Stable ∞-categories

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1 Foundations

Let $\mathcal{C}$ be a pointed ∞-category. Recall that a *triangle* in $\mathcal{C}$ is a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

where 0 is a zero object of $\mathcal{C}$. A triangle in $\mathcal{C}$ is a *fiber (resp. cofiber) sequence* if it is a pullback (resp. pushout) diagram, in which case we say that X is a *fiber* of g (resp. Z is a *cofiber* of f).

Definition 1.1 (HA 1.1.1.9)**.** An ∞-category is *stable* if it satisfies the following conditions:

(1) There exists a zero object $0 \in \mathcal{C}$.

(2) Every morphism in $\mathcal{C}$ admits a fiber and a cofiber.

(3) A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Notice that stability is a property and not extra structure; the homotopy category of a stable ∞-category $\mathcal{C}$ carries a canonical triangulated structure —which is a whole lot of structure— induced by the cofiber sequences in $\mathcal{C}$.

The two fundamental examples of stable ∞-categories are the derived ∞-category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ (with homotopy category the classical derived category) and the ∞-category of spectra (with homotopy category the classical stable homotopy category). In fact, it can be shown that stable ∞-categories are canonically enriched over the ∞-category of spectra. Unfortunately we do not have the time to develop all the technology required to even make sense of this last statement.

1.1 Functoriality of cofibers

There is a proposition in Lurie's *Higher Topos Theory*, the infamous 4.3.2.15, that is used extensively and without much comment on the part of the author both in the remainder of the book and in its sequel *Higher Algebra*. The statement is given in the book in the full generality of the relative setting, resulting in a mouthful in which it is easy to get lost in technicalities. Also I think there is not a single worked out example of a typical use of it anywhere, so let me hence do precisely that.

Proposition 1.2 (Simplified HTT 4.3.2.15)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $\mathcal{C}^0$ be a full subcategory of $\mathcal{C}$. Let $\mathcal{K} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the full subcategory spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ which are left Kan extensions of $F|_{\mathcal{C}^0}$. Let $\mathcal{K}' \subseteq \mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$ be the full subcategory spanned by those functors $F_0 : \mathcal{C}^0 \to \mathcal{D}$ with the property that, for each object $C \in \mathcal{C}$, the induced diagram $\mathcal{C}_{/C} \times_{\mathcal{C}} \mathcal{C}^0 =: \mathcal{C}^0_{/C} \to \mathcal{D}$ has a colimit. Then the restriction functor $\mathcal{K} \to \mathcal{K}'$ is a trivial fibration of simplicial sets.*

We have not even defined Kan extensions in the ∞-category world, but just take it for granted that a suitable generalization of the classical concept exists. Recall that 1-categorical left Kan extensions can be calculated as colimits of certain functors into the target category: the existence of these colimits is then equivalent to the existence of a left Kan extension. Moreover, left Kan extensions are essentially unique. Proposition HTT 4.3.2.15 can now be read as the ∞-categorical version of this statement. Notice that $\mathcal{K} \to \mathcal{K}'$ being a trivial fibration says that its fibers are contractible Kan complexes, which we know is the correct uniqueness statement in this framework.

We now want to use this proposition (twice) to define a cofiber functor $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D}$ in a pointed ∞-category $\mathcal{D}$. For the first step, we take:

$$\begin{aligned} \mathcal{C}^0 = \Delta^1 \quad &\rightsquigarrow \quad \mathrm{Fun}(\Delta^1, \mathcal{D}) = \{\bullet \to \bullet\} \\ \mathcal{C} = \Delta^1 \vee \Delta^1 \quad &\rightsquigarrow \quad \mathrm{Fun}(\Delta^1 \vee \Delta^1, \mathcal{D}) = \left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \right\} \end{aligned}$$

Each of the slice categories in the (dual) statement of the theorem has an initial object, so the necessary limits always exist. On the other hand, right Kan extensions along inclusions of full subcategories which are also sieves are extensions by the final object (this is obvious in the classical setting, so it is easy to believe it for ∞-categories). The proposition then says that the restriction functor

$$\left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ 0 & \end{tikzcd} \right\} = \mathcal{K} \longrightarrow \mathcal{K}' = \{\bullet \to \bullet\}$$

is a trivial fibration of simplicial sets. For the second step let

$$\begin{aligned} \mathcal{C}^0 = \Delta^1 \vee \Delta^1 \quad &\rightsquigarrow \quad \mathrm{Fun}(\Delta^1 \vee \Delta^1, \mathcal{D}) = \left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \right\} \\ \mathcal{C} = \Delta^1 \times \Delta^1 \quad &\rightsquigarrow \quad \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{D}) = \left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \right\} \end{aligned}$$

Now $\mathcal{K}'$ is the full subcategory on diagrams of the above shape that can be completed to a pushout square, while $\mathcal{K}$ consists of precisely those pushout squares. Hence,

$$\left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \bullet \end{tikzcd} \right\} = \mathcal{K} \longrightarrow \mathcal{K}' = \left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, dotted] \\ \bullet \arrow[r, dotted] \arrow[ur, phantom, "\lrcorner"] & \bullet \end{tikzcd} \right\}$$

is also a trivial fibration. Pulling back the latter along the fully faithful functor

$$\left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, dotted] \\ 0 \arrow[r, dotted] \arrow[ur, phantom, "\lrcorner"] & \bullet \end{tikzcd} \right\} \hookrightarrow \left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, dotted] \\ \bullet \arrow[r, dotted] \arrow[ur, phantom, "\lrcorner"] & \bullet \end{tikzcd} \right\}$$

we get yet one more trivial fibration:

$$\left\{\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near end] & \bullet \arrow[d] \\ 0 \arrow[r] & \bullet \end{tikzcd}\right\} \longrightarrow \left\{\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near end] & \bullet \arrow[d, dotted] \\ 0 \arrow[r, dotted] & \bullet \end{tikzcd}\right\}$$

All together we have the diagram

$$\left\{\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near end] & \bullet \arrow[d] \\ 0 \arrow[r] & \bullet \end{tikzcd}\right\} \longrightarrow \left\{\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near end] & \bullet \arrow[d, dotted] \\ 0 \arrow[r, dotted] & \bullet \end{tikzcd}\right\} \hookrightarrow \left\{\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ 0 & \end{tikzcd}\right\} \longrightarrow \{\bullet \to \bullet\}$$

where the first and third arrows are trivial fibrations and the second one is an embedding of a full subcategory. It is now trivial to check that the composition has the right lifting property with respect to the generating trivial cofibrations of the Quillen model structure (the maps $\Lambda^n_k \hookrightarrow \Delta^n$ for $n \geq 1$, $0 \leq k \leq n$); moreover, if every morphism in $\mathcal{D}$ admits a cofiber, the embedding is actually the identity functor and the composition is a trivial fibration. We can thus define a functorial cofiber by composing a section of the latter with the forgetful functor

$$\left\{\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near end] & \bullet \arrow[d] \\ 0 \arrow[r] & \bullet \end{tikzcd}\right\} \longrightarrow \{\bullet\}$$

that assigns to a cofiber sequence its lower right corner.

In the context of triangulated categories, it is well known that cones are not functorial. However, we have just proven that if a triangulated category arises as the homotopy category of a stable ∞-category, cones in it are indeed functorial at the ∞-level.

One thing to notice which will prove useful later is that this cofiber functor can be realized as a left adjoint to the left Kan extension functor $\mathcal{D} \simeq \mathrm{Fun}(\{1\}, \mathcal{D}) \to \mathrm{Fun}(\Delta^1, \mathcal{D})$ which associates to each object $X \in \mathcal{D}$ a zero morphism $0 \to X$ (that this is a left Kan extension is another easy application of HTT 4.3.2.15). Consequently, cofib preserves colimits.

1.2 The triangulated structure on the homotopy category of a stable ∞-category

Definition 1.3 (Verdier)**.** A *triangulated category* consists of the following data:

(1) An additive category $\mathcal{D}$.

(2) A translation functor $\mathcal{D} \to \mathcal{D}$ which is an equivalence of categories. We denote this functor by $X \mapsto X[1]$.

(3) A collection of *distinguished triangles*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

These data are required to satisfy the following axioms:

(TR1) (a) Every morphism $f : X \to Y$ in $\mathcal{D}$ can be extended to a distinguished triangle in $\mathcal{D}$.

(b) The collection of distinguished triangles is stable under isomorphism.

(c) Given an object $X \in \mathcal{D}$ the diagram

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$$

is a distinguished triangle.

(TR2) A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle if and only if the rotated diagram

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle.

(TR3) Given a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

in which both horizontal rows are distinguished triangles, there exists a dotted arrow rendering the entire diagram commutative.

(TR4) Suppose given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]$$

$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1]$$

in $\mathcal{D}$. Ther exists a fourth distinguished triangle

$$Y/X \xrightarrow{\phi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} Y/X[1]$$

such that the diagram

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"] & & Z \arrow[rr, "v"] \arrow[dr, "w"] & & Z/Y \arrow[rr, "\theta"] \arrow[dr, "d'"] & & Y/X[1] \\
& Y \arrow[ur, "g"] \arrow[dr, "u"] & & Z/X \arrow[ur, "\psi"] \arrow[dr, "d''"] & & Y[1] \arrow[ur, "u[1]"] & \\
& & Y/X \arrow[ur, "\phi"] \arrow[rr, "d"] & & X[1] \arrow[ur, "f[1]"] & &
\end{tikzcd}$$

commutes.

Let $\mathcal{C}$ be a stable ∞-category. We consider the full subcategory $\mathcal{M}^{\Omega} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by those diagrams

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0' \arrow[r] & Y
\end{tikzcd}$$

which are pushout squares, with 0 and $0'$ zero objects of $\mathcal{C}$. Applying HTT 4.3.2.15 (again, twice), we conclude that evaluation at the initial vertex induces a trivial fibration $\mathcal{M}^{\Sigma} \to \mathcal{C}$. Let $s : \mathcal{C} \to \mathcal{M}^{\Sigma}$ be a section of this trivial fibration, and let $e : \mathcal{M}^{\Sigma} \to \mathcal{C}$ denote the functor given by evaluation at the final vertex. The composition $e \circ s$ provides a *suspension functor* $\Sigma : \mathcal{C} \to \mathcal{C}$. Dually, we can construct a *loop functor* $\Omega : \mathcal{C} \to \mathcal{C}$ by considering the full subcategory $\mathcal{M}^{\Omega} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ of diagrams as above that are pullback squares. Since $\mathcal{C}$ is stable, $\mathcal{M}^{\Sigma} = \mathcal{M}^{\Omega}$ and (Σ, Ω) constitute an adjoint pair of equivalences.

Remark 1.4. In order to define a suspension functor (resp. loop space functor) we only needed to know that $\mathcal{C}$ is pointed and that it admits finite colimits (resp. limits). In the case where $\mathcal{C}$ is pointed and admits both finite limits and colimits but we do not know if it is stable, we can still say that (Σ, Ω) form an adjoint pair.

We can thus define translation functors $[n] : \mathcal{C} \to \mathcal{C}$ as the nth power of the suspension functor for $n \geq 0$ and as the $(-n)$th power of the loop functor for $n \leq 0$. In addition, we have natural homotopy equivalences of simplicial sets

$$\mathrm{Map}_{\mathcal{C}}(X, Y[-n]) \simeq \Omega^n \mathrm{Map}_{\mathcal{C}}(X, Y), \qquad n \geq 0,$$

which implies that

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y) \cong \pi_2 \mathrm{Map}_{\mathcal{C}}(X, Y[2])$$

has the structure of an abelian group. To finish the proof that $\mathrm{h}\mathcal{C}$ is additive, we only need to show that it admits finite coproducts. In fact, we can prove that $\mathcal{C}$ itself admits finite coproducts. Since $\mathcal{C}$ has an initial object, it is enough to show that pairwise coproducts exists. This can be deduced from the following facts:

- $X \simeq \mathrm{cofib}(X[-1] \xrightarrow{u} 0)$,
- $Y \simeq \mathrm{cofib}(0 \xrightarrow{v} Y)$,
- u and v admit a coproduct in $\mathrm{Fun}(\Delta^1, \mathcal{C})$, namely $X[-1] \xrightarrow{0} Y$, and
- cofib preserves colimits.

Definition 1.5 (HA 1.1.2.10)**.** Let $\mathcal{C}$ be a stable ∞-category. Suppose given a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in the homotopy category $\mathrm{h}\mathcal{C}$. We will say that this diagram is a *distinguished triangle* if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$ as shown

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\tilde{g}"] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r, "\tilde{h}"] & W \end{tikzcd}$$

satisfying the following conditions:

(i) The objects $0, 0' \in \mathcal{C}$ are zero.

(ii) Both squares are pushout diagrams in $\mathcal{C}$.

(iii) The morphisms $\tilde{f}$ and $\tilde{g}$ represent f and g, respectively.

(iv) The map $h : Z \to X[1]$ is the composition of the homotopy class of $\tilde{h}$ with the equivalence $W \simeq X[1]$ determined by the outer rectangle.

One can check that this definition of distinguished triangles in $\mathrm{h}\mathcal{C}$ satisfy all of Verdier's axioms. For example, consider the full subcategory $\mathcal{E} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^2, \mathcal{C})$ spanned by diagrams of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r] & W \end{tikzcd}$$

and let $e : \mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the restriction to the upper left horizontal arrow. Repeated applications of HTT 4.3.2.15 prove that e is a trivial fibration, establishing (TR1)(a).

1.3 An alternative characterization of stable ∞-categories

On our way to the proof that the homotopy category of a stable ∞-category $\mathcal{C}$ is triangulated we showed that $\mathcal{C}$ itself admits finite coproducts. It turns out that it also admits coequalizers, since they can be written in terms of cofibers:

$$\operatorname{coeq}\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y \right) = \operatorname{cofib}\left(X \xrightarrow{f-g} Y \right)$$

This proves part (1) of the "only if" direction of the following proposition,while part (2) uses (yes, you guessed right!) HTT 4.3.2.15. The "if" direction is trivial.

Proposition 1.6 (HA 1.1.3.4)**.** *Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is stable if and only if the following conditions are satisfied:*

1. *The ∞-category $\mathcal{C}$ admits finite limits and colimits.*
2. *A square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

in $\mathcal{C}$ is a pushout if and only if it is a pullback.

2 t-structures

Definition 2.1. Let $\mathcal{D}$ be a triangulated category. A *t-structure* on $\mathcal{D}$ is defined to be a pair of full subcategories $\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0}$ stable under isomorphism and having the following properties:

1. For $X \in \mathcal{D}_{\geq 0}$ and $Y \in \mathcal{D}_{\leq 0}$, we have $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$.
2. We have inclusions $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}[-1] \subseteq \mathcal{D}_{\leq 0}$.
3. For any $X \in \mathcal{D}$, there exists a fiber sequence $X' \to X \to X''$ where $X' \in \mathcal{D}_{\geq 0}$ and $X'' \in \mathcal{D}_{\leq 0}[-1]$.

Notation 2.2. We write $\mathcal{D}_{\geq n}$ for $\mathcal{D}_{\geq 0}[n]$, and $\mathcal{D}_{\leq n}$ for $\mathcal{D}_{\leq 0}[n]$

Remark 2.3. Notice that in the previous definition any one of $\mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}$ determines the other as the left or right orthogonal up to a translation.

Definition 2.4 (HA 1.2.1.4)**.** Let $\mathcal{C}$ be a stable ∞-category. A *t-structure* on $\mathcal{C}$ is a t-structure on its homotopy category $\mathrm{h}\mathcal{C}$. We denote by $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ the full subcategories spanned by $(\mathrm{h}\mathcal{C})_{\geq n}$ and $(\mathrm{h}\mathcal{C})_{\leq n}$, respectively.

Given a stable ∞-category $\mathcal{C}$ equipped with a t-structure, the inclusion functor $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ possesses a left adjoint, which we denote $\tau_{\leq n}$. Similarly, the inclusion $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$ has a right adjoint $\tau_{\geq n}$. Moreover, compositions of these commute up to homotopy, i.e., the functors $\tau_{\leq m} \circ \tau_{\geq n}$ and $\tau_{\geq n} \circ \tau_{\leq m}$ are equivalent.

Definition 2.5 (HA 1.2.1.11)**.** Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. The heart $\mathcal{C}^{\heartsuit}$ of $\mathcal{C}$ is the full subcategory $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$. For each $n \in \mathbb{Z}$, we let $\pi_0 : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ denote the functor $\tau_{\geq 0} \circ \tau_{\leq 0}$ and $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ the composition of π_0 with the shift functor $X \mapsto X[-n]$.

For any two objects $X, Y \in \mathcal{C}^{\heartsuit}$ we have $\pi_n \mathrm{Map}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y[-n]) = 0$ for $n > 0$, so that $\mathcal{C}^{\heartsuit}$ is equivalent to the nerve of its homotopy category $\mathrm{h}(\mathcal{C}^{\heartsuit})$, which is itself the heart $(\mathrm{h}\mathcal{C})^{\heartsuit}$ of the t-structure on $\mathrm{h}\mathcal{C}$, hence an abelian category. This fact opens the door to the use of spectral sequences as a calculational tool.

3 Spectra and stabilization

Definition 3.1 (HA 1.4.2.1)**.** Let $\mathcal{C}$ be a pointed ∞-category. A *prespectrum object* of $\mathcal{C}$ is a functor $X : N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$ with the following property: for every pair of integers $i \neq j$, the value of $X(i, j)$ is a zero object of $\mathcal{C}$. We let $\mathrm{PSp}(\mathcal{C})$ denote the fullsubcategory of $\mathrm{Fun}(N(\mathbb{Z} \times \mathbb{Z}), \mathcal{C})$ spanned by the prespectrum objects of $\mathcal{C}$.

That is, a prespectrum object of $\mathcal{C}$ is a diagram that looks as follows:

$$\begin{tikzcd} & 0 \arrow[r] & X(n+1,n+1) \\ 0' \arrow[r] & X(n,n) \arrow[r] \arrow[u] & 0'' \arrow[u] \\ X(n-1,n-1) \arrow[r] \arrow[u] & 0''' \arrow[u] & \end{tikzcd}$$

Definition 3.2 (HA 1.4.2.4)**.** Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits. A prespectrum object is said to be a *spectrum below* n if the canonical map $X(m-1, m-1) \to \Omega X(m, m)$ is an equivalence for each $m \leq n$. If it is a spectrum below n for every $n \in \mathbb{Z}$, then we say that X is a *spectrum object* of $\mathcal{C}$. We denote by $\mathrm{Sp}(\mathcal{C})$ (resp. $\mathrm{Sp}_n(\mathcal{C})$) full subcategory of $\mathrm{PSp}(\mathcal{C})$ spanned by the spectrum objects (resp. spectrum objects below n) of $\mathcal{C}$.

Remark 3.3. Although we will not make use of it here, it is important to know that, under certain appropriate conditions, the categories $\mathrm{PSp}(\mathcal{C})$, $\mathrm{Sp}_n(\mathcal{C})$ and $\mathrm{Sp}(\mathcal{C})$ are succesive localizations (in a suitable ∞-categorical sense) of $\mathrm{Fun}(N(\mathbb{Z}\times\mathbb{Z}), \mathcal{C})$. A left adjoint to the inclusion $\mathrm{Sp}(\mathcal{C}) \to \mathrm{PSp}(\mathcal{C})$ bears the name of *spectrification* (HA §1.5.1).

Proposition 3.4 (HA 1.4.2.6)**.** *Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits. Then the ∞-category* $\mathrm{Sp}(\mathcal{C})$ *can be identified with homotopy limit of the tower*

$$\cdots \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$$

Idea of proof. Let $\{X_n\}_{n\geq 0}$ be an element of the homotopy limit. This means that $X_n \simeq \Omega X_{n+1}$ for all $n \geq 0$. We can then construct a spectrum object of $\mathcal{C}$ by taking $X(n, n) = X_n$ for $n \geq 0$ and $X(n, n) = \Omega^{-n} X_0$ for $n < 0$. $\square$

We can read this proposition as saying that $\mathrm{Sp}(\mathcal{C})$ is the ∞-category obtained from $\mathcal{C}$ by inverting the loop space functor. The following should then not be surprising.

Proposition 3.5 (HA 1.4.2.18)**.** *Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits. Then the ∞-category* $\mathrm{Sp}(\mathcal{C})$ *is stable.*

If an ∞-category $\mathcal{C}$ admits finite limits, it has a final object $* \in \mathcal{C}$. The slice category of objects under it, denoted $\mathcal{C}_*$, is then pointed. Notice that if $\mathcal{C}$ was already pointed, we have $\mathcal{C} \simeq \mathcal{C}_*$.

Definition 3.6 (HA 1.4.4.1)**.** If $\mathcal{C}$ is an ∞-category which admits finite limits, we define its *stabilization* $\mathrm{Stab}(\mathcal{C})$ as $\mathrm{Sp}(\mathcal{C}_*)$.

3.1 The ∞-category of spectra

Of course, the most important example of stabilization is obtained by considering the case in which $\mathcal{C}$ is the ∞-category of spaces $\mathcal{S}$, defined, for example, as the simplicial nerve of the simplicial category whose objects are Kan complexes and whose simplicial sets of morphisms are the simplicial sets of morphisms of simplicial sets between them (a nice tongue twister!).

Definition 3.7 (HA 1.4.3.1)**.** A *spectrum* is a spectrum object in the ∞-category of pointed spaces $\mathcal{S}_*$. We let $\mathrm{Sp} = \mathrm{Sp}(\mathcal{S}_*)$ denote the ∞-category of spaces.

It turns out that the ∞-category of spectra comes equipped with a very interesting t-structure. In order to define it, let $\Omega^{\infty-n} : \mathrm{PSp}(\mathcal{C}) \to \mathcal{C}$ denote the so-called *nth space functor* that assigns to a prespectrum object of $\mathcal{C}$ the object $X(n,n)$.

Proposition 3.8 (HA 1.4.3.3)**.** *Let $\mathcal{C}$ be a presentable pointed ∞-category, and let $\mathrm{Sp}_{\leq 1}(\mathcal{C})$ be the full subcategory of $\mathrm{Sp}(\mathcal{C})$ spanned by those objects such that $\Omega^\infty(X)$ is a final object of $\mathcal{C}$. Then $\mathrm{Sp}_{\leq 1}(\mathcal{C})$ determines a t-structure on $\mathrm{Sp}(\mathcal{C})$.*

We will talk later about presentable ∞-categories. For the moment, just take for granted that the ∞-category of spaces is presentable, so that we do have the t-structure alluded to in the statement of the proposition. The importance of its existence stems from the following result.

Proposition 3.9 (HA 1.4.3.5 (3))**.** *The heart $\mathrm{Sp}^\heartsuit$ is canonically equivalent to the nerve of the category of abelian groups.*

Idea of proof. One observes that a spectrum X belongs to $\mathrm{Sp}_{\leq m}$ if and only if each $X(n,n)$ is $(n+m)$-truncated; similarly $X \in \mathrm{Sp}_{\geq m}$ if and only if each $X(n,n)$ is $(n+m)$-connective. In particular, X lies in the heart of Sp if and only if each $X(n,n)$ is an Eilenberg-MacLane space of degree n. But for $n \geq 2$ these correspond to abelian groups. $\square$

With this, we see that the functors $\pi_n = \tau_{\leq 0} \circ \tau_{\geq 0} \circ [-n]$ that we defined when we were discussing t-structures take values in the category of abelian groups, and in fact coincide with the classical stable homotopy groups of spectra. It is now clear that a spectrum lies in the heart of Sp if and only if its nth stable homotopy group vanishes for all $n \neq 0$ —these are called discrete spectra. This fact should be reminiscent of how sets sit inside of the category of simplicial sets as those that are discrete, and explains the central role that the ∞-category of spectra plays in higher algebraic constructions.

The category of abelian groups is symmetric monoidal. The ∞-category of spectra also carries the structure of a symmetric monoidal ∞-category, for which the tensor product receives the name of *smash product of spectra.* In the remainder of this talk I will try to sketch Lurie's construction of it.

4 The category of stable ∞-categories

We know that $(\infty,1)$-categories should organize themselves into an $(\infty,2)$-category. By forgetting the noninvertible 2-morphisms, we can instead get an $(\infty,1)$-category of $(\infty,1)$-categories. In the particular model of ∞-categories, we have the following

Definition 4.1 (HTT 3.0.0.1)**.** The simplicial category $\mathcal{C}at_\infty^\Delta$ is defined as follows:

(1) The objects of $\mathcal{C}at_\infty^\Delta$ are (small) ∞-categories.

(2) Given ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we define $\mathrm{Map}_{\mathcal{C}at_\infty^\Delta}(\mathcal{C},\mathcal{D})$ to be the largest Kan complex contained in the ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$.

We let $\mathcal{C}at_\infty$ denote the simplicial nerve $N(\mathcal{C}at_\infty^\Delta)$. We will refer to $\mathcal{C}at_\infty$ as the ∞-category of (small) ∞-categories.

Remark 4.2. $\mathcal{C}at_\infty$ admits all small limits and colimits. One can prove this directly by giving an explicit construction (HTT 3.3.3) or by realizing $\mathcal{C}at_\infty^\Delta$ as the simplical category of cofibrant and fibrant objects of a combinatorial simplicial model category (HTT 4.2.4.8). The Joyal model structure on simplicial sets does not provide such a structure, but it is Quillen equivalent (HTT 3.1.5.3) to one that is: the so-called Cartesian model structure on marked simplicial sets (HTT 3.1.3.7)

Remark 4.3 (HTT 3.0.0.5)**.** In the following, we will need to talk about ∞-categories that are not necessarily small. We will denote by $\widehat{\mathcal{C}at_\infty}$ the simplicial nerve of the simplicial category of not necessarily small ∞-categories.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. If F carries zero objects into zero objects, it follows immediately that it carries triangles into triangles. If, in addition, F carries (co)fiber sequences to (co)fiber sequences, then we say that F is exact.

Proposition 4.4 (HA 1.1.4.1)**.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between stable ∞-categories. The following conditions are equivalent:*

(1) The functor F is left exact. That is, F commutes with finite limits.

(2) The functor F is right exact. That is, F commutes with finite colimits.

(3) The functor F is exact

Proof. (2) $\Leftrightarrow$ (3) can be easily deduced from the fact that finite colimits can be expressed in terms of cofibers. The dual argument proves that (1) $\Leftrightarrow$ (3), $\square$

The identity functor from any stable ∞-category to itself is exact, and a composition of exact functors is exact. Hence, there exists a subcategory $\mathcal{C}at_\infty^{\mathrm{Ex}} \subseteq \mathcal{C}at_\infty$ in which the objects are stable ∞-categories and the morphisms are the exact functors between them. It can be shown that $\mathcal{C}at_\infty^{\mathrm{Ex}}$ admits small limits and small filtered colimits, and that the inclusion into the ∞-category of ∞-categories preserves them.

5 Presentable ∞-categories and the smash product of spectra

This section will be very sketchy, mainly because the theory of presentable ∞-categories occupies more than 200 pages in *Higher Topos Theory* and the smash product of spectra does not appear until page 637 of *Higher Algebra*.

5.1 Presentable ∞-categories

The role that the category of sets plays in the 1-categorical world is taken up in the framework of ∞-categories by the ∞-category of spaces $\mathcal{S}$. Consequently, instead of talking about categories of presheaves of sets we talk about ∞-categories of presheaves of spaces: $\mathcal{P}(-) := \mathrm{Fun}(-^{\mathrm{op}}, \mathcal{S})$. In the same way that categories of presheaves of sets are free cocompletions of their domain categories, so can we think of ∞-categories of presheaves of spaces as freely adjoining (homotopy or ∞-) colimits to the domain ∞-category. Doing this takes a small ∞-category into a large one, but one that is determined by the original small ∞-category.

An ∞-category is *presentable* (HTT 5.5.0.1) if it is accesible and cocomplete. *Accesible* (HTT 5.4.2.1) means that it is generated by a small subcategory under κ-filtered colimits, for κ a regular cardinal. Alternatively, we have the following characterization.

Theorem 5.1 (HTT 5.5.1.1)**.** *An ∞-category is presentable if and only if there exists a small ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is an accesible (i.e., commuting with κ-filtered colimits for some regular cardinal κ) localization of $\mathcal{P}(\mathcal{D})$.*

Here are two results that highlight how nice presentable ∞-categories are.

Proposition 5.2 (HTT 5.5.2.2)**.** *Let $\mathcal{C}$ be a presentable ∞-category and let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be a functor. The following are equivalent:*

(1) The functor F is representable by an object $C \in \mathcal{C}$.

(2) The functor F preserves small limits.

Proposition 5.3 (HTT 5.5.2.9, Special Adjoint Functor Theorem)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.*

(1) The functor F has a right adjoint if and only if it preserves small colimits.

(2) The functor F has a left adjoint if and only if it is accessible and preserves small limits.

5.2 The monoidal structure

The definition of a (symmetric) monoidal structure on an ∞-category $\mathcal{C}$ is involved and would take us quite some time. We evidently need a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, but there is a whole collection of coherence conditions that it must satisfy that are impossible to spell out explicitly; they are usually hidden in the statement that a certain map of ∞-categories is some kind of fibration. We will content ourselves here with giving the bifunctor and assuming that the coherence conditions hold.

It is easy to believe that the ∞-category of ∞-categories $\mathcal{C}at_\infty$ (and $\widehat{\mathcal{C}at_\infty}$ too) carries a closed symmetric monoidal structure given by the cartesian product of simplicial sets. We can restrict this monoidal structure to the ∞-category $\mathcal{C}at_\infty(\mathcal{K})$ (resp. $\widehat{\mathcal{C}at_\infty}(\mathcal{K})$) of ∞-categories admitting colimits indexed by some collection $\mathcal{K}$ of simplicial sets, with morphisms given by the functors that preserve those colimits (HA 6.3.1.1, HA 6.3.1.4).

Let now $\mathcal{K}$ be the collection of all small simplicial sets, and let $\mathcal{P}r^L$ denote the full subcategory of $\widehat{\mathcal{C}at_\infty}(\mathcal{K})$ spanned by the presentable ∞-categories.

Proposition 5.4 (HA 6.3.1.14)**.** *The ∞-category $\mathcal{P}r^L$ of presentable ∞-categories is closed under tensor products in $\widehat{\mathcal{C}at_\infty}(\mathcal{K})$, and therefore inherits a closed symmetric monoidal structure.*

The unit object of $\widehat{\mathcal{C}at_\infty}(\mathcal{K})$ is given by $\mathcal{P}(\Delta^0) \simeq \mathcal{S}$ (HA 6.3.1.14), which is a presentable ∞-category (HTT 5.5.1.8). Hence it is also the unit object for the induced monoidal structure on $\mathcal{P}r^L$. A more explicit description of the latter is given by the following result.

Lemma 5.5 (HA 6.3.1.15, 6.3.1.16)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories. Then* $\mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ *(i.e., the category of those functors that preserve limits) is a presentable ∞-category, and there is a canonical equivalence $\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$.*

Remark 5.6. This lemma provides a proof of the representability criterion of proposition 5.2:

$$\mathcal{C} \simeq \mathcal{C} \otimes \mathcal{S} \simeq \mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

Remembering that the stabilization of an ∞-category can be identified with the infinite loop objects of its ∞-category of pointed objects, we see that

$$\mathcal{C} \otimes \mathrm{Sp} \simeq \mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{holim}\{\mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_*)\} \simeq \mathrm{holim}\{\mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_*\} \simeq \mathrm{holim}\{\mathcal{C}_*\} \simeq \mathrm{Stab}(\mathcal{C})$$

Taking for granted that stabilization is an idempotent operation, this calculation hints at the fact that there is an induced closed symmetric monoidal structure on the ∞-category of presentable stable ∞-categories with Sp as its monoidal unit. In particular, the morphism

$$\mathrm{Sp} \otimes \mathrm{Sp} \to \mathrm{Sp}$$

provides a tensor product on the ∞-category of spectra —the *smash product*— which can be extended to a closed symmetric monoidal structure on it.

The monoidal unit of this smash product is given by the fabled sphere spectrum, which can be constructed as the image of the 0-sphere under the left adjoint $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$ to the 0th space functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$, whose existence is ensured by the Special Adjoint Functor Theorem.

References

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