Let $\tilde{X} \to X$ be a conical symplectic resolution. Let $H_2(\tilde{X};\mathbb{Z})_{\text{free}}$ denote the quotient of $H_2(\tilde{X};\mathbb{Z})$ by its torsion subgroup. Let $QH^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})$ be the equivariant quantum cohomology ring of $\tilde{X}$, with the quantum product shifted by the canonical theta characteristic. The underlying graded vector space of $QH^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})$ is equal to the tensor product of $H^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})$ with the completion of the semigroup ring of the semigroup of effective curve classes in $H_2(\tilde{X};\mathbb{Z})_{\text{free}}$. Let $\star$ denote the quantum product. In [Oko, §2.3.4], Okounkov conjectures that there exists a finite set $\Delta_+ \subset H_2(\tilde{X};\mathbb{Z})_{\text{free}}$ and an element $L_\alpha \in H^{2\dim X}(\tilde{X} \times_X \tilde{X};\mathbb{C})$ for each $\alpha \in \Delta_+$ such that, for all $u \in H^2_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})$, we have

$$u \star \cdot = u \cup \cdot + \hbar \sum_{\alpha \in \Delta_+} \langle \alpha, \bar{u} \rangle \frac{q^\alpha}{1 - q^\alpha} L_\alpha(\cdot),$$

where $\hbar$ is the standard generator of $H^2_{\mathbb{C}^\times}(pt;\mathbb{C})$ and L_α acts via convolution. We will assume that this conjecture holds. The minimal such subset Δ_+ is called the set of **positive Kähler roots**.

Let $QH^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})_{\text{pol}}$ be the subring of $QH^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})$ generated by $z \otimes 1$ for all $z \in H^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})$ and $1 \otimes q^\alpha$ for all positive Kähler roots α. Let $I \subset QH^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})_{\text{pol}}$ be the ideal generated by $1 - q^\alpha$ for each positive Kähler roots α, and let $J := \{f \mid \hbar f \in I\}$. We will be interested in the quotient ring $QH^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})_{\text{pol}}/J$.

By the decomposition theorem, there is a natural inclusion of vector spaces from $IH^*_{\mathbb{C}^\times}(X;\mathbb{C})$ into $H^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})$. The following conjecture appears in [MP15, 2.5].

Conjecture 1. *Suppose that the positive Kähler roots are all primitive. The composition*

$$IH^*_{\mathbb{C}^\times}(X;\mathbb{C}) \hookrightarrow H^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C}) \hookrightarrow QH^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})_{\text{pol}} \twoheadrightarrow QH^*_{\mathbb{C}^\times}(\tilde{X};\mathbb{C})_{\text{pol}}/J$$

is an isomorphism. In particular, $IH^*_{\mathbb{C}^\times}(X;\mathbb{C})$ has a canonical ring structure.

Remark 2. The original conjecture was missing the primitivity hypothesis, but the Hilbert scheme of n points in the plane provides a counterexample. We want the ideal I to be the "worst possible" specialization of the quantum parameters, and in the case of the Hilbert scheme, each of the n^{th} roots of unity is equally bad. Our hope is that the primitivity hypothesis will prevent this sort of behavior.

Remark 3. Conjecture 1 is proved for hypertoric varieties in [MP15]. The ring structure on the intersection cohomology of an affine hypertoric variety had already been discovered [BP09], but Conjecture 1 provides a more conceptual explanation for the existence of this ring structure. All other examples remain open. The two simplest unresolved cases are probably the cotangent bundles of the variety of flags in $\mathbb{C}^3$ and the Grassmannian of planes in $\mathbb{C}^4$.

References

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