Let M be a matroid. The Kazhdan-Lusztig polynomial $P_M(t) \in \mathbb{Z}[t]$ was introduced in [EPW16], and the closely related Z-polynomial $Z_M(t) \in \mathbb{Z}[t]$ was introduced in [PXY18]. Kazhdan-Lusztig polynomials of matroids are neither special cases nor generalizations of classical Kazhdan-Lusztig polynomials. Rather, both classes of polynomials are special cases of Kazhdan-Lusztig-Stanley polynomials; see [Pro18] for more details. The following conjecture appears in [GPY17b, 3.2] and [PXY18, 5.1].

Conjecture 1. *The polynomials $P_M(t)$ and $Z_M(t)$ are real rooted.*

Remark 2. We also have various conjectures that say that the roots of the Kazhdan-Lusztig polynomials or Z-polynomials of various matroids should interlace. For the conjectural statement about Kazhdan-Lusztig polynomials, see [GPY17b, 3.4 and 3.5]. The Z-polynomial statement should roughly say that $Z_M(t)$ and $Z_{M/e}(t)$ have interlacing roots, but one has to rule out degenerate examples. For example, if M is the thagomizer matroid of rank 4 and e is the distinguished element of the ground set, then M/e is Boolean, thus $Z_{M/e}(t) = (1+t)^3$. But this does not interlace with $Z_M(t) = 1 + 11t + 21t^2 + 11t^3 + t^4$.

Remark 3. If there is a finite group W acting on M, then these polynomials have equivariant analogues $P_M^W(t)$ [GPY17a] and $Z_M^W(t)$ [PXY18], whose coefficients isomorphism classes of representations of W. I've made various attempt to formulate equivariant versions of Conjecture 1 (involving minors of the Toeplitz matrix) and Remark 2 (involving minors of the Bézout matrix), but I keep finding counterexamples. For instance, $Z^{S_n}_{U_{n-1,n}}(t)$ fails to be real rooted or to equivariantly interlace with $Z^{S_n}_{U_{n,n+1}}(t)$ when n is large.

References

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