xi_1 = S_{\rho(\mathcal{M})}m\xi_1$$

for each $m \in \mathcal{M}$, showing $US_{\mathcal{M}}U^* = S_{\rho(\mathcal{M})}$. Therefore, we have $U\overline{S_{\mathcal{M}}}U^* = \overline{US_{\mathcal{M}}U^*} = \overline{S_{\rho(\mathcal{M})}}$ by passing to the closures and hence we conclude

$$UJU^* = J_{\rho(\mathcal{M})} \tag{6.1}$$

by the uniqueness of the polar decomposition.

Proof. The contragredient $\mathcal{M}$-$\mathcal{M}$ bimodule $\overline{\mathcal{H}_\rho}$ is the opposite Hilbert space $\overline{L^2(\mathcal{M})}$ equipped with the $\mathcal{M}$-$\mathcal{M}$ action

$$m_1 \cdot \overline{\xi} \cdot m_2 = \overline{m_2^* \cdot \xi \cdot m_1^*} = \overline{\rho(m_2^*)Jm_1J\xi}.$$

On the other hand, the $\mathcal{M}$-$\mathcal{M}$ bimodule $\mathcal{H}_{\bar{\rho}}$ associated with $\bar{\rho} \in Sect(\mathcal{M})$ is the Hilbert space $L^2(\mathcal{M})$ with the $\mathcal{M}$-$\mathcal{M}$ action $\xi \mapsto \bar{\rho}(m_1)Jm_2^*J\xi$. Before the lemma we saw $\bar{\rho} = \rho^{-1} \circ \gamma = Ad(U^*J_{\rho(\mathcal{M})}J)$, and hence this action is

$$\begin{aligned}
\bar{\rho}(m_1)Jm_2^*J\xi &= U^*J_{\rho(\mathcal{M})}Jm_1JJ_{\rho(\mathcal{M})}UJm_2^*J\xi \\
&= JU^*Jm_1JUJJm_2^*J\xi \\
&= JU^*Jm_1JUm_2^*J\xi \\
&= JU^*Jm_1J\rho(m_2^*)UJ\xi \\
&= JU^*\rho(m_2^*)Jm_1JUJ\xi.
\end{aligned}$$

Here the last equality comes from the fact $\rho(m_2^*) \in \rho(\mathcal{M})$ ($\subseteq \mathcal{M}$) and $Jm_1J \in J\mathcal{M}J = \mathcal{M}'$ commute. Now it is plain to see that the surjective isometry $\xi \in L^2(\mathcal{M}) \mapsto \overline{UJ\xi} \in \overline{L^2(\mathcal{M})}$ gives rise to the desired unitary equivalence. □

Passing to the contragredient bimodule twice obviously means that one come back to the original bimodule so that we get

Corollary 6.3. *We have* $\bar{\bar{\rho}} = \rho$ *as sectors.*

What makes the use of bimodules so useful is the notion of relative tensor products. Namely, from two $\mathcal{M}$-$\mathcal{M}$ bimodules ${}_{\mathcal{M}}\mathcal{X}_{\mathcal{M}}, {}_{\mathcal{M}}\mathcal{Y}_{\mathcal{M}}$ one can construct a new $\mathcal{M}$-$\mathcal{M}$ bimodule ${}_{\mathcal{M}}\mathcal{X}_{\mathcal{M}} \otimes_{\mathcal{M}} {}_{\mathcal{M}}\mathcal{Y}_{\mathcal{M}}$, called the relative tensor product (over $\mathcal{M}$). The rigorous definition of this notion is based on the spatial theory ([5]) and hence somewhat involved (see [62]), however it is relatively easy to show (and it is intuitively trivial) that the standard Hilbert space ${}_{\mathcal{M}}L^2(\mathcal{M})_{\mathcal{M}}$ (i.e., $\mathcal{H} = \mathcal{H}_{id}$) with the standard left and right $\mathcal{M}$-actions (i.e., $m_1 \cdot \xi \cdot m_2 = m_1 J m_2^* J\xi$) is the multiplicative unit for the operation $\otimes_{\mathcal{M}}$:

$$ {}_{\mathcal{M}}L^2(\mathcal{M})_{\mathcal{M}} \otimes_{\mathcal{M}} {}_{\mathcal{M}}\mathcal{X}_{\mathcal{M}} = {}_{\mathcal{M}}\mathcal{X}_{\mathcal{M}} \otimes_{\mathcal{M}} {}_{\mathcal{M}}L^2(\mathcal{M})_{\mathcal{M}} = {}_{\mathcal{M}}\mathcal{X}_{\mathcal{M}}. $$

What is relevant to define the underlying Hilbert space structure for the relative tensor product are the "inside" $\mathcal{M}$-actions (i.e. $\mathcal{X}_{\mathcal{M}} \otimes_{\mathcal{M}} {}_{\mathcal{M}}\mathcal{Y}$), and the "outside" $\mathcal{M}$-actions are used to introduce an $\mathcal{M}$-$\mathcal{M}$ action on the relative tensor product. Anyway, the standard left and right $\mathcal{M}$-actions give us

$$ L^2(\mathcal{M})_{\mathcal{M}} \otimes_{\mathcal{M}} {}_{\mathcal{M}}L^2(\mathcal{M}) = L^2(\mathcal{M}), $$

and this is the only thing we need in what follows.

In the sector picture taking a relative tensor product simply corresponds the usual composition operation, which is a great advantage of the sector theory.

Lemma 6.4. *For* $\eta, \zeta \in Sect(\mathcal{M})$ *we have*

$$ \mathcal{H}_\eta \otimes_{\mathcal{M}} \mathcal{H}_\zeta = \mathcal{H}_{\zeta\eta} \quad (\textit{as } \mathcal{M}\text{-}\mathcal{M} \textit{ bimodules}). $$

Proof. For $\rho \in Sect(\mathcal{M})$, by $\mathcal{H}^\rho$ ($= \mathcal{H}$ as a Hilbert space) we denote the $\mathcal{M}$-$\mathcal{M}$ bimodule equipped with the following "opposite" action

$$ m_1 \cdot \xi \cdot m_2 = m_1 J\rho(m_2^*) J\xi. $$

We at first claim that $\mathcal{H}^{\bar{\rho}}$ and $\mathcal{H}_\rho$ are unitarily equivalent. In fact, by recalling

$\bar{\rho} = AdU^* J_{\rho(\mathcal{M})} J$, we see that the $\mathcal{M}$-$\mathcal{M}$ action on $\mathcal{H}^{\bar{\rho}}$ is given by

$$\begin{aligned} m_1 J \bar{\rho}(m_2^*) J \xi &= m_1 J U^* J_{\rho(\mathcal{M})} J m_2^* J J_{\rho(\mathcal{M})} U J \xi \\ &= m_1 J J U^* J m_2^* J U J J \xi \quad \text{(by (6.1))} \\ &= m_1 U^* J m_2^* J U \xi \\ &= U^* \rho(m_1) J m_2^* J U \xi, \end{aligned}$$

where the last equality comes from $\rho = AdU$. Therefore, the implementing unitary U (for ρ) gives rise to the desired unitary equivalence, and the claim is proved.

From the claim we see

$$\mathcal{H}_\eta \otimes_{\mathcal{M}} \mathcal{H}_\zeta = \mathcal{H}_\eta \otimes_{\mathcal{M}} \mathcal{H}^{\bar{\zeta}},$$

and note that the Hilbert spaces here are all $L^2(\mathcal{M})$ and the inside $\mathcal{M}$-actions in the above right side are the standard ones. Therefore, wee see that $\mathcal{H}_\eta \otimes_{\mathcal{M}} \mathcal{H}^{\bar{\zeta}}$ is the standard Hilbert space $L^2(\mathcal{M})$ equipped with the $\mathcal{M}$-$\mathcal{M}$ action

$$\eta(m_1) J \bar{\zeta}(m_2^*) J \xi$$

(see the paragraph before the lemma). Let V be the implementing unitary for ζ with $VJ = J_{\zeta(\mathcal{M})} V$ (see (6.1)). We repeat the computations in the first half to get

$$\begin{aligned} &\eta(m_1) J \bar{\zeta}(m_2^*) J \xi = \eta(m_1) J V^* J_{\zeta(\mathcal{M})} J m_2^* J J_{\zeta(\mathcal{M})} V J \xi \\ &\quad = \eta(m_1) J J V^* J m_2^* J V J J \xi = \eta(m_1) V^* J m_2^* J V \xi = V^* (\zeta\eta)(m_1) J m_2^* J V \xi. \end{aligned}$$

showing that V gives us a unitary equivalence between $\mathcal{H}_\eta \otimes_{\mathcal{M}} \mathcal{H}^{\bar{\zeta}}$ and $\mathcal{H}_{\zeta\eta}$. □

We point out that $\mathcal{H}_\eta \otimes_{\mathcal{M}} \mathcal{H}^{\bar{\zeta}}$ appearing in the above proof is also unitarily equivalent to $\mathcal{H}^{\bar{\eta}\bar{\zeta}}$. In fact, let W be the implementing unitary for η with $WJ = J_{\eta(\mathcal{M})} W$ (see (6.1)), and hence the $\mathcal{M}$-$\mathcal{M}$ action of $\mathcal{H}^{\bar{\eta}\bar{\zeta}}$ is

$$\begin{aligned} m_1 J (\bar{\eta}\bar{\zeta})(m_2^*) J \xi &= m_1 J W^* J_{\eta(\mathcal{M})} J \bar{\zeta}(m_2^*) J J_{\eta(\mathcal{M})} W J \xi \\ &= m_1 W^* J \bar{\zeta}(m_2^*) J W \xi \\ &= W^* \eta(m_1) J \bar{\zeta}(m_2^*) J W \xi. \end{aligned}$$

Note that this action is unitarily equivalent (via W) to the action appeared in the second half of the above proof (i.e., the one for $\mathcal{H}_\eta \otimes_{\mathcal{M}} \mathcal{H}^{\bar{\zeta}}$).

We notice that the formula $\mathcal{H}^{\eta} \otimes_{\mathcal{M}} \mathcal{H}^{\zeta} = \mathcal{H}_{\bar{\eta}} \otimes_{\mathcal{M}} \mathcal{H}^{\zeta} = \mathcal{H}^{\eta\zeta}$ is valid, by changing η, ζ to their conjugates (see Corollary 6.3). We then observe

$$\mathcal{H}_{\overline{\bar{\zeta}\bar{\eta}}} = \mathcal{H}^{\bar{\zeta}\bar{\eta}} = \mathcal{H}^{\bar{\zeta}} \otimes_{\mathcal{M}} \mathcal{H}^{\bar{\eta}} = \mathcal{H}_{\zeta} \otimes_{\mathcal{M}} \mathcal{H}_{\eta} = \mathcal{H}_{\eta\zeta}.$$

This means $\overline{\bar{\zeta}\bar{\eta}} = \mathcal{H}_{\eta\zeta}$ (as sectors), and by taking the conjugates of the both sides (recall Corollary 6.3), we get

Lemma 6.5. *We have $\bar{\zeta}\bar{\eta} = \overline{\eta\zeta}$ for $\eta, \zeta \in Sect(\mathcal{M})$.*

This result corresponds to the well-known fact $\overline{\mathcal{X} \otimes_{\mathcal{M}} \mathcal{Y}} = \overline{\mathcal{Y}} \otimes_{\mathcal{M}} \overline{\mathcal{X}}$ for bimodules.

The algebra of self-intertwiners of an $\mathcal{M}$-$\mathcal{M}$ bimodule $\mathcal{H}_{\rho}$ ($\rho \in Sect(\mathcal{M})$) is the relative commutant

$$\begin{aligned} Hom(\mathcal{H}_{\rho}, \mathcal{H}_{\rho}) &= \{T \in \mathcal{B}(\mathcal{H}_{\rho});\ T(m_1 \cdot \xi \cdot m_2) = m_1 \cdot T\xi \cdot m_2\} \\ &= \mathcal{M} \cap \rho(\mathcal{M})'. \end{aligned}$$

In fact, the compatibility between the right $\mathcal{M}$-actions forces $T \in (J\mathcal{M}J)' = J\mathcal{M}'J = \mathcal{M}$ as was pointed out before, and then the compatibility between the left $\mathcal{M}$-action implies $T\rho(m) = \rho(m)T$ for each $m \in \mathcal{M}$, i.e., $T \in \mathcal{M} \cap \rho(\mathcal{M})'$. The above fact means that $\mathcal{H}_{\rho}$ is irreducible as an $\mathcal{M}$-$\mathcal{M}$ bimodule if and only if $\mathcal{M} \cap \rho(\mathcal{M})' = \mathbf{C}1$. In this case, ρ is called an irreducible sector.

From now on we just consider sectors ρ such that the inclusion $\mathcal{M} \supseteq \rho(\mathcal{M})$ admits a normal conditional expectation E (and most often we consider the case when $IndE < \infty$). The square root of the minimal index $[\mathcal{M} : \rho(\mathcal{M})]_0$ (see §3.3) of the above inclusion is called the statistical dimension of ρ

$$d\rho = \sqrt{[\mathcal{M} : \rho(\mathcal{M})]_0} \ \left(\in \left\{ 2\cos\left(\frac{\pi}{n}\right); n = 3, 4, \cdots \right\} \cup [2, +\infty] \right),$$

which is by definition the dimension $\dim \mathcal{H}_{\rho}$ of the $\mathcal{M}$-$\mathcal{M}$ bimodule $\mathcal{H}_{\rho}$. Let $\mathcal{M} \supseteq \mathcal{N} \supseteq \mathcal{L}$ be a two-step inclusions of factors, and $E : \mathcal{M} \longrightarrow \mathcal{N}$, $F : \mathcal{N} \longrightarrow \mathcal{L}$ be the minimal expectations. The composition $F \circ E$ being minimal as pointed out in §3.3, the minimal index is multiplicative

$$[\mathcal{M} : \mathcal{L}]_0 = Ind\ (F \circ E) = (Ind\ E) \times (Ind\ F) = [\mathcal{M} : \mathcal{N}]_0 [\mathcal{N} : \mathcal{L}]_0.$$

Let $\rho_1, \rho_2 \in Sect(\mathcal{M})$ (or more precisely their representatives from $End(\mathcal{M})$) so that $\mathcal{M} \supseteq \rho_1(\mathcal{M}) \supseteq \rho_1\rho_2(\mathcal{M})$. The second inclusion is conjugate to $\mathcal{M} \supseteq \rho_2(\mathcal{M})$

(via ρ_1), and they obviously have the same minimal index. Therefore, we get the multiplicativity of statistical dimensions

$$d(\rho_1\rho_2) = d\rho_1\, d\rho_2.$$

6.1. Irreducible Decomposition of Sectors. At first we explain what the (direct) sum of sectors is. For $\rho_1, \rho_2 \in Sect(\mathcal{M})$,

$$m \in \mathcal{M} \longrightarrow \begin{pmatrix} \rho_1(m) & 0 \\ 0 & \rho_2(m) \end{pmatrix} \in \mathcal{M} \otimes M_2(\mathbf{C})$$

is a homomorphism. Let $v_1, v_2 \in \mathcal{M}$ be isometries with orthogonal ranges summing up to I. Of course they give rise to the usual identification map $\Phi : \mathcal{M} \otimes M_2(\mathbf{C}) \longrightarrow \mathcal{M}$ given by $[y_{ij}] \mapsto \sum_{i,j} v_i y_{ij} v_j^*$ (with the inverse $\Phi^{-1}(x) = [v_i^* x v_j]$). The above homomorphism followed by Φ is

$$\rho :\ m \in \mathcal{M} \longrightarrow v_1\rho_1(m)v_1^* + v_2\rho_2(m)v_2^* \in \mathcal{M},$$

which of course gives us an element in $End(\mathcal{M})$. The class of ρ depends only on the classes $[\rho_1], [\rho_2]$ (and not on the choice of v_i's). The sum $\rho_1 \oplus \rho_2$ is defined as (the class of) ρ. Via Φ the inclusion $\mathcal{M} \supseteq \rho(\mathcal{M})$ is conjugate to

$$\mathcal{M} \otimes M_2(\mathbf{C}) \supseteq \left\{ \begin{pmatrix} \rho_1(m) & 0 \\ 0 & \rho_2(m) \end{pmatrix} ;\ m \in \mathcal{M} \right\}$$

(so that the minimal index of this inclusion is $(d\rho)^2$). Notice that the projections $p_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are in the relative commutant. It is obvious that the reduced inclusions arising from p_1, p_2 are $\mathcal{M} \supseteq \rho_1(\mathcal{M})$ and $\mathcal{M} \supseteq \rho_2(\mathcal{M})$ respectively. As was seen at the end of §3.3 (see (3.8)), we have $d\rho_i = d\rho \times E(p_i)$ $(i = 1, 2)$ (where E is the relevant expectation) and get the additivity of statistical dimensions

$$d(\rho_1 \oplus \rho_2) = d\rho_1 + d\rho_2.$$

When $d\rho < \infty$, the inclusion $\mathcal{M} \supseteq \rho(\mathcal{M})$ is of finite index so that the relative commutant $\mathcal{M} \cap \rho(\mathcal{M})'$ (i.e., the algebra of self-intertwiners of $\mathcal{H}_\rho$) is finite-dimensional, and hence (as in the representation theory) one can perform the irreducible decomposition for ρ as will be seen below.

Let $\{p_i\}_{i=1,2,\cdots,n}$ be minimal projections in $\mathcal{M} \cap \rho(\mathcal{M})'$ summing up to I. Then we choose and fix isometries $v_i \in \mathcal{M}$ $(i = 1, 2, \cdots, n)$ satisfying $v_i v_i^* = p_i$. We set

$\rho_i = v_i^*\rho(\cdot)v_i \in Sect(\mathcal{M})$ (it is an endomorphism since $p_i = v_iv_i^*$ is in the relative commutant). Assume $x \in \mathcal{M}$ satisfies the intertwining property $x\rho_i(m) = \rho_i(m)x$ for each $m \in \mathcal{M}$. Then, we have

$$\begin{aligned} xv_i^*\rho(m)v_i = v_i^*\rho(m)v_ix &\implies v_ixv_i^*\rho(m)p_i = p_i\rho(m)v_ixv_i^* \\ &\implies v_ixv_i^*\rho(m) = \rho(m)v_ixv_i^*, \end{aligned}$$

showing $v_ixv_i^* \in {}_{p_i}(\mathcal{M} \cap \rho(\mathcal{M})')_{p_i}$. Thus, $v_ixv_i^* = \lambda p_i$ ($\lambda \in \mathbf{C}$) by the minimality of p_i and $x = v_i^*(v_ixv_i^*)v_i = \lambda I$. Therefore, we have seen that each ρ_i is an irreducible sector. The obvious intertwining property $v_i\rho_i(\cdot) = \rho(\cdot)v_i$ shows

$$\sum_{i=1}^n v_i\rho_i(m)v_i^* = \sum_{i=1}^n \rho(m)v_iv_i^* = \rho(m),$$

which means

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n.$$

When p_i, p_j belong to different central summands in $\mathcal{M} \cap \rho(\mathcal{M})'$, then they are disjoint (i.e, admit no non-trivial intertwiner). In fact, an intertwiner x between ρ_i and ρ_j satisfies $v_ixv_j^*$ $(= p_iv_ixv_j^* = v_ixv_j^*p_j) \in \mathcal{M} \cap \rho(\mathcal{M})'$ as before. Then, the assumption forces $v_ixv_j^* = 0$ and hence $x = 0$. On the other hand, when p_i, p_j belong to the same central summand, on can find a unitary $u \in \mathcal{M} \cap \rho(\mathcal{M})'$ satisfying $up_iu^* = p_j$. The isometries uv_i and v_j have the same range p_j and $(uv_i)^*\rho(\cdot)(uv_i) = v_i^*\rho(\cdot)v_i = \rho_i$. Notice also that $U = v_j^*(uv_i)$ is a unitary and $U(uv_i)^*\rho(\cdot)(uv_i)U^* = v_j^*\rho(\cdot)v_j = \rho_j$ so that we have seen $\rho_i = \rho_j$ as sectors.

Note that an irreducible sector η appears in the irreducible decomposition of ρ (the standard notation $\eta \prec \rho$ will be used) if and only if one finds a non-zero intertwiner $v \in \mathcal{M}$: $v\eta(m) = \rho(m)v$ ($m \in \mathcal{M}$). In fact, when this condition is met, by taking unitaries $m \in \mathcal{M}$ we observe

$$\begin{aligned} vv^* &= v\eta(m)\eta(m^*)v^* = \rho(m)vv^*\rho(m^*), \\ v^*v &= v^*\rho(m^*)\rho(m)v = \eta(m^*)v^*v\eta(m). \end{aligned}$$

Thus, we see $vv^* \in \mathcal{M} \cap \rho(\mathcal{M})'$ and v^*v is a scalar (due to the irreducibility of η). Hence, when v is normalized ($||v|| = 1$), it is an isometry with the range projection in $\mathcal{M} \cap \rho(\mathcal{M})'$ and we get $\eta \prec \rho$ (i.e., one of the above ρ_i's is the same as η as a sector).

6.2. Basic Extensions. Let $\mathcal{M} \supseteq \mathcal{N}$ be an inclusion of (type III) factors with finite index acting on the standard Hilbert space $L^2(\mathcal{M})$. Recall that the basic extension is $\mathcal{M}_1 = J\mathcal{N}'J = JJ_{\mathcal{N}}\mathcal{N}J_{\mathcal{N}}J$, and the next basic extension (i.e., the one for $\mathcal{M}_1 \supseteq \mathcal{M}$) is

$$\mathcal{M}_2 = J_{\mathcal{M}_1}\mathcal{M}'J_{\mathcal{M}_1} = J_{\mathcal{M}_1}J\mathcal{M}JJ_{\mathcal{M}_1}.$$

Let ξ_0 be a common cyclic and separating vector for $\mathcal{M} \supseteq \mathcal{N}$, and we may assume $J = J_{\mathcal{M},\xi_0}$ and $J_{\mathcal{N}} = J_{\mathcal{N},\xi_0}$. As noted above, we have $\mathcal{M}_1 = U\mathcal{N}U^*$ with $U = JJ_{\mathcal{N}}$. Since $U\xi_0 = \xi_0$, it is also a cyclic and separating vector for $\mathcal{M}_1$. By considering the S-operator associated with $(\mathcal{M}_1, \xi_0)$ and the polar decomposition of the closure, we see $J_{\mathcal{M}_1} = UJ_{\mathcal{N}}U^*$, that is,

$$J_{\mathcal{M}_1} = JJ_{\mathcal{N}}J.$$

Hence, we compute

$$\mathcal{M}_2 = JJ_{\mathcal{N}}J(J\mathcal{M}J)JJ_{\mathcal{N}}J = JJ_{\mathcal{N}}\mathcal{M}J_{\mathcal{N}}J.$$

Similarly, we see $J_{\mathcal{M}_2} = (JJ_{\mathcal{N}})J(J_{\mathcal{N}}J)$ and

$$J_{\mathcal{M}_2}J_{\mathcal{M}_1} = (JJ_{\mathcal{N}}JJ_{\mathcal{N}}J)(JJ_{\mathcal{N}}J) = JJ_{\mathcal{N}}.$$

Therefore, we get

$$\begin{aligned}\mathcal{M}_3 &= J_{\mathcal{M}_2}(\mathcal{M}_1)'J_{\mathcal{M}_2} = J_{\mathcal{M}_2}J_{\mathcal{M}_1}\mathcal{M}_1J_{\mathcal{M}_1}J_{\mathcal{M}_2}\\ &= J_{\mathcal{M}_2}J_{\mathcal{M}_1}(JJ_{\mathcal{N}}\mathcal{M}J_{\mathcal{N}}J)J_{\mathcal{M}_1}J_{\mathcal{M}_2} = Ad(JJ_{\mathcal{N}})^2\mathcal{M},\end{aligned}$$

and so on. In this way we observe that the Jones tower is described by

$$\begin{array}{lllllll}\mathcal{N} & \subseteq & \mathcal{M} & \subseteq & \mathcal{M}_1 = Ad(JJ_{\mathcal{N}})(\mathcal{N}) & \subseteq & \mathcal{M}_2 = Ad(JJ_{\mathcal{N}})(\mathcal{M})\\ & & & \subseteq & \mathcal{M}_3 = Ad(JJ_{\mathcal{N}})^2(\mathcal{N}) & \subseteq & \mathcal{M}_4 = Ad(JJ_{\mathcal{N}})^2(\mathcal{M})\\ & & & \subseteq & \mathcal{M}_5 = Ad(JJ_{\mathcal{N}})^3(\mathcal{N}) & \subseteq & \mathcal{M}_6 = Ad(JJ_{\mathcal{N}})^3(\mathcal{M})\\ & & & & \cdots . & & \end{array}$$

Down-ward basic extensions are

$$\begin{aligned}&\mathcal{N}_1 = J_{\mathcal{N}}\mathcal{M}'J_{\mathcal{N}} = J_{\mathcal{N}}J\mathcal{M}JJ_{\mathcal{N}} = \gamma(\mathcal{M}),\\ &\mathcal{N}_2 = J_{\mathcal{N}}\mathcal{M}_1'J_{\mathcal{N}} = J_{\mathcal{N}}JJ_{\mathcal{N}}\mathcal{N}'J_{\mathcal{N}}JJ_{\mathcal{N}} = J_{\mathcal{N}}J\mathcal{N}JJ_{\mathcal{N}} = \gamma(\mathcal{N}),\\ &\mathcal{N}_3 = J_{\mathcal{N}}\mathcal{M}_2'J_{\mathcal{N}} = J_{\mathcal{N}}JJ_{\mathcal{N}}\mathcal{M}'J_{\mathcal{N}}JJ_{\mathcal{N}} = J_{\mathcal{N}}JJ_{\mathcal{N}}J\mathcal{M}JJ_{\mathcal{N}}JJ_{\mathcal{N}} = \gamma^2(\mathcal{M}),\end{aligned}$$

and so on. Therefore, down-ward basic extensions give us the following tunnel of factors:

$$\mathcal{M} \supseteq \mathcal{N} \supseteq \gamma(\mathcal{M}) \supseteq \gamma(\mathcal{N}) \supseteq \gamma^2(\mathcal{M}) \supseteq \gamma^2(\mathcal{N}) \supseteq \cdots .$$

We now assume $\rho(\mathcal{M}) = \mathcal{N}$ so that $\gamma = \rho\bar{\rho}$ and the above tunnel becomes

$$\mathcal{M} \supseteq \mathcal{N} = \rho(\mathcal{M}) \supseteq \rho\bar{\rho}(\mathcal{M}) \supseteq \rho\bar{\rho}\rho(\mathcal{M}) \supseteq \rho\bar{\rho}\rho\bar{\rho}(\mathcal{M}) \supseteq \cdots .$$

(In particular, we have $d\bar{\rho} = d\rho$.) The decomposition rules for $\rho,\ \rho\bar{\rho},\ \rho\bar{\rho}\rho,\ \rho\bar{\rho}\rho\bar{\rho},\ \cdots$ are described by the following algebras (of self-intertwiners):

$$\begin{array}{rcccl} \mathcal{M} \cap \rho(\mathcal{M})' & = & \mathcal{M} \cap \mathcal{N}' & \longleftrightarrow & \mathcal{M}_1 \cap \mathcal{M}' \\ \mathcal{M} \cap \rho\bar{\rho}(\mathcal{M})' & = & \mathcal{M} \cap \mathcal{N}_1' & \longleftrightarrow & \mathcal{M}_2 \cap \mathcal{M}' \\ \mathcal{M} \cap \rho\bar{\rho}\rho(\mathcal{M})' & = & \mathcal{M} \cap \mathcal{N}_2' & \longleftrightarrow & \mathcal{M}_3 \cap \mathcal{M}' \\ \cdots . & & & & \end{array}$$

via AdJ. Similarly, the decomposition rules for $\bar{\rho},\ \bar{\rho}\rho,\ \bar{\rho}\rho\bar{\rho},\ \bar{\rho}\rho\bar{\rho}\rho, \cdots$ correspond to

$$\begin{array}{rcccl} \mathcal{M} \cap \bar{\rho}(\mathcal{M})' & \cong & \rho(\mathcal{M}) \cap \rho\bar{\rho}(\mathcal{M})' & \longleftrightarrow & \mathcal{M} \cap \mathcal{N}' \\ \mathcal{M} \cap \bar{\rho}\rho(\mathcal{M})' & \cong & \rho(\mathcal{M}) \cap \rho\bar{\rho}\rho(\mathcal{M})' & \longleftrightarrow & \mathcal{M}_1 \cap \mathcal{N}' \\ \mathcal{M} \cap \bar{\rho}\rho\bar{\rho}(\mathcal{M})' & \cong & \rho(\mathcal{M}) \cap \rho\bar{\rho}\rho\bar{\rho}(\mathcal{M})' & \longleftrightarrow & \mathcal{M}_2 \cap \mathcal{N}' \\ \cdots . & & & & \end{array}$$

via $AdJ_{\mathcal{N}}$ (and the first identification $\cong$ is given by ρ).

From the discussions so far, we conclude that the decomposition rules for

$$\bar{\rho},\ \bar{\rho}\rho,\ \bar{\rho}\rho\bar{\rho},\ \bar{\rho}\rho\bar{\rho}\rho,\ \cdots ,$$

$$\rho,\ \rho\bar{\rho},\ \rho\bar{\rho}\rho,\ \rho\bar{\rho}\rho\bar{\rho},\ \cdots$$

are described by the principal and dual principal graphs respectively.

In Ocneanu's approach on subfactor analysis ([54, 55]), the following four kinds of bimodules are important:

$${}_{\mathcal{N}}L^2(\mathcal{M}_k)_{\mathcal{N}},\ {}_{\mathcal{M}}L^2(\mathcal{M}_k)_{\mathcal{N}},\ {}_{\mathcal{N}}L^2(\mathcal{M}_k)_{\mathcal{M}}, {}_{\mathcal{M}}L^2(\mathcal{M}_k)_{\mathcal{M}},$$

where $\{\mathcal{M}_k\}$ denotes the Jones tower. We point out that they correspond to

$$(\bar{\rho}\rho)^k,\ (\bar{\rho}\rho)^k\bar{\rho},\ (\rho\bar{\rho})^k\rho,\ (\rho\bar{\rho})^k$$

respectively. We note that $\mathcal{H}_\gamma$ arising from $\rho\bar{\rho} = \gamma \in Sect(\mathcal{M})$ is unitarily equivalent to the $\mathcal{M}$-$\mathcal{M}$ bimodule ${}_{\mathcal{M}}L^2(\mathcal{M}_1)_{\mathcal{M}}$. In fact, since $\mathcal{M}_1$ is also acting on $\mathcal{H} = L^2(\mathcal{M})$

standardly, we have $\mathcal{H} = L^2(\mathcal{M}_1)$. Since $J_{\mathcal{M}_1} = JJ_{\mathcal{N}}J$ as was seen before, the $\mathcal{M}$-$\mathcal{M}$ action on $\mathcal{H} = L^2(\mathcal{M}_1)$ is given by

$$m_1 \cdot \xi \cdot m_2 = m_1 J_{\mathcal{M}_1} m_2^* J_{\mathcal{M}_1} \xi = m_1 JJ_{\mathcal{N}} J m_2^* JJ_{\mathcal{N}} J\xi = JJ_{\mathcal{N}} \gamma(m_1) J m_2^* JJ_{\mathcal{N}} J\xi$$

because of $\gamma = Ad(J_{\mathcal{N}}J)$. Obviously this action is unitarily equivalent (via $J_{\mathcal{N}}J$) to that of $\mathcal{H}_\gamma$. We then see

$${}_{\mathcal{M}}L^2(\mathcal{M}_2)_{\mathcal{M}} = {}_{\mathcal{M}}L^2(\mathcal{M}_1)_{\mathcal{M}} \otimes_{\mathcal{M}} {}_{\mathcal{M}}L^2(\mathcal{M}_1)_{\mathcal{M}} = \mathcal{H}_\gamma \otimes_{\mathcal{M}} \mathcal{H}_\gamma = \mathcal{H}_{\gamma^2},$$

and so on ([57]). In this way, we see that $(\rho\bar{\rho})^k$ corresponds to ${}_{\mathcal{M}}L^2(\mathcal{M}_k)_{\mathcal{M}}$. Similar things can be done for other three kinds of bimodules with the identification between $\mathcal{M}$ and $\mathcal{N}$ via ρ, and full details are left to the reader.

We remark that $\rho\bar{\rho}$ $(= \gamma)$ always contains the identity sector. A "sector-theoretical" proof can be found in [51], but this happens because of the presence of the Jones projection. The Jones projection $e_{\mathcal{M}}$ $(= J_{\mathcal{M}_1} e_{\mathcal{M}} J_{\mathcal{M}_1})$ sits in $\mathcal{M}'$, which means that $e_{\mathcal{M}}$ is an intertwiner for the $\mathcal{M}$-$\mathcal{M}$ bimodule ${}_{\mathcal{M}}L^2(\mathcal{M}_1)_{\mathcal{M}}$. Note $e_{\mathcal{M}}(L^2(\mathcal{M}_1)) = L^2(\mathcal{M})$, which is nothing but the definition of the Jones projection. Note that the restriction of the $\mathcal{M}$-$\mathcal{M}$ action $m_1 J_{\mathcal{M}_1} m_2^* J_{\mathcal{M}_1}$ (on $L^2(\mathcal{M}_1)$) to the subspace is obviously the standard one on $L^2(\mathcal{M})$. Thus, the $\mathcal{M}$-$\mathcal{M}$ bimodule ${}_{\mathcal{M}}L^2(\mathcal{M}_1)_{\mathcal{M}}$ contains the trivial bimodule ${}_{\mathcal{M}}L^2(\mathcal{M})_{\mathcal{M}} = \mathcal{H}_{id}$. By using this fact for $\bar{\rho}$, we have $id \prec \bar{\rho}\rho$, i.e., there is an isometry $v \in \mathcal{M}$ satisfying the intertwining property $vm = \bar{\rho}\rho(m)v$ $(m \in \mathcal{M})$. A crucial observation in [51] is that from the intertwining property the map

$$m \in \mathcal{M} \longrightarrow \rho(v^* \bar{\rho}(m) v) \in \rho(\mathcal{M}) \tag{6.2}$$

is a conditional expectation.

6.3. Frobenius Reciprocity. As explained so far sectors arise naturally from a subfactor, and decomposition rules for them (fusion rules) contain enormous amount of information. A very useful fact on fusion rules is the following Frobenius reciprocity:

Lemma 6.6. *Let η, ζ, $\rho \in Sect(\mathcal{M})$ be irreducible sectors of finite statistical dimensions. Then $\rho \prec \eta\zeta$ if and only if $\zeta \prec \bar{\eta}\rho$.*

Proof. The statement is symmetric so that we just prove one direction, and let us assume the existence of an isometry u satisfying $u\rho(m) = \eta\zeta(m)u$ for each $m \in \mathcal{M}$. Consider the two-step inclusions

$$\mathcal{M} \supseteq \eta(\mathcal{M}) \supseteq \eta\zeta(\mathcal{M})$$

with (unique) conditional expectations E_1 and E_2. From the projection $p = uu^* \in \mathcal{M}\cap(\eta\zeta(\mathcal{M}))'$ we get the reduced inclusion $p\mathcal{M}p \supseteq (\eta\zeta(\mathcal{M}))p$, which can be rewritten as

$$u\mathcal{M}u^* \supseteq (\eta\zeta(\mathcal{M}))uu^* = u(\rho(\mathcal{M}))u^*.$$

Note that it is conjugate to $\mathcal{M} \supseteq \rho(\mathcal{M})$ via Adu^* so that the index of the reduced inclusion (i.e., the local index) is $(d\rho)^2$. The local index formula (§3.3) thus says $E_2\circ E_1(p) = \frac{d\rho}{d(\eta\zeta)} = \frac{d\rho}{d\eta d\zeta}$. Since $E_1(p)$ sits in $\eta(\mathcal{M})\cap(\eta\zeta(\mathcal{M}))' = \eta(\mathcal{M}\cap\zeta(\mathcal{M})') = \mathbf{C}I$, we actually have $E_1(p) = \frac{d\rho}{d\eta d\zeta}$.

Let v be an isometry satisfying $vm = \bar{\eta}\eta(m)v$ $(m \in \mathcal{M})$ so that we have $E_1(m) = \eta(v^*\bar{\eta}(m)v)$. We thus have

$$\frac{d\rho}{d\eta d\zeta} = E_1(uu^*) = \eta(v^*\bar{\eta}(uu^*)v).$$

Since η is faithful, this means $\frac{d\rho}{d\eta d\zeta} = v^*\bar{\eta}(u)\bar{\eta}(u^*)v$, and $\bar{\eta}(u^*)v$ is an isometry multiplied by the square root of $\frac{d\rho}{d\eta d\zeta}$. This is an intertwiner between ζ and $\bar{\eta}\rho$ by the following computation based on the intertwining property for u and that for v:

$$v^*\bar{\eta}(u)\bar{\eta}\rho(m) = v^*\bar{\eta}(u\rho(m)) = v^*\bar{\eta}(\eta\zeta(m)u) = v^*\bar{\eta}(\eta\zeta(m))\bar{\eta}(u) = \zeta(m)v^*\bar{\eta}(u).$$

□

The explicit construction of intertwiners in the above proof is important to deal with connections in the paragroup theory ([54, 55]).

Remark 6.7. The fact $\rho_1 \prec \rho_2 \iff \bar{\rho}_1 \prec \bar{\rho}_2$ (where the irreducibility of ρ_2 is not assumed) is obvious in the bimodule picture. But we can show this by repeating similar arguments as in the above proof, and we indeed construct an intertwiner in an explicit fashion. By symmetry we assume $u\rho_1(m) = \rho_2(m)u$ for an isometry $u \in \mathcal{M}$. With isometries $v, w \in \mathcal{M}$ satisfying

$$vm = \rho_1\bar{\rho}_1(m)v \quad \text{and} \quad wm = \bar{\rho}_2\rho_2(m)w,$$

we compute $uvm = u\rho_1\bar{\rho}_1(m)v = \rho_2\bar{\rho}_1(m)uv$. By applying $\bar{\rho}_2$ to the both sides, we get $\bar{\rho}_2(uv)\bar{\rho}_2(m) = \bar{\rho}_2\rho_2\bar{\rho}_1(m)\bar{\rho}_2(uv)$. Then, by hitting w^* from the left, we get

$$w^*\bar{\rho}_2(uv)\bar{\rho}_2(m) = w^*\bar{\rho}_2\rho_2\bar{\rho}_1(m)\bar{\rho}_2(uv) = \bar{\rho}_1(m)w^*\bar{\rho}_2(uv).$$

Hence, $\bar{\rho}_2(v^*u^*)w$ is an intertwiner between $\bar{\rho}_1$ and $\bar{\rho}_2$, and it remains to show that this intertwiner is non-zero. To see this, we recall that the isometry uv intertwines id and $\rho_2\bar{\rho}_1$. We set $q = uvv^*u^* \in \mathcal{M} \cap \rho_2\bar{\rho}_1(\mathcal{M})'$ and note $\rho_2\bar{\rho}_1(\mathcal{M})q = uv\mathcal{M}v^*u^*$. Therefore, the reduced inclusion $q\mathcal{M}q \supseteq \rho_2\bar{\rho}_1(\mathcal{M})q$ is trivial and hence $E_2E_1(q) = \frac{1}{d\rho_1 d\rho_2}$ by the local index formula, where $E_1 :\ \mathcal{M} \longrightarrow \rho_2(\mathcal{M})$ and $E_2 :\ \rho_2(\mathcal{M}) \longrightarrow \rho_2\bar{\rho}_1(\mathcal{M})$ are minimal expectations. Since $\rho_2(\mathcal{M}) \supseteq \rho_2\bar{\rho}_1(\mathcal{M})$ is irreducible, $E_1(q)$ is a scalar and $E_1(q) = \frac{1}{d\rho_1 d\rho_2}$. By making use of the expression (6.2) for E_1, we get

$$\frac{1}{d\rho_1 d\rho_2} = E_1(uvv^*u^*) = \rho_2(w^*\bar{\rho}_2(uvv^*u^*)w),$$

showing $\frac{1}{d\rho_1 d\rho_2} = w^*\bar{\rho}_2(uvv^*u^*)w$. Therefore, $\sqrt{d\rho_1 d\rho_2}\ \bar{\rho}_2(v^*u^*)w$ is an isometry intertwining $\bar{\rho}_1$ and $\bar{\rho}_2$.

Thanks to the remark, we also have the following dual version:

$$\rho \prec \eta\zeta \quad \Longleftrightarrow \quad \eta \prec \rho\bar{\zeta}.$$

In the principal and dual principal graphs, a vertex appearing at a certain level is known to appear in two steps below. In the sector (or bimodule) picture going two steps further down means to hit ρ and then $\bar{\rho}$ from the same side (or the other way around). Therefore, this phenomenon is exactly the Frobenius reciprocity. Also note that (if ρ is irreducible) the conjugate sector $\bar{\rho}$ is characterized as an irreducible sector η satisfying $id \prec \rho\eta$ (or equivalently $id \prec \eta\rho$).

Several versions of Frobenius reciprocity are known (see [67, 68] for example, where bimodules are considered). The following version for (not necessarily irreducible) sectors is quite handy:

Theorem 6.8. *For sectors $\zeta, \eta, \rho \in Sect(\mathcal{M})$ of finite statistical dimension, we have*

$$\dim Hom(\zeta\eta, \rho) = \dim Hom(\zeta, \rho\bar{\eta}) = \dim Hom(\eta, \bar{\zeta}\rho).$$

Here,

$$Hom(\rho_1, \rho_2) = \{x \in \mathcal{M};\ x\rho_1(m) = \rho_2(m)x \text{ for each } m \in \mathcal{M}\}$$

is the space of intertwiners.

The result readily follows from Lemma 6.6 since dim $Hom(\cdot, \cdot)$ is obviously additive in both variables (with respect to $\oplus$).

6.4. Applications of the Sector Theory. Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of factors with finite index. To compute many invariants for subfactor analysis, we may and do assume that $\mathcal{N}$ and $\mathcal{M}$ are isomorphic (type III) factors. In fact, one can find a factor $\mathcal{L}$ such that $\mathcal{N} \otimes \mathcal{L}$ and $\mathcal{M} \otimes \mathcal{L}$ are isomorphic (see Lemma 2.3, [52]) and then one can consider $\mathcal{N} \otimes \mathcal{L} \subseteq \mathcal{M} \otimes \mathcal{L}$ instead. In the rest we assume that $\mathcal{N}, \mathcal{M}$ are isomorphic so that $\mathcal{N} = \rho(\mathcal{M})$ with some $\rho \in End(\mathcal{M})$.

1. Non-occurence of E_7 and D_{2n+1}
When index is less than 4, the Dynkin diagrams A_n $(n \geq 3), D_{2n}$ $(n \geq 2), E_6, E_8$ occur as principal graphs. However, the Dynkin diagrams D_{2n+1}, E_7 cannot occur, and this fact can be easily seen from the sector theory.

Assume that $\mathcal{N} = \rho(\mathcal{M}) \subseteq \mathcal{M}$ possesses the Dynkin diagram D_5

Figure 10 (Dynkin diagram D_5)

Since the statistical dimension $d\rho$ is the Perron-Frobenius eigenvalue of the corresponding incidence matrix, we know

$$d\rho = 2\cos\left(\frac{\pi}{8}\right) \ \left(= \sqrt{2+\sqrt{2}}\right)$$

(see [16]). Since $\rho\bar{\rho} = id \oplus \alpha$, we see

$$d\alpha = (d\rho)^2 - 1 = 1 + \sqrt{2}.$$

We notice

$$\alpha\rho = \rho + \beta + \gamma, \ \beta\bar{\rho} = \alpha, \ \gamma\bar{\rho} = \alpha$$

(From the last two relations we see $d\beta = d\gamma = \frac{d\alpha}{d\rho} = \frac{(d\rho)^2-1}{d\rho}$. Then, the first relation says $((d\rho)^2 - 1)d\rho = d\rho + \frac{2((d\rho)^2-1)}{d\rho}$, from which we easily get $d\rho = \sqrt{2+\sqrt{2}}$.) Notice

$$d\beta = \frac{d\alpha}{d\rho} = \frac{1+\sqrt{2}}{\sqrt{2+\sqrt{2}}} = 1.306\cdots < 2$$

Therefore, this value has to be one of $2\cos\left(\frac{\pi}{n}\right)$'s ($n = 3, 4, \cdots$). However, this is impossible since the value is smaller than $\sqrt{2} = 2\cos\left(\frac{\pi}{4}\right)$, and hence actually the Dynkin diagram D_5 cannot occur.

The above simple but powerful argument was first used in [50], and is known as the "$2\cos(\pi/n)$-rule". We can also eliminate the Dynkin diagram E_7 based on the same rule. With a little bit more careful argument one can actually see that D_{2n+1}'s are also impossible, and details can be found in [29].

2. Goldman type theorems

Assume that a factor $\mathcal{M}$ is equipped with an outer action $\alpha : G \longrightarrow Aut(\mathcal{M})$ of a finite group G, and we consider the irreducible inclusion $\mathcal{N} = \mathcal{M}^G \subseteq \mathcal{M}$. Then, the basic extension $\mathcal{M}_1$ is the crossed product $\mathcal{M} \rtimes_\alpha G$ and $\mathcal{M}_2 \cap \mathcal{M}' \cong \ell^\infty(G)$ (see Theorem 4.3). Actually we have the converse.

Theorem 6.9. *Let $\mathcal{N} \subseteq \mathcal{M}$ be an irreducible inclusion of factors with index n. If $\mathcal{M}_2 \cap \mathcal{M}'$ is an n-dimensional abelian algebra, then we can construct an outer action on $\mathcal{M}$ of a group G of order n such that $\mathcal{N} = \mathcal{M}^G$.*

Proof. Let $\mathcal{N} = \rho(\mathcal{M})$ ($\rho \in End(\mathcal{M})$) be as usual, and recall that $\mathcal{M}_2 \cap \mathcal{M}'$ is the algebra of self-intertwiners of $\rho\bar{\rho}$. The assumption means

$$\rho\bar{\rho} = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n.$$

Since $d(\rho\bar{\rho}) = n$, each α_i is of statistical dimension 1, i.e., $\alpha_i \in Aut(\mathcal{M})$ (or more precisely $[\alpha_i] \in Aut(\mathcal{M})/Int(\mathcal{M})$). Generally, (when ρ is irreducible) a one-dimensional sector α satisfies $\alpha \prec \rho\bar{\rho}$ if and only if $\alpha\rho = \rho$ (thanks to the Frobenius reciprocity). From this characterization, we see that these one-dimensional α's form a (finite) group. Hence, the above $[\alpha_i]$'s form a group G of order n and we write

$$\rho\bar{\rho} = \sum_{g \in G}{}^{\oplus} [\alpha_g] \quad (\text{with } [\alpha_{g_1}][\alpha_{g_2}] = [\alpha_{g_1 g_2}]).$$

Since $[\alpha_g \rho] = [\rho] \in Sect(\mathcal{M})$, after inner perturbation we may and do choose a representative $\alpha_g \in Aut(\mathcal{M})$ in such a way that $\alpha_g \rho = \rho \in End(\mathcal{M})$ (and $\alpha_e = id_{\mathcal{M}}$). We claim that $\alpha : \ g \in G \longrightarrow \alpha_g \in Aut(\mathcal{M})$ is an (outer) action. In fact, we note

$\alpha_{g_1}\alpha_{g_2} = \alpha_{g_1g_2}$ as sectors so that $\alpha_{g_1}\alpha_{g_2} = Adu \circ \alpha_{g_1g_2}$ as endomorphisms (with some unitary $u \in \mathcal{M}$). From the choice of α_g's, for each $m \in \mathcal{M}$ we compute

$$u\rho(m)u^* = u\alpha_{g_1g_2}\rho(m)u^* = \alpha_{g_1}\alpha_{g_2}\rho(m) = \alpha_{g_1}\rho(m) = \rho(m).$$

Thus, $u \in \mathcal{M} \cap \rho(\mathcal{M})'$ is a scalar so that we see $\alpha_{g_1}\alpha_{g_2} = \alpha_{g_1g_2}$ as desired. Note $\mathcal{N} \subseteq \mathcal{M}^{(\alpha,G)} \subseteq \mathcal{M}$, and we conclude $\mathcal{N} = \mathcal{M}^{(\alpha,G)}$ (by comparing indices). □

When $[\mathcal{M} : \mathcal{N}] = 2$, we have $\mathcal{M} \cap \mathcal{N}' = \mathbf{C}I$ and $\mathcal{M}_2 \cap \mathcal{M}' = \mathbf{C} \oplus \mathbf{C}$. Therefore, the assumption in the theorem is automatically satisfied with $n = 2$ and hence $\mathcal{N} = \mathcal{M}^{\mathbf{Z}_2}$ or equivalently $\mathcal{M} = \mathcal{N} \rtimes \mathbf{Z}_2$, which is known as the Goldman theorem [15] (see [28, 30] for more sophisticated Goldman type results).

In the above example $\mathcal{N} = \mathcal{M}^G$, we had $\rho\bar{\rho} = \sum_{g\in G}{}^{\oplus}\alpha_g$ and $\alpha_g\rho = \rho$. Hence all the irreducible components appearing in powers of $(\rho\bar{\rho})$ already appear in $\rho\bar{\rho}$. Such an inclusion is called an inclusion of depth 2. When $\mathcal{N} \subseteq \mathcal{M}$ is an irreducible inclusion of depth 2, then $\mathcal{N}$ is the fixed-point subalgebra of $\mathcal{M}$ under a Kac algebra (whose dimension is $[\mathcal{M} : \mathcal{N}]$) action. This result is due to A. Ocneanu and proofs can be found in [12, 31, 53, 65].

3. Non-strongly outer automorphisms

Importance of automorphisms in $(\rho\bar{\rho})^n$ for subfactor analysis was first pointed out in [29] (see also [30, 32]). Here, we characterize automorphisms appearing in $(\rho\bar{\rho})^n$.

Let $\mathcal{N} \subseteq \mathcal{M}$ be as usual with a minimal expectation E. When an automorphism $\theta \in Aut(\mathcal{M}, \mathcal{N})$ is given, we have $\theta \circ E = E \circ \theta$ by the uniqueness of a minimal expectation. Thus θ is uniquely extended to an automorphism of the basic extension $\mathcal{M}_1$ (still denoted by θ) subject to the condition $\theta(e_{\mathcal{N}}) = e_{\mathcal{N}}$. Similarly, θ is extended to all the basic extensions $\{\mathcal{M}_n\}_{n=1,2\cdots}$. Let $\theta:\ G \longrightarrow Aut(\mathcal{M};\mathcal{N})$ be an (outer) action of a (discrete) group G, and consider the inclusion

$$\mathcal{B} = \mathcal{N} \rtimes_\theta G \subseteq \mathcal{A} = \mathcal{M} \rtimes_\theta G.$$

As was seen in Proposition 3.13, the n-th basic extension $\mathcal{A}_n$ is given by $\mathcal{A}_n = \mathcal{M}_n \rtimes_\theta G$ so that n-th relative commutant is

$$\mathcal{A}_n \cap \mathcal{B}' = (\mathcal{M}_n \rtimes_\theta G) \cap (\mathcal{N} \rtimes_\theta G)'.$$

We begin by computing $(\mathcal{M}_n \rtimes_\theta G) \cap \mathcal{N}'$. An element $x = \sum_{g \in G} x_g \lambda_g \in \mathcal{M}_n \rtimes_\theta G$ belongs to this relative commutant if and only if

$$\sum_{g \in G} n x_g \lambda_g = \sum_{g \in G} x_g \lambda_g n = \sum_{g \in G} x_g \theta_g(n) \lambda_g$$

for each $n \in \mathcal{N}$, that is, for each $g \in G$ we have $n x_g = x_g \theta_g(n)$ $(n \in \mathcal{N})$.

Definition 6.10. ([1, 60]) *An automorphism $\theta \in Aut(\mathcal{M}; \mathcal{N})$ is called strongly outer if the following condition is satisfied for each k: we must have $x = 0$*

(6.3) *whenever $x \in \mathcal{M}_k$ satisfies $nx = x\theta(n)$ for each $n \in \mathcal{N}$.*

An action $\theta : G \longrightarrow Aut(\mathcal{M}; \mathcal{N})$ is called strongly outer if each automorphism θ_g $(g \neq e)$ is strongly outer.

The above discussion shows that when the action θ is strongly outer we have $(\mathcal{M}_n \rtimes_\theta G) \cap \mathcal{N}' = \mathcal{M}_n \cap \mathcal{N}'$ and hence

$$\mathcal{A}_n \cap \mathcal{B}' = (\mathcal{M}_n \cap \mathcal{N}')^\theta.$$

We begin with an even integer $k = 2n$ and assume that $x \in \mathcal{M}_{2n}$ satisfies (6.3), that is, $x\theta\rho(y) = \rho(y)x$ for $y \in \mathcal{M}$. Since $\mathcal{M}_{2n} = Ad(JJ_{\mathcal{N}})^n(\mathcal{M})$ (see §6.2), this means that $\tilde{x} = Ad(J_{\mathcal{N}}J)^n(x)$ in $\mathcal{M}$ satisfying $\tilde{x}(\rho\bar{\rho})^n\theta\rho(y) = (\rho\bar{\rho})^n\rho(y)\tilde{x}$, i.e., $\tilde{x} \in Hom((\rho\bar{\rho})^n\theta\rho, (\rho\bar{\rho})^n\rho)$. The Frobenius reciprocity says

$$\dim Hom((\rho\bar{\rho})^n\theta\rho, (\rho\bar{\rho})^n\rho) = \dim Hom((\rho\bar{\rho})^n\theta, (\rho\bar{\rho})^{n+1}) = \dim Hom(\theta, (\rho\bar{\rho})^{2n+1}).$$

Hence, the condition in Definition 6.10 breaks for $\mathcal{M}_{2n}$ if and only if $\theta \prec (\rho\bar{\rho})^{2n+1}$.

The above characterization remains valid regardless of the parity of k. In fact, let us assume that $x \in \mathcal{M}_{2n+1}$ satisfies (6.3). Then $\tilde{x} = Ad(J_{\mathcal{N}}J)^{n+1}(x) \in \mathcal{N}$ satisfies

$$\tilde{x}(\rho\bar{\rho})^{n+1}\theta\rho(y) = (\rho\bar{\rho})^{n+1}\rho(y)\tilde{x} \quad \text{(for each } y \in \mathcal{M}).$$

By hitting ρ^{-1} to the above both sides, we see that $\tilde{\tilde{x}} = \rho^{-1}(\tilde{x}) \in \mathcal{M}$ satisfies

$$\tilde{\tilde{x}}(\bar{\rho}\rho)^n\bar{\rho}\theta\rho(y) = (\bar{\rho}\rho)^{n+1}(y)\tilde{\tilde{x}},$$

that is, $\tilde{\tilde{x}} \in Hom((\bar{\rho}\rho)^n\bar{\rho}\theta\rho, (\bar{\rho}\rho)^{n+1})$. Once again the Frobenius reciprocity shows that the condition in Definition 6.10 breaks if and only if $\theta \prec (\rho\bar{\rho})^{2n+2}$ thanks to

$$\begin{aligned}\dim Hom((\bar{\rho}\rho)^n\bar{\rho}\theta\rho, (\bar{\rho}\rho)^{n+1}) &= \dim Hom(\theta\rho, \rho(\bar{\rho}\rho)^{2n+1}) \\ &= \dim Hom(\theta, \rho(\bar{\rho}\rho)^{2n+1}\bar{\rho}) \\ &= \dim Hom(\theta, (\rho\bar{\rho})^{2n+2}).\end{aligned}$$

Therefore, we have shown

Theorem 6.11. ([1, 40, 42]) *An automorphism $\theta \in Aut(\mathcal{M}; \mathcal{N} = \rho(\mathcal{M}))$ is strongly outer if and only if θ does not appear in $\sqcup_k(\rho\bar{\rho})^k$. More precisely, the strong outerness of θ breaks at $\mathcal{M}_k$ if and only if $\theta \prec (\rho\bar{\rho})^{k+1}$.*

See [35] for “analytic characterization” of non-strongly outer automorphims and [60] for classification of strongly outer actions.

Appendix A. Conditional Expectations and Operator Valued Weights

For the reader's convenience basic facts on conditional expectations and operator valued weights are summarized here.

1. Connes' Radon-Nikodym cocycles

Let φ, φ_0 be (fns) weights on a von Neumann algebra $\mathcal{M}$. Then one can construct a continuous one-parameter family $\{(D\varphi; D\varphi_0)_t\}_{t\in\mathbf{R}}$ of unitaries in $\mathcal{M}$ satisfying $\sigma_t^{\varphi} = (D\varphi; D\varphi_0)_t\sigma_t^{\varphi_0}(\cdot)(D\varphi; D\varphi_0)_t^*$ and the following σ^{φ_0}-cocycle property:

$$(D\varphi; D\varphi_0)_{t+s} = (D\varphi; D\varphi_0)_t\sigma_t^{\varphi_0}((D\varphi; D\varphi_0)_s) \quad \text{for each } t, s \in \mathbf{R}.$$

The construction of these unitaries are based on the celebrated 2×2-matrix trick due to Connes, and $\{(D\varphi; D\varphi_0)_t\}_{t\in\mathbf{R}}$ is called Connes' Radon-Nikodym cocycle. Moreover, the converse theorem is also valid. Namely, if $\{u_t\}_{t\in\mathbf{R}}$ is a continuous one-parameter family of unitaries in $\mathcal{M}$ with the above σ^{φ_0}-cocycle property, then there exists a unique weight φ on $\mathcal{M}$ such that $u_t = (D\varphi; D\varphi_0)_t$.

2. Extended positive part

The extended positive part $\hat{\mathcal{M}}_+$ ([18]) is the set of all lower semi-continuous (in the σ-weak topology) maps $m : \ \mathcal{M}_*^+ \longrightarrow [0,\infty]$ such that

$$m(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 m(\varphi_1) + \lambda_2 m(\varphi_2) \quad \text{for } \varphi_i \in \mathcal{M}_*^+ \text{ and } \lambda_i \geq 0$$

(with the convention $0\times\infty = 0$). When $\mathcal{M} = L^\infty(X)$, its extended positive part is the set of $[0,\infty]$-valued functions.

Each $x \in \mathcal{M}_+$ admits the spectral decomposition $x = \int_0^\infty \lambda de_\lambda$, and the map

$$m_x : \ \varphi \in \mathcal{M}_*^+ \longrightarrow \varphi(x) = \int_0^\infty \lambda d\varphi(e_\lambda) \in [0,\infty)$$

gives rise to a ("finite") element in $\hat{\mathcal{M}}_+$. Therefore, we get the natural imbedding $\mathcal{M}_+ \subseteq \hat{\mathcal{M}}_+$. Conversely, each $m \in \hat{\mathcal{M}}_*$ admits the spectral decomposition

$$m(\varphi) = \int_0^\infty \lambda d\varphi(e_\lambda) + \infty\times\varphi(e_\infty) \quad (\varphi \in \mathcal{M}_*^+).$$

Note that the above right side becomes the familiar quantity $\int_0^\infty \lambda d\langle e_\lambda\xi, \xi\rangle + \infty \times \langle e_\infty\xi, \xi\rangle$ when $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and φ is a vector state ω_ξ. The projection $1 - e_\infty$ (in $\mathcal{M}$) corresponds to the closure of the subspace $\{\xi \in \mathcal{H};\ m(\omega_\xi) < \infty\}$.

3. Conditional expectations

In the rest we assume that an inclusion $\mathcal{M} \supseteq \mathcal{N}$ of von Neumann algebras is given. Let φ be a weight on $\mathcal{M}$. Takesaki's theorem is: If $\varphi \mid_{\mathcal{N}}$ is semi-finite on $\mathcal{N}$ and $\sigma_t^{\varphi}(\mathcal{N}) = \mathcal{N}$ for each $t \in \mathbf{R}$, there exists a conditional expectation $E : \mathcal{M} \to \mathcal{N}$ satisfying $\varphi \circ E = \varphi$. We have $\sigma_t^{\varphi} \mid_{\mathcal{N}} = \sigma_t^{\varphi|_{\mathcal{N}}}$, and furthermore E is uniquely determined by this requirement.

Let ψ be a weight on $\mathcal{M}$. Then the restriction of the modular automorphism $\sigma_t^{\psi\circ E}$ to $\mathcal{N}$ is σ_t^{ψ}. Thus, $\sigma_t^{\psi\circ E}$ leaves $\mathcal{N}$ invariant and induces an automorphism of $\mathcal{M} \cap \mathcal{N}'$. The restriction of $\sigma_t^{\psi\circ E}$ to $\mathcal{M} \cap \mathcal{N}'$ does not depend on ψ, and it is denoted by σ_t^{E}. For another conditional expectation E_1, the Radon-Nikodym cocycle $(D(\psi \circ E_1); D(\psi \circ E))_t$ belongs to $\mathcal{M} \cap \mathcal{N}'$ and it does not depend on ψ. This Radon-Nikodym cocycle is denoted by $(DE_1; DE)_t \in \mathcal{M} \cap \mathcal{N}'$.

Assume that there exists a conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. Let $\mathcal{E}(\mathcal{M},\mathcal{N})$ be the set of all conditional expectations from $\mathcal{M}$ onto $\mathcal{N}$. For each $E \in \mathcal{E}(\mathcal{M},\mathcal{N})$, its restriction to $\mathcal{M} \cap \mathcal{N}'$ is obviously a conditional expectation from $\mathcal{M} \cap \mathcal{N}'$ onto $\mathcal{Z}(\mathcal{N})$. The map

$$E \in \mathcal{E}(\mathcal{M},\mathcal{N}) \longrightarrow E \mid_{\mathcal{M}\cap\mathcal{N}'} \in \mathcal{E}(\mathcal{M} \cap \mathcal{N}', \mathcal{Z}(\mathcal{N}))$$

is bijective (Combes-Delaroche, [2]). In particular, when $\mathcal{N}$ is a factor, $\mathcal{E}(\mathcal{M},\mathcal{N})$ is parameterized by the (faithful) state space of $\mathcal{M} \cap \mathcal{N}'$. Hence, when $\mathcal{M} \cap \mathcal{N}' = \mathbf{C}I$ in addition, we have a unique conditional expectation (if any).

4. Operator valued weights

An operator valued weight $F : \mathcal{M}_+ \longrightarrow \hat{\mathcal{N}}_+$ (roughly speaking, an unbounded generalization of a conditional expectation) means an additive homogeneous map satisfying the bimodule property

$$F(ymy^*) = yF(m)y^* \quad \text{for } m \in \mathcal{M}_+ \text{ and } y \in \mathcal{N}.$$

Here, $\hat{\mathcal{N}}_+$ is the extended positive part in **2**. Notions such as normality, semi-finiteness, and faithfulness can be introduced as in the theory of weights, and we consider only normal semi-finite and faithful operator valued weights. Note that if $F(1) = \lambda I$ with $\lambda > 0$ then $\lambda^{-1}F(\cdot)$ is a conditional expectation (after linearly extended to $\mathcal{M}$ in the usual way). For a weight ψ on $\mathcal{N}$ the composition $\psi \circ F$ is a weight on $\mathcal{M}$ and $\sigma_t^{\psi \circ F} = \sigma_t^{\psi}$ on $\mathcal{N}$. The following important result is due to Haagerup ([18]): Let φ, ψ be weights on $\mathcal{M}, \mathcal{N}$ respectively. If $\sigma_t^{\varphi}(y) = \sigma_t^{\psi}(y)$ for $y \in \mathcal{N}$, then there exists a unique operator valued weight F satisfying $\varphi = \psi \circ F$.

For weights ψ, ψ_1 on $\mathcal{N}$ we have $\sigma_t^{\psi \circ F}(y) = \sigma_t^{\psi}(y)$ for each $y \in \mathcal{N}$ and $(D(\psi \circ F); D(\psi_1 \circ F))_t = (D\psi; D\psi_1)_t \in \mathcal{N}$. The modular automorphism $\sigma_t^F \in Aut(\mathcal{M} \cap \mathcal{N}')$ and the Radon-Nikodym cocycle $(DF_1; DF)_t \in \mathcal{M} \cap \mathcal{N}'$ are defined analogously.

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