A constructive definition of a hypertoric variety first appears in the work of Bielawski and Dancer [BD00], with foundational work also appearing in papers of Konno [Kon00] and Hausel and Sturmfels [HS02]. In each of these papers, one starts with a hyperplane arrangement and constructs a variety, projective over its affinization, with certain properties. The right analogy to think of is a constructive definition of (projective) toric varieties that involves starting with a rational polytope and using it to build a projective variety explicitly (using any one of many equivalent constructions). It is desirable to have an abstract definition of a class of varieties along with a combinatorial classification, just as we do for toric varieties. Such a definition appears in [AP16]. Let T be a torus.

Definition 1. A **T-hypertoric variety** is a conical symplectic variety of dimension $2\dim T$ equipped with an effective Hamiltonian action of T that commutes with the conical action of $\mathbb{G}_m$. That is, it is a conical symplectic variety with the largest possible abelian symmetry group.

Let N be the cocharacter lattice of T. An **integral zonotope** in $N_\mathbb{R}$ is an integral translation of a polytope of the form

$$\sum_{i=1}^{n}[-1,1]\cdot a_i,$$

where $a_i \in N$ for all i. If Z is an integral zonotope in $N_\mathbb{R}$ A **zonotopal tiling** of Z is a finite set $\mathcal{T}$ of integral zonotopes in $N_\mathbb{R}$, closed under intersections, such that any two elements of $\mathcal{T}$ intersect along a common face, and the union of all of the elements of $\mathcal{T}$ is equal to Z. Every integral zonotope Z admits a trivial tiling consisting of Z and all of its faces.

If Z is an integral zonotope in $N_\mathbb{R}$ containing the origin in its interior and $\mathcal{T}$ is a tiling of Z, we construct a T-hypertoric variety $Y(\mathcal{T})$ [AP16, 5.7]. Our main conjecture is that every hypertoric variety arises in this way [AP16, 5.8].

Conjecture 2. *Every T-hypertoric variety is isomorphic as a Poisson $(T\times\mathbb{G}_m)$-variety to $Y(\mathcal{T})$ for some tiling $\mathcal{T}$ of an integral zonotope Z in $N_\mathbb{R}$ with the origin in its interior.*

Remark 3. We write $Y(Z)$ to denote the hypertoric variety associated with the trivial tiling of Z, which is always affine. If $\mathcal{T}$ is a tiling of Z, then $Y(\mathcal{T})$ is a partial resolution of $Y(Z)$. The map from $Y(\mathcal{T})$ to $Y(Z)$ is projective if and only if the tiling $\mathcal{T}$ is **regular** [AP16, 6.8], which means that it comes from a hyperplane arrangement. In this case, $Y(\mathcal{T})$ coincides with the variety constructed in the papers of Bielawski-Dancer, Konno, and Hausel-Sturmfels. Thus Conjecture 2 implies that every hypertoric variety that is projective (rather than just proper) over its affinization arises via the original construction.

Remark 4. Conjecture 2 breaks naturally into two parts:

1. Every affine T-hypertoric variety is isomorphic to $Y(Z)$ for some integral zonotope Z in $N_\mathbb{R}$ with the origin in its interior.

2. Every hypertoric variety with affinization $Y(Z)$ is isomorphic to $Y(\mathcal{T})$ for some tiling $\mathcal{T}$ of Z.

Both of these statements are open. The one thing that we do know is that every hypertoric variety that is projective over $Y(Z)$ is isomorphic to $Y(\mathcal{T})$ for some regular tiling $\mathcal{T}$ of Z; this follows fairly easily from the work of Namikawa [Nam15].

Remark 5. Continuing the analogy with toric varieties, a zonotope corresponds to a rational cone, a tiling of the zonotope corresponds to a fan supported on the cone, and a regular tiling (which may be thought of as normal to a hyperplane arrangement) corresponds to the normal fan to a polyhedron.

Remark 6. Another conjectural attempt at an abstract definition and classification of hypertoric varieties appeared in [Pro08, 1.4.3]. In this formulation, the conical action does not appear and the affinization map is required to be projective. While there does not appear to be any direct relationship between this conjecture and Conjecture 2, I think that Conjecture 2 is a much more natural and satisfying statement.

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