Hyperplane arrangements and K-theory[1]

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Abstract. We study the $\mathbb{Z}_2$-equivariant K-theory of $\mathcal{M}(\mathcal{A})$, where $\mathcal{M}(\mathcal{A})$ is the complement of the complexification of a real hyperplane arrangement, and $\mathbb{Z}_2$ acts on $\mathcal{M}(\mathcal{A})$ by complex conjugation. We compute the rational equivariant K- and KO-rings of $\mathcal{M}(\mathcal{A})$, and we give two different combinatorial descriptions of a subring $Line(\mathcal{A})$ of the integral equivariant KO-ring, where $Line(\mathcal{A})$ is defined to be the subring generated by equivariant line bundles.

1 Introduction

Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^d$, and let $\mathcal{M}(\mathcal{A})$ denote the complement of $\mathcal{A}$ in $\mathbb{C}^d$. It is a fundamental problem in the study of hyperplane arrangements to investigate the extent to which the topology of $\mathcal{M}(\mathcal{A})$ is determined by the combinatorics (more precisely the pointed matroid) of $\mathcal{A}$. Perhaps the first major theorem in the subject is the celebrated result of Orlik and Solomon [OS], in which the cohomology ring of $\mathcal{M}(\mathcal{A})$ is shown to have a combinatorial presentation in terms of the pointed matroid. Our goal is to give a combinatorial description of the K-theory of $\mathcal{M}(\mathcal{A})$.

We will work only with hyperplane arrangements which are defined over the real numbers. Though restrictive, this hypothesis allows for more subtle constructions in both combinatorics and topology. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$, where H_i is the zero set of an affine linear map $\omega_i : \mathbb{R}^d \to \mathbb{R}$, and let $H_i^{\pm} = \omega_i^{-1}(\mathbb{R}^{\pm})$ be the corresponding *open* half-spaces in $\mathbb{R}^d$. On the combinatorial side, a real hyperplane arrangement determines a *pointed oriented matroid* [BLSWZ]. The pointed oriented matroid of $\mathcal{A}$ is characterized by two types of combinatorial data:

1. which subsets $S \subseteq \{1, \ldots, n\}$ have the property that $\bigcap_{i\in S} H_i$ is nonempty with codimension less than $|S|$, and

2. which pairs of subsets $S^+, S^- \subseteq \{1, \ldots, n\}$ have the property that $\bigcap_{i\in S^+} H_i^+ \cap \bigcap_{j\in S^-} H_j^- = \varnothing$.

On the topological side, the complement $\mathcal{M}(\mathcal{A})$ of the complexified arrangement carries an action of $\mathbb{Z}_2$, given by complex conjugation. This allows us to consider not only the ordinary algebraic invariants, but their $\mathbb{Z}_2$-equivariant analogues as well. The equivariant fundamental group and the equivariant cohomology ring have been studied in [Hu] and [Pr], respectively. In [Pr], we extend a theorem of Salvetti [Sa] to show that the pointed oriented matroid determines the equivariant homotopy type of $\mathcal{M}(\mathcal{A})$, hence the "extra" combinatorics and "extra" topology arising from the real structure on $\mathcal{A}$ go hand in hand.

Our main result is to give two combinatorial descriptions of the ring $Line(\mathcal{A})$, which we define to be the subring of the degree zero equivariant KO-ring $KO_{\mathbb{Z}_2}\big(\mathcal{M}(\mathcal{A})\big)$ generated by line bundles. We first present $Line(\mathcal{A})$ as a quotient of a polynomial ring, in a manner similar to our presentation of the equivariant cohomology ring of $\mathcal{M}(\mathcal{A})$ in [Pr]. One important difference is that the equivariant cohomology ring is only well behaved with coefficients in $\mathbb{Z}_2$, whereas $Line(\mathcal{A})$ is both interesting and computable over the integers. We then give a second description of $Line(\mathcal{A})$ as a subring of the equivariant KO-ring of the fixed point set $\mathcal{C}(\mathcal{A})$, the complement of the real arrangement. Since $\mathcal{C}(\mathcal{A})$ is a finite disjoint union of contractible spaces, its equivariant KO-ring is simply a direct sum of equivariant KO-rings of points.

[1] *MSC 2000 Subject Classification:* Primary 52C35, Secondary 19L47
[2] Partially supported by the Clay Mathematics Institute Liftoff Program

In Section 2, we also compute the more familiar rings $KO_{\mathbb{Z}_2}\big(\mathcal{M}(\mathcal{A})\big)$ and $K_{\mathbb{Z}_2}\big(\mathcal{M}(\mathcal{A})\big)$ after tensoring with the rational numbers (see Proposition 2.3, Remark 2.4, and Corollary 2.5). A dimension count reveals that these rings are strictly larger than the tensor product of $Line(\mathcal{A})$ with $\mathbb{Q}$; in other words, they are *not* entirely generated by line bundles. We find that these rings may be described purely in terms of the ordinary cohomology rings of $\mathcal{M}(\mathcal{A})$ and $\mathcal{C}(\mathcal{A})$, thus the only truly new invariants come from working over $\mathbb{Z}$.

Acknowledgments. I am grateful to Laurent Bartholdi, Allen Knutson, André Henriques, Greg Landweber, and Alexander Varchenko for helpful discussions.

2 Equivariant K-theory

Let X be a topological space equipped with an action of a group G. The *equivariant K-ring* $K_G(X)$ is defined to be the Grothendieck ring of G-equivariant complex vector bundles on X. More precisely, $K_G(X)$ is additively generated by G-equivariant complex vector bundles over X, modulo the ideal generated by elements of the form $\sum_{i=0}^{m}(-1)^i E^i$ for every exact sequence $0 \to E^1 \to E^2 \to \ldots \to E^m \to 0$. The multiplicative structure is given by the tensor product, and the trivial line bundle is the multiplicative identity. Then K_G is a contravariant functor from G-spaces to rings, and is constant on G-equivariant homotopy equivalence classes of G-spaces. Since every G-space X maps G-equivariantly to a point, $K_G(X)$ is naturally a module over $K_G(pt)$, the representation ring of G. Similarly, we may define the contravariant functor KO_G from G-spaces to rings, which takes a space X to its Grothendieck ring of G-equivariant *real* vector bundles, which is a module over the real representation ring $KO_G(pt)$.

We will say that a (not necessarily exact) complex of bundles $0 \to E^1 \to E^2 \to \ldots \to E^m \to 0$ *represents* the element $\sum_{i=0}^{m}(-1)^i E^i$ in either real or complex K-theory. Given two complexes, we may tensor them together and then add up the diagonals to make a third complex, and the class represented by the tensor product of the two complexes is equal to the product of the classes represented by each complex. If $E^\bullet$ and $F^\bullet$ are complexes of G-equivariant vector bundles, then the locus of points in X over which the tensor product $(E \otimes F)^\bullet$ fails to be exact is contained in the intersection of the loci over which $E^\bullet$ and $F^\bullet$ individually fail to be exact. In particular, given any two K-theory classes which may be represented by complexes that fail to be exact on disjoint sets, their product is equal to zero. This will be our principal means of identifying relations in $KO_G(X)$ (see Example 3.1 and Theorem 3.10).

In this paper we will be concerned only with the case $G = \mathbb{Z}_2$. Furthermore, we will restrict our attention in later sections to a subring $Line(X) \subseteq KO_{\mathbb{Z}_2}(X)$, which we define to be the subring additively generated by line bundles. Though not part of a generalized cohomology theory, $Line$ is a contravariant functor from $\mathbb{Z}_2$-spaces to rings, and $Line(X)$ is always a module over $Line(pt)$. The ring $Line(pt)$ is additively generated by the unit element 1, and the element $N \in Line(pt)$ representing the unique nontrivial one-dimensional representation of $\mathbb{Z}_2$, subject to the relation $N^2 = 1$. We will write $x = 1 - N$, so that we have

$$Line(pt) = \mathbb{Z}[x]/x(2-x).$$

For an arbitrary $\mathbb{Z}_2$-space X, we will abuse notation by writing $x \in Line(X)$ to denote the image of $x \in Line(pt)$.

Real $\mathbb{Z}_2$-equivariant line bundles on X are classified by the equivariant cohomology group $H^1_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$, with the isomorphism given by the first equivariant Stiefel-Whitney class. (The completely analogous statement for complex line bundles is proven in [GGK, C.6.3] as well as [HL, A.1].) Hence $Line(X)$ is isomorphic to a quotient of the group ring $\mathbb{Z}\big[H^1_{\mathbb{Z}_2}(X;\mathbb{Z}_2)\big]$ by relations that arise when two different sums of equivariant line bundles are isomorphic. The obvious advantage of working with this subring is that the group $H^1_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$ is often computable.

Despite the relative intractability of computing the more familiar rings $K_{\mathbb{Z}_2}(X)$ and $KO_{\mathbb{Z}_2}(X)$, it is not so hard to compute their rationalizations $K_{\mathbb{Z}_2}(X)_{\mathbb{Q}} := K_{\mathbb{Z}_2}(X)\otimes\mathbb{Q}$ and $KO_{\mathbb{Z}_2}(X)_{\mathbb{Q}} := KO_{\mathbb{Z}_2}(X)\otimes\mathbb{Q}$, especially in the case where X is the complement of a hyperplane arrangement. We include this computation here, though it will not be relevant to the rest of the paper.

Let $\sigma : X \to X$ be the involution given by the $\mathbb{Z}_2$-action. By definition, a $\mathbb{Z}_2$-equivariant vector bundle on X is an ordinary bundle E along with a choice of isomorphism $E \cong \sigma^* E$, hence the image of the forgetful map $fo : K_{\mathbb{Z}_2}(X)_{\mathbb{Q}} \to K(X)_{\mathbb{Q}}$ is equal to the invariant ring $K(X)^{\sigma^*}_{\mathbb{Q}}$. Let $x = 1 - N_{\mathbb{C}} \in K_{\mathbb{Z}_2}(X)_{\mathbb{Q}}$, where $N_{\mathbb{C}}$ is the complexification of the real line bundle N defined above.

Lemma 2.1 *The kernel of the forgetful map $fo : K_{\mathbb{Z}_2}(X)_{\mathbb{Q}} \to K(X)_{\mathbb{Q}}$ is generated by x.*

Proof: The element $x = 1 - N$ is clearly contained in the kernel of the forgetful map. To prove the other containment, we observe the fact that an equivariant bundle on a free G-space carries the same data as an ordinary bundle on the quotient, hence the nonequivariant K-ring $K(X)$ may be identified with the equivariant ring $K_{\mathbb{Z}_2}(X\times\mathbb{Z}_2)$, where $\mathbb{Z}_2$ acts diagonally (and therefore freely) on $X\times\mathbb{Z}_2$. In this picture, the forgetful map gets identified with the pullback along the projection $\pi : X\times\mathbb{Z}_2 \to X$.

Consider the pushforward $\pi_* : K_{\mathbb{Z}_2}(X\times\mathbb{Z}_2)_{\mathbb{Q}} \to K_{\mathbb{Z}_2}(X)_{\mathbb{Q}}$, taking a bundle E on $X\times\mathbb{Z}_2$ to $E|_{X\times\{1\}} \oplus E|_{X\times\{-1\}}$. It is easy to check that the equivariant structure on E defines a natural equivariant structure on $\pi_*(E)$, and that this pushforward satisfies the projection formula $\pi_*(\beta\cdot\pi^*\alpha) = \pi_*(\beta)\cdot\alpha$. Suppose that $\alpha \in \ker \pi^*$. Then

$$(2-x)\cdot\alpha = \pi_*(1)\cdot\alpha = \pi_*\left(\pi^*(\alpha)\right) = 0,$$

hence

$$\alpha = \frac{1}{2}(2-x+x)\alpha = \frac{1}{2}x\alpha$$

is a multiple of x. □

Remark 2.2 To prove Lemma 2.1 we did not really have to work over the rationals, we only had to invert 2. The analogous statement over the integers is false.

Proposition 2.3 *There is a ring isomorphism $K_{\mathbb{Z}_2}(X)_{\mathbb{Q}} \;\cong\; H^{2*}(X;\mathbb{Q})^{\sigma^*} \oplus\; K_{\mathbb{Z}_2}(X^{\sigma})_{\mathbb{Q}}/\langle x-2\rangle$.*

Remark 2.4 Suppose that $X = \mathcal{M}(\mathcal{A})$ is the complement of the complexification of a real hyperplane arrangement $\mathcal{A}$, and let σ be the involution given by complex conjugation. Then the cohomology ring of X is isomorphic to the Orlik-Solomon algebra of $\mathcal{A}$ [OS]. The involution σ^* acts by negation on the generators of the Orlik-Solomon algebra, and therefore the invariant ring $H^{2*}(X;\mathbb{Q})^{\sigma^*}$ is simply the even degree part of the Orlik-Solomon algebra. Let $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})^{\sigma}$ be the complement of the real arrangement. This space is a disjoint union of contractible pieces, hence $K_{\mathbb{Z}_2}(X^{\sigma})_{\mathbb{Q}}/\langle x-2\rangle$ is isomorphic a product of copies of $\mathbb{Q} = K_{\mathbb{Z}_2}(pt)_{\mathbb{Q}}/\langle x-2\rangle$ for each component. This ring is also known as the Varchenko-Gelfand ring $VG(\mathcal{A};\mathbb{Q})$ of locally constant $\mathbb{Q}$-valued functions on $\mathcal{C}(\mathcal{A})$.

Proof of Proposition 2.3: We begin by considering the map

$$K_{\mathbb{Z}_2}(X)_{\mathbb{Q}} \to K_{\mathbb{Z}_2}(X)_{\mathbb{Q}}/\langle x\rangle \oplus K_{\mathbb{Z}_2}(X)_{\mathbb{Q}}/\langle x-2\rangle$$

given by the two projections. This map is surjective because the generator of the kernel of the first projection maps to a unit in the second factor. It is also injective, because any element of the kernel is annihilated both by $2-x$ and by x, and therefore also by 2. By Lemma 2.1, $K_{\mathbb{Z}_2}(X)_{\mathbb{Q}}/\langle x\rangle$ is isomorphic to $K(X)_{\mathbb{Q}}^{\sigma^*}$, and $H^{2*}(X;\mathbb{Q})$ is isomorphic to $K(X)_{\mathbb{Q}}$ via the Chern character. Hence the first factor of $K_{\mathbb{Z}_2}(X)_{\mathbb{Q}}$ is isomorphic to $H^{2*}(X;\mathbb{Q})^{\sigma^*}$. The fact that $K_{\mathbb{Z}_2}(X)_{\mathbb{Q}}/\langle x-2\rangle \cong K_{\mathbb{Z}_2}(X^\sigma)/\langle x-2\rangle$ is a consequence of the localization theorem [AS, 3.4.1]. $\square$

Corollary 2.5 *There is a ring isomorphism* $KO_{\mathbb{Z}_2}(X)_{\mathbb{Q}} \cong H^{4*}(X;\mathbb{Q})^{\sigma^*} \oplus KO_{\mathbb{Z}_2}(X^\sigma)/\langle x-2\rangle$.

Proof: The complexification map from $KO_{\mathbb{Z}_2}(X)_{\mathbb{Q}}$ to $K_{\mathbb{Z}_2}(X)_{\mathbb{Q}}$ is injective, and its image may be identified with the fixed point set under the involution taking a complex vector bundle to its conjugate [Bo, p. 74]. (This involution is not to be confused with the involution σ^*.) On $H^{2k}(X)$, this involution translates into multiplication by $(-1)^k$, hence the invariant ring is $H^{4*}(X;\mathbb{Q})^{\sigma^*}$. $\square$

3 The quotient description of $\boldsymbol{Line(\mathcal{A})}$

Let $\omega_1,\ldots,\omega_n$ be a collection of affine linear functionals on $\mathbb{R}^d$, and let $\mathcal{A}=\{H_1,\ldots,H_n\}$ be the associated cooriented hyperplane arrangement. By the word cooriented, we mean that we have not only a collection of hyperplanes, but also a collection of positive open half spaces $H_i^+=\omega_i^{-1}(\mathbb{R}^+)$, along with their negative counterparts $H_i^-=\omega_i^{-1}(\mathbb{R}^-)$. Let $\mathcal{C}(\mathcal{A})=\mathbb{R}^d\smallsetminus\cup_{i=1}^n H_i$ be the complement of $\mathcal{A}$ in $\mathbb{R}^d$, and let $\mathcal{M}(\mathcal{A})=\mathbb{C}^d\smallsetminus\cup_{i=1}^n H_i^{\mathbb{C}}$ be its complexification. Then $\mathcal{M}(\mathcal{A})$ carries an action of $\mathbb{Z}_2$ given by complex conjugation, with fixed point set $\mathcal{C}(\mathcal{A})$. For each i, the complexification of ω_i restricts to a map $\mathcal{M}(\mathcal{A})\to\mathbb{C}^*$. We will abuse notation by calling this map ω_i as well.

The purpose of this section is to give a combinatorial presentation of the ring $Line(\mathcal{A}) := Line\big(\mathcal{M}(\mathcal{A})\big)$. We begin with the most basic example, where $\mathcal{A}$ consists of a single point on a line, and therefore $\mathcal{M}(\mathcal{A})=\mathbb{C}^*$. This example will be fundamental to understanding the general case, as all line bundles on a general $\mathcal{M}(\mathcal{A})$ will be constructed as tensor products of pullbacks of line bundles on $\mathbb{C}^*$ along the maps $\omega_i:\mathcal{M}(\mathcal{A})\to\mathbb{C}^*$.

Example 3.1 Let $\mathcal{A}$ consist of one point in $\mathbb{R}$, so that $\mathcal{M}(\mathcal{A})\cong\mathbb{C}^*$. Let N be the topologically trivial real line bundle on $\mathbb{C}^*$ with the nontrivial $\mathbb{Z}_2$-action at every fixed point (the pullback of the nontrivial $\mathbb{Z}_2$ line bundle over a point), so that $x=1-N$. Let L be the Möbius line bundle on $\mathbb{C}^*$, equipped with the $\mathbb{Z}_2$-action that restricts to the trivial action over $\mathbb{R}^-$ and the nontrivial action over $\mathbb{R}^+$, and put $e=1-L\in Line(\mathbb{C}^*)$. The equivariant Stiefel-Whitney classes of N and L generate $H^1_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$ [Pr], hence x and e generate $Line(\mathcal{A})$. The relations $N^2=L^2=1$ translate into $x^2=2x$ and $e^2=2e$. To obtain another relation, consider a pair of complexes

$$0\to 1\xrightarrow{g} L\to 0 \qquad\text{and}\qquad 0\to N\xrightarrow{g'} L\to 0$$

representing e and $x-e$, respectively. The map g is forced to be zero over $\mathbb{R}^+$, but we may choose it to be injective elsewhere. Similarly, we may choose g' to vanish only on $\mathbb{R}^-$. Tensoring these two complexes together, we obtain an exact complex representing $e(x-e)$, hence this class is trivial in $Line(\mathbb{C}^*)$. In Theorem 3.10 we will prove that these are all of the relations.

Let $\eta_i=\omega_i^* e\in Line(\mathcal{A})$. Equivariant line bundles on $\mathcal{M}(\mathcal{A})$ are classified by the group $H^1_{\mathbb{Z}_2}\big(\mathcal{M}(\mathcal{A});\mathbb{Z}_2\big)$, which is generated by the pullbacks of the equivariant Stiefel-Whitney classes of L and N along the various maps ω_i [Pr]. Then by naturality of the equivariant Stiefel-Whitney class, $Line(\mathcal{A})$ is generated multiplicitively by $\eta_1, \ldots, \eta_n$ and x.

Remark 3.2 We may rephrase this observation by saying that the pullback $\omega^* : Line\big((\mathbb{C}^*)^n\big) \to Line\big(\mathcal{M}(\mathcal{A})\big)$ along the map $\omega = (\omega_1, \ldots, \omega_n) : \mathcal{M}(\mathcal{A}) \to (\mathbb{C}^*)^n$ is surjective. Note that if $\operatorname{rk} \mathcal{A} = d$, then ω is an embedding, and ω^* is simply the restriction map. It seems reasonable to conjecture that the pullback $\omega^* : KO_G\big((\mathbb{C}^*)^n\big) \to KO_G\big(\mathcal{M}(\mathcal{A})\big)$ is surjective as well.

For any connected component $C \subseteq \mathcal{C}(\mathcal{A})$, let $h_C : Line(\mathcal{A}) \to Line(C) = \mathbb{Z}[x]/x(2-x)$ be the map given by restriction to C.

Lemma 3.3 *For all C, h_C takes η_i to x if $C \subseteq H_i^+$, and to 0 if $C \subseteq H_i^-$.*

Proof: Restricting to the real locus commutes with pulling back along ω_i, hence it is enough to see that $e|_{\mathbb{R}^+} = x$ and $e|_{\mathbb{R}^-} = 0$. This observation follows from the representation of e as a complex in Example 3.1. □

Definition 3.4 Let $P(\mathcal{A})$ be the ring $\mathbb{Z}[e_1, \ldots, e_n, x]/\mathcal{I}_\mathcal{A}$, where $\mathcal{I}_\mathcal{A}$ is generated by the following five families of relations. [3]

1) $x(2-x)$

2) $e_i(2-e_i) \quad \text{for} \quad i \in \{1, \ldots, n\}$

3) $e_i(e_i - x) \quad \text{for} \quad i \in \{1, \ldots, n\}$

4) $\displaystyle\prod_{i \in S^+} e_i \times \prod_{j \in S^-} (e_j - x) \quad \text{if} \quad \bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing$

5) $\displaystyle x^{-1}\left(\prod_{i \in S^+} e_i \times \prod_{j \in S^-} (e_j - x) - \prod_{i \in S^+} (e_i - x) \times \prod_{j \in S^-} e_j\right) \text{ if } \bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing$
and $\displaystyle\bigcap_{i \in S} H_i$ is nonempty with codimension less than $|S|$, where $S = S^+ \sqcup S^-$.

Remark 3.5 For the fourth and fifth families of generators of $\mathcal{I}_\mathcal{A}$, it is sufficient to consider only pairs of subsets $S^+, S^- \subseteq \{1, \ldots, n\}$ which are minimal with respect to the given conditions; the other relations are generated by these.

Remark 3.6 The notation that we have chosen is slightly abusive, as the ideal $\mathcal{I}_\mathcal{A}$ and the ring $P(\mathcal{A})$ depend not just on the hyperplane arrangement, but also on the choice of linear forms ω_i used to define the hyperplanes. If ω_i is scaled by a positive real number, nothing changes, but if it is scaled by a negative real number, the roles of H_i^+ and H_i^- are reversed. Let $\mathcal{A}^i$ be the same arrangement as $\mathcal{A}$ with the sign of ω_i reversed. Then $P(\mathcal{A})$ is isomorphic to $P(\mathcal{A}^i)$ via the map $e_i \mapsto x - e_i$, which justifies the abuse. This point will be revisited in the beginning of the proof of Theorem 3.10.

[3] Note that all of these relations are polynomial; the x^{-1} in the fifth family of relations cancels with a factor of x.

Remark 3.7 The generators of $\mathcal{I}_\mathcal{A}$ in Definition 3.4 are similar to the relations given by Varchenko and Gelfand in their presentation of the ring $VG(\mathcal{A};\mathbb{Z})$ [VG]. In fact, the ring $P(\mathcal{A})/\langle x-2\rangle$ is isomorphic to the subring of $VG(\mathcal{A};\mathbb{Z})$ generated by two times the Heaviside functions. We give an abstract characterization of this subring for simple arrangements in Section 4.

Definition 3.8 A *circuit* is a minimal set S such that $\cap_{i\in S}H_i$ is nonempty with codimension less than $|S|$. All circuits admit a unique decomposition $S = S^+ \sqcup S^-$ (up to permutation of the two pieces) such that

$$\bigcap_{i\in S^+} H_i^+ \cap \bigcap_{j\in S^-} H_j^- = \varnothing.$$

A set T is called a *broken circuit* if there exists i with $i<j$ for all $j\in T$ such that $T\cup\{i\}$ is a circuit. An *nbc-set* $A\subseteq\{1,\ldots,n\}$ is a set such that $\cap_{i\in A}H_i$ is nonempty and A does not contain a broken circuit.

For any subset $A\subseteq\{1,\ldots,n\}$, let $e_A=\prod_{i\in A}e_i$.

Lemma 3.9 *The ring $P(\mathcal{A})$ is additively a free abelian group of rank $R+1$, where R is the number of connected components of $\mathcal{C}(\mathcal{A})$.*

Proof: The set $\{x\}\cup\{e_A \mid A \text{ an nbc-set}\}$ is an additive basis for $P(\mathcal{A})$. The monomials indexed by nbc-sets also form a basis for the Orlik-Solomon algebra $A(\mathcal{A};\mathbb{Z})$, which is free-abelian of rank R (see for example [Yu, §2]). Hence $P(\mathcal{A})$ is free abelian of rank $R+1$. $\square$

In the following theorem, we show that the relations between the K-theory classes $\eta_1,\ldots,\eta_n,x\in Line(\mathcal{A})$ are exactly given by the ideal $\mathcal{I}_\mathcal{A}$.

Theorem 3.10 *The homomorphism $\phi:\mathbb{Z}[e_1,\ldots,e_n,x]\to Line(\mathcal{A})$ given by $\phi(e_i)=\eta_i$ and $\phi(x)=x$ is surjective with kernel $\mathcal{I}_\mathcal{A}$, hence $Line(\mathcal{A})$ is isomorphic to $P(\mathcal{A})$.*

Proof: We begin by reducing to the case in which the polyhedron $\Delta=\cap_{i=1}^n H_i^-$ is nonempty. To do this, let $\mathcal{A}^i$ be as in Remark 3.6. We then obtain a diagram as follows,

$$\begin{tikzcd} P(\mathcal{A}) \arrow[rr] \arrow[dr, "\Phi"'] & & P(\mathcal{A}^i) \arrow[dl, "\Phi^i"] \\ & Line(\mathcal{A}) & \end{tikzcd}$$

which commutes because $(-\omega_i)^*e = x-\eta_i$. This tells us that Φ is an isomorphism if and only if Φ^i is an isomorphism. By changing the signs of enough of the linear forms, we may achieve the condition that Δ is nonempty.

To see that $\mathcal{I}_\mathcal{A}$ is contained in the kernel of ϕ, we must show that each of the families of generators maps to zero. The images under ϕ of the first three families are all pullbacks of relations in $Line(\mathbb{C}^*)$, and are therefore zero in $Line(\mathcal{A})$.

Let

$$Y_i^+=\omega_i^{-1}(\mathbb{R}^+) \text{ and } Y_i^-=\omega_i^{-1}(\mathbb{R}^-)\subseteq\mathcal{M}(\mathcal{A}).$$

We have already observed that $e\in Line(\mathbb{C}^*)$ may be represented by a complex which is exact away from $\mathbb{R}^+$, therefore $\eta_i=\omega_i^*(e)$ may be represented by a complex which is exact away from Y_i^+. Similarly,

$\eta_i - x = \omega_i^*(e - x)$ may be represented by a complex which is exact away from Y_i^-. Suppose that

$$p \in \bigcap_{i \in S^+} Y_i^+ \cap \bigcap_{j \in S^-} Y_j^-.$$

Then the real part

$$\mathrm{Re}(p) \in \bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^-,$$

hence

$$\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing \;\;\Rightarrow\;\; \bigcap_{i \in S^+} Y_i^+ \cap \bigcap_{j \in S^-} Y_j^- = \varnothing.$$

In this case $\prod_{i \in S^+} \eta_i \times \prod_{j \in S^-} (\eta_j - x)$ is represented by an exact complex, and is therefore equal to zero. This accounts for the fourth family of generators of $\mathcal{I}_{\mathcal{A}}$.

Now suppose given a circuit $S = S^+ \sqcup S^- \subseteq \{1, \ldots, n\}$ with $\left(\bigcap_{i \in S^+} H_i^+\right) \cap \left(\bigcap_{j \in S^-} H_j^-\right) = \varnothing$, and consider the arrangement $\mathcal{A}_S = \{H_i^{\mathbb{C}} \mid i \in S\}$. The space

$$\mathcal{M}(\mathcal{A}_S) = \mathbb{C}^d \smallsetminus \bigcup_{i \in S} H_i$$

contains the space $\mathcal{M}(\mathcal{A})$, and we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[e_i, x]_{i \in S} & \xrightarrow{\phi_S} & \mathit{Line}(\mathcal{A}_S) \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}[e_1, \ldots, e_n, x] & \xrightarrow{\phi} & \mathit{Line}(\mathcal{A}) \end{array}$$

where the map from $\mathit{Line}(\mathcal{A}_S)$ to $\mathit{Line}(\mathcal{A})$ is given by restriction. Hence to show that the class

$$x^{-1}\left(\prod_{i \in S^+} e_i \times \prod_{j \in S^-} (e_j - x) - \prod_{i \in S^+} (e_i - x) \times \prod_{j \in S^-} e_j\right)$$

is in the kernel of ϕ, it will suffice to show that it is in the kernel of ϕ_S. Dividing by the vector space $\bigcap_{i \in S} H_i$, which is a factor of $\mathcal{M}(\mathcal{A}_S)$, we obtain a homotopy equivalent space $\mathcal{M}(\hat{\mathcal{A}}_S)$, where $\hat{\mathcal{A}}_S$ is a central, essential arrangement of $|S|$ hyperplanes in a vector space of dimension $|S| - 1$. Thus we have reduced to the special case where $\mathcal{A}$ is a central arrangement of $d + 1$ generic hyperplanes in $\mathbb{R}^d$.

Let us number our hyperplanes $H_0, \ldots, H_d$. We have a splitting $\{0, \ldots, d\} = S^+ \sqcup S^-$ such that

$$\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing,$$

and the fact that $\Delta = \bigcap_{i=0}^d H_i^-$ is nonempty implies that either S^+ or S^- is a singleton consisting of the unique hyperplane that is not a facet of Δ. Without loss of generality, let us assume that $S^- = \{0\}$. Let $d\mathcal{A}$ be the decone of $\mathcal{A}$ with respect to H_0. More explicitly, $d\mathcal{A}$ is the cooriented arrangement of d affine hyperplanes in the affine space $V = \{p \in \mathbb{C}^d \mid \omega_0(p) = -1\}$ whose hyperplanes are cut out by the restrictions of $\omega_1, \ldots, \omega_d$ to V. (We ask that $\omega_0(p) = -1$ rather than 1 so that $\Delta \cap \mathcal{M}(d\mathcal{A})$ will be nonempty.)

The ring $\mathit{Line}(d\mathcal{A})$ is generated by $\nu_1, \ldots, \nu_d$ and x, where the generator ν_i corresponding to the hyperplane $H_i \cap V$ is equal to the restriction of η_i to $\mathcal{M}(d\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$. We have

$$\bigcap_{i=1}^{d}(H_i^+ \cap V) = (-\Delta) \cap V = \varnothing,$$

hence we have the relation $\prod_{i=1}^{d} \nu_i = 0 \in Line(d\mathcal{A})$.

Consider the map $f : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(d\mathcal{A})$ given by the formula $f(p) = -p/\omega_0(p)$.

Lemma 3.11 *For all $i \in \{1, \ldots, d\}$, $f^*(\nu_i) = \eta_0 + \eta_i - \eta_0\eta_i$.*

Proof: We will prove the equivalent statement that $1 - f^*(\nu_i) = (1 - \eta_0)(1 - \eta_i)$. From the definitions of x, ν_i, and η_i, we see that both sides of this equation can be represented by honest equivariant line bundles (rather than virtual bundles), hence we may interpret the statement as an equation in the Picard group of $\mathbb{Z}_2$-equivariant line bundles on $\mathcal{M}(\mathcal{A})$. This group is isomorphic to the cohomology group $H^1_{\mathbb{Z}_2}(\mathcal{M}(\mathcal{A}); \mathbb{Z}_2)$, which injects into $H^1_{\mathbb{Z}_2}(\mathcal{C}(\mathcal{A}); \mathbb{Z}_2)$ by the restriction map [Pr, 2.4 & 2.5]. Since the isomorphism between the Picard group and the first equivariant cohomology commutes with restriction, it is enough to prove that

$$h_C\left(1 - f^*(\nu_i)\right) = h_C\Big((1 - \eta_0)(1 - \eta_i)\Big)$$

for all components $C \subseteq \mathcal{C}(\mathcal{A})$. By Lemma 3.3, and the observation that $(1 - x)^2 = 1$, we have

$$h_C\Big((1 - \eta_0)(1 - \eta_i)\Big) = \begin{cases} 1 - x & \text{if } \omega_0 \text{ and } \omega_i \text{ take values of opposite sign on } C \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand, $f^*(\nu_i) = \left(-\frac{\omega_i}{\omega_0}\right)^*(e)$. Using the fact that restriction to the real locus commutes with pulling back, and the fact that $e|_{\mathbb{R}^+} = x$ and $e|_{\mathbb{R}^-} = 0$, we obtain the desired equality. $\square$

By Lemma 3.11, we have

$$0 = \prod_{i=1}^{d} f^*(\nu_i) = \prod_{i=1}^{d}(\eta_0 + \eta_i - \eta_0\eta_i) = \prod_{i=1}^{d}\Big(\eta_i(1 - \eta_0) + \eta_0\Big) = \sum_{A \subseteq \{1,\ldots,d\}} (1 - \eta_0)^{|A|} \cdot \eta_0^{|A^c|} \cdot \eta_A, \tag{1}$$

where $\eta_A = \prod_{i \in A} \eta_i$. Since $(1 - \eta_0)^2 = 1$ and $(1 - \eta_0) \cdot \eta_0 = -\eta_0$, we also have

$$(1 - \eta_0)^{|A|} \cdot \eta_0^{|A^c|} = \begin{cases} 1 - \eta_0 & \text{if } A = \{1, \ldots, d\} \text{ and } d \text{ is odd} \\ (-1)^{|A|} \cdot \eta_0^{|A^c|} & \text{otherwise.} \end{cases} \tag{2}$$

On the other hand, consider the expression

$$x^{-1}\left(\eta_0 \prod_{i=1}^{d}(\eta_i - x) - (\eta_0 - x)\prod_{i=1}^{d}\eta_i\right),$$

which may be rewritten as

$$\sum_{A \subseteq \{1,\ldots,d\}} (-\eta_0)^{|A^c|} \cdot \eta_A \;=\; (-1)^d \cdot \sum_{A \subseteq \{1,\ldots,d\}} (-1)^{|A|} \cdot \eta_0^{|A^c|} \cdot \eta_A.$$

By Equations (1) and (2), this expression is equal to zero if d is even, and otherwise equal to

$$\prod_{i=1}^{d}\eta_i + (1-\eta_0)\prod_{i=1}^{d}\eta_i \;=\; (2-\eta_0)\prod_{i=1}^{d}\eta_i \;=\; (x-\eta_0)\prod_{i=1}^{d}\eta_i.$$

But $H_0^- \cap \bigcap_{i=1}^{d} H_i^+ = \varnothing$, therefore $(x-\eta_0)\prod_{i=1}^{d}\eta_i = 0$ from the fourth family of relations. Hence we have shown that

$$x^{-1}\left(\eta_0\prod_{i=1}^{d}(\eta_i - x) - (\eta_0 - x)\prod_{i=1}^{d}\eta_i\right) = 0,$$

and therefore that all of the generators of $\mathcal{I}_{\mathcal{A}}$ are contained in the kernel of ϕ.

Our work up to this point implies that ϕ descends to a surjection $\hat{\phi} : P(\mathcal{A}) \to \mathit{Line}(\mathcal{A})$; it remains to show that $\hat{\phi}$ is injective. We prove instead the following stronger statement. Let $h : \mathit{Line}(\mathcal{A}) \to \mathit{Line}\big(\mathcal{C}(\mathcal{A})\big)$ be the restriction to the fixed point set. (The ring $\mathit{Line}\big(\mathcal{C}(\mathcal{A})\big)$ is a direct sum one copy of $\mathbb{Z}[x]/x(2-x)$ for each component $C \subseteq \mathcal{C}(\mathcal{A})$, and h is the direct sum of the maps h_C.)

Lemma 3.12 *The composition $h \circ \hat{\phi} : P(\mathcal{A}) \to \mathit{Line}\big(\mathcal{C}(\mathcal{A})\big)$ is injective.*

Proof: By Lemma 3.9, it is enough to prove injectivity after tensoring with the rational numbers $\mathbb{Q}$. Given any component $C \subseteq \mathcal{C}(\mathcal{A})$, choose a pair of subsets $S^+, S^- \subseteq \{1, \ldots, n\}$ such that $\left(\bigcap_{i\in S^+} H_i^+\right) \cap \left(\bigcap_{j\in S^-} H_j^-\right) = C$. Then for any other component $D \subseteq \mathcal{C}(\mathcal{A})$, Lemma 3.3 tells us that

$$h_D\left(\prod_{i\in S^+}\eta_i \cdot \prod_{j\in S^-}(x-\eta_j)\right) = \delta_{CD}\cdot x^{|S^+\cup S^-|},$$

hence

$$h\left(\prod_{i\in S^+}\eta_i \cdot \prod_{j\in S^-}(x-\eta_j)\right)$$

is supported on a single component of $\mathcal{C}(\mathcal{A})$. The R elements obtained this way, along with the trivial vector bundle 1, generate an $(R+1)$-dimensional subspace of $\mathit{Line}\big(\mathcal{C}(\mathcal{A})\big) \otimes \mathbb{Q}$. Since $\dim P(\mathcal{A}) \otimes \mathbb{Q} = R+1$, $h \circ \hat{\phi}$ must be injective. □

Injectivity of $h \circ \hat{\phi}$ implies injectivity of $\hat{\phi}$, therefore $\hat{\phi} : P(\mathcal{A}) \to \mathit{Line}(\mathcal{A})$ is an isomorphism. This completes the proof of Theorem 3.10. □

4 The subring description of *Line*($\mathcal{A}$)

An arrangement $\mathcal{A}$ is called *simple* if $\operatorname{codim} \cap_{i\in S} H_i = |S|$ for any S such that $\cap_{i\in S} H_i$ is nonempty. By Theorem 3.10 and Lemma 3.12, we know that $P(\mathcal{A}) \cong \mathit{Line}(\mathcal{A})$ is isomorphic to a subring of $\mathit{Line}\big(\mathcal{C}(\mathcal{A})\big)$. In this section we give a combinatorial interpretation of that subring in the special case where the arrangement $\mathcal{A}$ is simple, which we will assume for the rest of the section.

The arrangement $\mathcal{A}$ divides $\mathbb{R}^d$ into a polytopal complex $|\mathcal{A}|$ whose maximal faces are the connected components $C \subseteq \mathcal{C}(\mathcal{A})$, and whose smaller faces are the open faces of the polytopes $\bar{C}$. Given any face

$F \in |\mathcal{A}|$, we let $\mathfrak{C}_F$ denote the set of maximal faces C containing F in their closure, and we choose a sign function $\epsilon_F : \mathfrak{C}_F \to \{\pm 1\}$ such that any two maximal faces separated by a single hyperplane receive a different sign.

Definition 4.1 Let $B(\mathcal{A})$ be the subgroup of $\mathit{Line}\big(\mathcal{C}(\mathcal{A})\big) \cong \bigoplus_{C \subseteq \mathcal{C}(\mathcal{A})} \mathbb{Z}[x]/x(2-x)$ defined by the following condition.

$$\mu \in B(\mathcal{A}) \text{ if and only if for all faces } F \in |\mathcal{A}|, \sum_{C \in \mathfrak{C}_F} \epsilon_F(C)\mu_C \in \left\langle x^{\operatorname{codim} F} \right\rangle \subseteq \mathbb{Z}[x]/x(2-x).$$

It is clear that $B(\mathcal{A})$ is a subgroup. It is not so obvious that it is also a subring; this fact will follow from Theorem 4.5.

Proposition 4.2 *The image of the restriction map $h : \mathit{Line}(\mathcal{A}) \to \mathit{Line}\big(\mathcal{C}(\mathcal{A})\big)$ is contained in $B(\mathcal{A})$.*

Proof: We need only check that $h(\eta_A) \in B(\mathcal{A})$ for all subsets $A \subseteq \{1, \ldots, n\}$. Choose a face $F \in |A|$, and a component $C \in \mathfrak{C}_F$. Lemma 3.3 tells us that

$$h_C(\eta_A) = \begin{cases} x^{|A|} & \text{if } C \subseteq \bigcap_{i \in A} H_i^+ \\ 0 & \text{otherwise.} \end{cases}$$

If $|A| \geq \operatorname{codim} F$, we are done. If not, then by simplicity, there must be an index j such that $F \subseteq H_j$ but $j \notin A$. In this case, $h_C(\eta_A) = h_D(\eta_A)$, where $D \in \mathfrak{C}_F$ is the component separated from C by H_j. Since $\epsilon_F(C) = -\epsilon_F(D)$, these two terms of $\sum_{C \in \mathfrak{C}_F} \epsilon_F(C) h_C(\eta_A)$ will cancel with each other. Thus every term will cancel the contribution of another term, and the total sum will be zero. $\square$

Given a cooriented arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in $\mathbb{R}^d$, let

$$\mathcal{A}' = \{H_1, \ldots, H_{n-1}\}$$

denote the arrangement obtained by deleting H_n, and let

$$\mathcal{A}'' = \{H_i \cap H_n \mid i < n \text{ and } H_i \cap H_n \neq \varnothing\}$$

denote the arrangement of hyperplanes in H_n given by restriction. If $H_n \cap \Delta \neq \varnothing$, then $\mathcal{A}'$ and $\mathcal{A}''$ remain in the class of arrangements with Δ nonempty.

Proposition 4.3 *We have an exact sequence of groups*

$$0 \to P(\mathcal{A}') \xrightarrow{\alpha} P(\mathcal{A}) \xrightarrow{\beta} P(\mathcal{A}'') \xrightarrow{\gamma} \mathbb{Z} \to 0.$$

The map α is the ring homomorphism taking e_i to e_i and x to x. We define β on the additive basis $\{x\} \cup \{e_A \mid A$ an nbc-set$\}$ by $\beta(x) = 0$ and

$$\beta(e_A) = \begin{cases} e_{A \smallsetminus \{n\}} & \textit{if } n \in A \\ 0 & \textit{otherwise.} \end{cases}$$

The third map γ is defined by extracting the coefficient of x in the corresponding basis for $P(\mathcal{A}'')$.

Proof: Injectivity of α is a consequence of the fact that every nbc-set for $\mathcal{A}'$ is also an nbc-set for $\mathcal{A}$. Similarly, exactness at $P(\mathcal{A})$ and $P(\mathcal{A}'')$ follow from the fact that if $n \in A$, then A is an nbc-set for $\mathcal{A}$ if and only if $A \smallsetminus \{n\}$ is an nbc-set for $\mathcal{A}''$. Surjectivity of γ is trivial. $\square$

Remark 4.4 Our proof of Proposition 4.3 holds for arbitrary arrangements, not just simple ones.

Suppose that $H_n \cap \Delta$ is nonempty, and consider the sequence

$$0 \to B(\mathcal{A}') \xrightarrow{a} B(\mathcal{A}) \xrightarrow{b} B(\mathcal{A}'') \xrightarrow{c} \mathbb{Z} \to 0$$

defined as follows. The map a is given by restriction, and c is given by taking the coefficient of x corresponding to the component $\Delta \cap H^n$ of $\mathcal{C}(\mathcal{A}'')$. Given an element $\mu \in B(\mathcal{A})$ and a component C'' of $\mathcal{C}(\mathcal{A}'')$, we put $b(\mu)|_{C''} = (\mu_C - \mu_D)/x$, where $C \subseteq H_n^+$ and $D \subseteq H_n^-$ are the two components of $\mathcal{C}(\mathcal{A})$ neighboring C''. The fact that $\mu_C - \mu_D$ is a multiple of x follows from the fact that $\mu \in B(\mathcal{A})$. There is an inherent ambiguity in dividing by x, owing to the fact that x is annihilated by $2 - x$. We resolve this ambiguity by requiring that $b(\mu)|_{\Delta \cap H^n} \in \mathbb{Z}$, and $b(\mu)|_{C''}$ is congruent to $b(\mu)|_{\Delta\cap H^n}$ modulo x for all components C''. This sequence is evidently a complex, and it is easy to check exactness at $B(\mathcal{A}')$, $B(\mathcal{A})$, and $\mathbb{Z}$. Exactness at $B(\mathcal{A}'')$ will fall out of the process of proving the following theorem.

Proposition 4.5 *The restriction map $h : Line(\mathcal{A}) \to B(\mathcal{A})$ is an isomorphism.*

Proof: By Theorem 3.10, it is sufficient to prove that the composition $h \circ \hat{\phi} : P(\mathcal{A}) \to B(\mathcal{A})$ is an isomorphism. We proceed by induction on the number of hyperplanes. The base case $n = 0$ is trivial. For the inductive step, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P(\mathcal{A}) \arrow[r, "\alpha"] \arrow[d] & P(\mathcal{A}) \arrow[r, "\beta"] \arrow[d] & P(\mathcal{A}'') \arrow[r, "\gamma"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B(\mathcal{A}') \arrow[r, "a"] & B(\mathcal{A}) \arrow[r, "b"] & B(\mathcal{A}'') \arrow[r, "c"] & \mathbb{Z} \arrow[r] & 0,
\end{tikzcd}$$

where the first three downward arrows are given by the composition $h \circ \hat{\phi}$, and the last is the identity map. Our inductive hypothesis tells us that the maps from $P(\mathcal{A}')$ to $B(\mathcal{A}')$ and $P(\mathcal{A}'')$ to $B(\mathcal{A}'')$ are isomorphisms. This, along with exactness of the top row at $P(\mathcal{A}'')$, implies the exactness of the bottom row at $B(\mathcal{A}'')$. Our Theorem then follows from the Five Lemma. $\square$

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