Notes for 'Representations of Finite Groups'

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Contents

1 Introduction

The following notes were written in preparation for the first talk of a week-long workshop on categorical representation theory. We focus on basic constructions in the representation theory of finite groups. The participants are likely familiar with much of the material in this talk; we hope that this review provides perspectives that will precipitate a better understanding of later talks of the workshop.

2 Functions on finite sets

Let X be a finite set of size n. Let $\mathbb{C}[X]$ denote the vector space of complex-valued functions on X. In what follows, $\mathbb{C}[X]$ will be endowed with various algebra structures, depending on the nature of X. The simplest algebra structure is pointwise multiplication, and in this case we can identify $\mathbb{C}[X]$ with the algebra $\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}$ (n times). To emphasize pointwise multiplication, we write $(\mathbb{C}[X], \text{ptwise})$.

A $\mathbb{C}[X]$-module is the same as X-graded vector space, or a vector bundle on X. To see this, let V be a $\mathbb{C}[X]$-module and let $\delta_x \in \mathbb{C}[X]$ denote the delta function at x. Observe that

$$\delta_x \cdot \delta_y = \begin{cases} \delta_x & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

It follows that each δ_x acts as a projection onto a subspace V_x of V and $V_x \cap V_y = 0$ if $x \neq y$. Since $1 = \sum_{x \in X} \delta_x$, we have that $V = \bigoplus_{x \in X} V_x$.

Let Y be another finite set and $\alpha : X \to Y$ a map of sets. The **pullback** α^* of α is defined by precomposition with α, while the **pushforward** α_* is defined using summation over the fibers of α:

$$\begin{aligned} \alpha^* : \mathbb{C}[Y] \to \mathbb{C}[X], &\quad \alpha^*(f) = f \circ \alpha \\ \alpha_* : \mathbb{C}[X] \to \mathbb{C}[Y], &\quad \alpha_*(f) : y \mapsto \sum_{x \in \alpha^{-1}(y)} f(x). \end{aligned}$$

For finite sets X and Y, consider the linear map $\Phi : \mathbb{C}[X \times Y] \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y])$ taking $A \in \mathbb{C}[X \times Y]$ to the integral transform $f \mapsto \sum_{x \in X} A(x, -) f(x)$. In terms of the projections

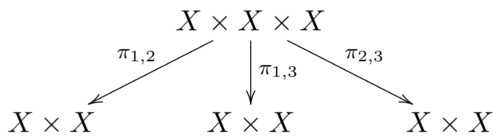
$$\begin{array}{ccccc} & & X \times Y & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ X & & & & Y \end{array}$$

the map Φ can be written as $A \mapsto (\pi_2)_*(A \cdot \pi_1^*(-))$. On the other hand, given any linear map $\psi \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y])$, define an element of $\mathbb{C}[X \times Y]$ by sending (x, y) to $\psi(\delta_x)(y)$; thus we have a map $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y]) \to \mathbb{C}[X \times Y]$. It is straightforward to check that these are mutual inverse maps, hence

$$\Phi : \mathbb{C}[X \times Y] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y])$$

is an isomorphism. It also true that $\mathbb{C}[X] \otimes \mathbb{C}[Y] \simeq \mathbb{C}[X \times Y]$.

In the case that $X = Y$, the set $\mathbb{C}[X \times X]$ can be given multiplication making Φ an isomorphism of algebras. Consider the projections



If A and B are complex-valued functions on $X \times X$, then define $A * B = (\pi_{1,3})_*(\pi_{1,2}^* A \cdot \pi_{2,3}^* B)$. To be explicit:

$$A * B(x, y) = \sum_{z \in X} A(x, z)B(z, y).$$

It is almost immediate from this last formula that $(\mathbb{C}[X \times X], *)$ is isomorphic to the matrix algebra $M_n(\mathbb{C}) \simeq \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X])$.

Recall that two rings R and S are **Morita equivalent** if the category R-Mod of R-modules is equivalent to the category S-Mod of S-modules.

Proposition 1. *The algebra* $(\mathbb{C}[X \times X], *)$ *is Morita equivalent to* $\mathbb{C}$.

Proof. We may assume that $X = \{1, 2, \ldots, n\}$ and identify the delta function at (i, j) in $\mathbb{C}[X \times X]$ with the matrix $E_{i,j} \in M_n(\mathbb{C})$ having 1 as its (i, j)-entry and zero elsewhere.

Consider the diagonal map $(\mathbb{C}[X], \text{ptwise}) \to (\mathbb{C}[X \times X], *)$ sending δ_i to $E_{i,i}$. Since this map is an algebra homomorphism, any $\mathbb{C}[X \times X]$-module V also has the structure of a $\mathbb{C}[X]$-module, i.e. a vector bundle on X. Write $V = \bigoplus V_i$. We leave it as an exercise to check that $E_{i,j}$ gives an isomorphism between V_i and V_j. Therefore, the data of a $\mathbb{C}[X \times X]$-module is given by a single vector space.

Conversely, given a vector space W, let $V = X \times W$ be the trivial vector bundle on X. Then V carries a natural action of $\mathbb{C}[X \times X]$. Specifically, the basis element $E_{i,j}$ of $\mathbb{C}[X \times X]$ acts as the composition

$$V \twoheadrightarrow V_i = \{i\} \times W \to \{j\} \times W = V_j \hookrightarrow V,$$

where the first map is the identity on the fiber V_i over i and sends all other fibers to $0 \in V_i \simeq W$, the second map is the identity on W, and the third map is inclusion. $\square$

In this way, we have proven the standard result that all matrix algebras over $\mathbb{C}$ are Morita equivalent to $\mathbb{C}$ using somewhat geometric techniques. Now we consider a generalization. Let $\alpha : X \to Y$ be a surjective function between finite sets. Consider the fiber product

$$X \times_Y X = \{(x_1, x_2) \in X \times X \ : \ \alpha(x_1) = \alpha(x_2)\}.$$

Replacing '$\times$' with '$\times_Y$' in the diagram displaying projections from $X \times X \times X$, we observe that $\mathbb{C}[X \times_Y X]$ is a subalgebra of $(\mathbb{C}[X \times X], *)$. The corresponding subalgebra of $M_n(\mathbb{C})$ consists of block diagonal matrices of the following form: there is one block for each element y of Y, and its size is given by the size of $\alpha^{-1}(y)$. Arguments similar to those in the proof of Proposition 1 can be used to prove the following:

Proposition 2. *The algebra* $(\mathbb{C}[X \times_Y X], *)$ *is Morita equivalent to* $\mathbb{C}[Y]$.

3 The group algebra $\mathbb{C}[G]$

Let G be a finite group. Consider the diagram

$$\begin{array}{ccccc} & & G \times G & & \\ & \overset{\pi_1}{\swarrow} & \downarrow m & \overset{\pi_2}{\searrow} & \\ G & & G & & G \end{array} \tag{1}$$

where the middle arrow is the multiplication map on G. Endow $\mathbb{C}[G]$ with a convolution product: $f_1 * f_2 = m_*(\pi_1^* f_1 \cdot \pi_2^* f_2)$, that is,

$$f_1 * f_2(g) = \sum_{xy=g} f_1(x) f_2(y) = \sum_{x \in G} f_1(x) f_2(x^{-1}g).$$

The **group algebra** of G is defined as $(\mathbb{C}[G], *)$. We make some elementary observations about the group algebra. A basis for $\mathbb{C}[G]$ is given by the delta functions $\{\delta_g : g \in G\}$, and these satisfy the relations $\delta_g * \delta_h = \delta_{gh}$. The multiplicative unit of $\mathbb{C}[G]$ is δ_e, where e is the identity element of G. In particular, every δ_g is invertible in $\mathbb{C}[G]$.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on a complex vector space V. Letting δ_g act by $\rho(g)$ and extending linearly, we see that V acquires the structure of a (left) $\mathbb{C}[G]$-module. Conversely, if V is a $\mathbb{C}[G]$-module, then V carries the structure of a representation of G, where g acts as δ_g. We conclude that there is a bijection between the $\mathbb{C}[G]$-module structures on V and the representations of G on V. In other words, there is an equivalence between the category $\mathbb{C}[G]$-Mod of $\mathbb{C}[G]$-modules and the category $\mathrm{Rep}(G)$ of complex representations of G. Moreover, this equivalence commutes with the forgetful functors:

$$\begin{array}{ccc} \mathbb{C}[G]\text{-Mod} & \xrightarrow{\ \simeq\ } & \mathrm{Rep}(G) \\ & {\scriptstyle \text{forget}} \searrow \quad \swarrow {\scriptstyle \text{forget}} & \\ & \mathrm{Vec}_{\mathbb{C}} & \end{array}.$$

Thinking of G as a finite set, recall that $\mathbb{C}[G \times G]$ has a matrix multiplication, which we now denote $(\mathbb{C}[G \times G], \text{matrix})$. The group G acts diagonally on $\mathbb{C}[G \times G]$ as $(g \cdot A)(h, k) = A(g^{-1}h, g^{-1}k)$. The following proposition allows us to realize the convolution product as a matrix multiplication.

Proposition 3. *The space $\mathbb{C}[G \times G]^G$ of G-invariant functions is a subalgebra of $(\mathbb{C}[G \times G], \text{matrix})$. There is an isomorphism of algebras $(\mathbb{C}[G], *)$ and $(\mathbb{C}[G \times G]^G, \text{matrix})$.*

Proof. The proof of the first statement is straightforward. We leave the reader to verify that the maps $\mathbb{C}[G] \to \mathbb{C}[G \times G]^G : f \mapsto [(h, k) \mapsto f(h^{-1}k)]$ and $\mathbb{C}[G \times G]^G \to \mathbb{C}[G] : \phi \mapsto [g \mapsto \phi(g^{-1}, 1)]$ are mutual inverses. □

Observe that G acts on $\mathbb{C}[G]$ by conjugation: $f^x(g) = f(x^{-1}gx)$. The **class functions** on G, denoted $\mathbb{C}[G]^G$ or $\mathbb{C}[G/G]$ or $\mathbb{C}[G/\mathrm{ad}G]$, are the fixed points of this action:

$$\mathbb{C}[G/G] = \{f \in \mathbb{C}[G] : f(xgx^{-1}) = f(g) \text{ for all } x, g \in G\}.$$

Recall that the **cocenter**, or **abelianization**, of an algebra A over $\mathbb{C}$ is defined as the A-module $A/[A,A]$ where $[A,A]$ is the subspace generated by all elements of the form $ab - ba$. The map $\pi : A \to A/[A,A]$ from A to the cocenter has the following universal property. Suppose V is a vector space and $f : A \to V$ is a linear map with the property that $f(ab) = f(ba)$ for all $a, b \in A$, that is, f is a trace map. Then f factors uniquely through π. For this reason, the quotient map π is called the **universal trace** of A. If $\tilde{\pi} : A \to C$ is another map satisfying the same universal property as π, then we can identify C with the cocenter of A. We leave the proof of the following proposition as an exercise.

Proposition 4. *The class functions $\mathbb{C}[G/G]$ are the center of the group algebra $\mathbb{C}[G]$. Moreover, the projection $\pi : \mathbb{C}[G] \to \mathbb{C}[G/G]$ defined on basis elements by*

$$\pi(\delta_g) = \frac{1}{|G|} \sum_{x \in G} \delta_{xgx^{-1}}.$$

is a universal trace, hence the class functions $\mathbb{C}[G/G]$ can be identified with the cocenter of the group algebra $\mathbb{C}[G]$.

Remark. In later talks we will see that the center of an algebra is its degree 0 Hochschild cohomology and the cocenter is its degree 0 Hochschild homology. Hence we have that $HH_0(\mathbb{C}[G]) = HH^0(\mathbb{C}[G]) = \mathbb{C}[G/G]$.

4 Induced representations

Let G be a finite group and K a subgroup of G. In this case, $\mathbb{C}[K]$ is a subalgebra of $\mathbb{C}[G]$ and any representation of G is a representation of K by restriction. Thus we have a functor

$$\mathrm{Res}_G^K : \mathrm{Rep}(G) \to \mathrm{Rep}(K).$$

Natural questions are: does the functor Res_G^K have a left adjoint? a right adjoint? The answer to both questions turns out to be yes.

A left adjoint to Res_G^K is given by the **induction** functor

$$\begin{aligned} \mathrm{Ind}_K^G : \mathrm{Rep}(K) &\to \mathrm{Rep}(G) \\ W &\mapsto \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W. \end{aligned}$$

Note that $\mathbb{C}[G]$ is a $\mathbb{C}[G]$-$\mathbb{C}[K]$-bimodule via multiplication in the group algebra and that Ind_K^G is an additive functor, i.e. $\mathrm{Ind}_K^G(W_1 \oplus W_2) \simeq \mathrm{Ind}_K^G(W_1) \oplus \mathrm{Ind}_K^G(W_2)$. Thinking of representations as modules for the group algebra, it is straightforward to verify that there are indeed isomorphisms

$$\mathrm{Hom}_K(W, \mathrm{Res}_G^K(V)) \simeq \mathrm{Hom}_G(\mathrm{Ind}_K^G(W), V)$$

functorial in $V \in \mathrm{Rep}(G)$ and $W \in \mathrm{Rep}(K)$.

For example, let $\mathbb{C}_{\mathrm{triv}}$ denote the trivial representation of K. The induced representation of G can be identified with the invariants of $\mathbb{C}[G]$ under the right action of K, or, equivalently, functions on the left cosets G/K. In symbols, $\mathrm{Ind}_K^G(\mathbb{C}_{\mathrm{triv}}) = \mathbb{C}[G/K]$.

Now we describe a right adjoint to the restriction functor. Let $\sigma : K \to \mathrm{GL}(W)$ be a representation of K and define the **coinduction** functor $\mathrm{Coind}_K^G : \mathrm{Rep}(K) \to \mathrm{Rep}(G)$ as

$$\mathrm{Coind}_K^G(W) = \{f : G \to W : f(kg) = \sigma(k)f(g) \ \text{ for all } \ g \in G, k \in K\}.$$

The action of G is given by $(g \cdot f)(x) = f(xg)$. Equivalently, $\mathrm{Coind}_K^G(W) = \mathrm{Hom}_K(\mathbb{C}[G], W)$, where K acts on $\mathbb{C}[G]$ by left multiplication and G acts on $f : \mathbb{C}[G] \to W$ as $(g \cdot f)(\delta_x) = f(\delta_{xg})$.

Proposition 5. *Let V be a representation of G and W a representation of K. Then*

$$\mathrm{Hom}_K(\mathrm{Res}_G^K(V), W) \simeq \mathrm{Hom}_G(V, \mathrm{Coind}_K^G(W)).$$

Consequently, Coind_K^G is a right adjoint to Res_G^K.

Proof. The vector space

$$\mathrm{Hom}(V, \mathrm{Hom}(\mathbb{C}[G], W)) \simeq \mathrm{Hom}(V \otimes \mathbb{C}[G], W) \simeq \mathrm{Hom}(\mathbb{C}[G], \mathrm{Hom}(V, W)) \tag{2}$$

admits a left action of G and a right action of K. The two actions commute; taking $G \times K$-invariants on the far left side of equation 2, we obtain:

$$\mathrm{Hom}(V, \mathrm{Hom}(\mathbb{C}[G], W))^{G\times K} = \mathrm{Hom}_G(V, \mathrm{Hom}_K(\mathbb{C}[G], W)) = \mathrm{Hom}_G(V, \mathrm{Coind}_K^G(W)).$$

Taking $G \times K$-invariants on the far right side of equation 2, we obtain:

$$\mathrm{Hom}(\mathbb{C}[G], \mathrm{Hom}(V, W))^{G\times K} = \mathrm{Hom}_G(\mathbb{C}[G], \mathrm{Hom}_K(\mathrm{Res}_G^K(V), W)) = \mathrm{Hom}_K(\mathrm{Res}_G^K(V), W).$$

The proposition now follows. □

Proposition 6. *As representations of G, $\mathrm{Ind}_K^G(W)$ and $\mathrm{Coind}_K^G(W)$ are isomorphic.*

Proof. Let $\sigma : K \to \mathrm{GL}(W)$ be the the group homomorphism giving the action of K on W. Define a linear map $\epsilon : \mathbb{C}[G] \times W \to \mathrm{Hom}(\mathbb{C}[G], W)$ by

$$\epsilon(\delta_x, w)(\delta_y) = \begin{cases} \sigma(yx)w & \text{if } yx \in K \\ 0 & \text{otherwise} \end{cases}$$

and extending linearly. One shows that the map $\epsilon(\delta_x, w)$ is K-equivariant and that ϵ is $\mathbb{C}[K]$-bilinear, so we obtain a map

$$\epsilon : \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W \to \mathrm{Hom}_{\mathbb{C}[K]}(\mathbb{C}[G], W).$$

It is not difficult to see that ϵ is G-equivariant.

Fix a set of left coset representatives $\{g_1, \ldots, g_n\}$ for K in G. Given $\phi \in \mathrm{Hom}_{\mathbb{C}[K]}(\mathbb{C}[G], W)$, a computation shows that $\epsilon(\sum_i \delta_{g_i} \otimes \phi(g_i^{-1})) = \phi$, and this proves that ϵ is surjective. Note that $\mathbb{C}[G] \otimes_{\mathbb{C}[K]} W = \bigoplus_i \delta_{g_i} \otimes W$; therefore, to show that ϵ is injective, it suffices to show that $\epsilon(\sum_i \delta_{g_i} \otimes w_i)$ is zero if and only if w_i is zero for all i. For each i, let $\alpha(i) \in \{1, \ldots, n\}$ be the unique index such that $g_i^{-1}K = g_{\alpha(i)}K$. Since the map $\epsilon(\sum_i \delta_{g_i} \otimes w_i)$ is K-equivariant, it is determined by its values on $\{\delta_{g_j}\}$. Direct computations verify that $\epsilon(\sum_i \delta_{g_i} \otimes w_i) = 0$ if and only if for all j we have

$$\sum_{\{i: g_j g_i \in K\}} \sigma(g_j g_i) w_i = 0$$

which happens if and only if $\sigma(g_j g_{\alpha(j)}) w_{\alpha(j)} = 0$ for all j. The fact that $\sigma(g_j g_{\alpha(j)})$ is invertible and α is a bijection imply that $w_i = 0$ for all i. □

Consequently, we have a single induction functor that is left and right adjoint to the restriction functor. These adjunctions are known as **Frobenius reciprocity**:

$$\mathrm{Hom}_G(\mathrm{Ind}_K^G(W), V) \simeq \mathrm{Hom}_K(W, \mathrm{Res}_G^K(V))$$
$$\mathrm{Hom}_G(V, \mathrm{Ind}_K^G(W)) \simeq \mathrm{Hom}_K(\mathrm{Res}_G^K(V), W).$$

In particular, both restriction and induction are exact functors.

Finally, we give a geometric picture of induced representations. Let W be a representation of K. Consider the trivial bundle $G \times W \to G$ on G. Let $G \times_K W$ be the 'balanced product' formed by taking the quotient of $G \times W$ by the equivalence relation $(g, w) \sim (gk, k^{-1} \cdot w)$ for any k in K. The map $G \times_K W \to G/K$ sending $[g, w]$ to the coset gK is well-defined and makes $G \times_K W$ a vector bundle over G/K. The induced representation $\mathrm{Ind}_K^G(W)$ can be defined as global sections of $G \times_K W$. Note that $G \times_K W \to G/K$ is trivial W-bundle over G/K once a complete set of coset representatives is chosen. The group G acts on the G/K by changing coset representatives, and this gives an action of G on the space of sections. We leave the details to the reader.

5 The Hecke algebra $\mathcal{H}(G, K)$

As in the previous section, let G be a finite group and K a subgroup of G. Let V be a representation of G and define V^K as the subspace of K-invariant vectors:

$$V^K = \{v \in V : k \cdot v = v \text{ for all } k \in K\}.$$

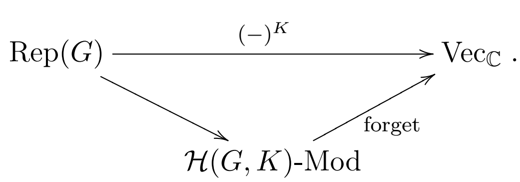
Observe that taking K-invariants is functorial, so we have a functor $(-)^K : \mathrm{Rep}(G) \to \mathrm{Vec}_{\mathbb{C}}$. By Frobenius reciprocity, this functor is representable by $\mathbb{C}[G/K]$:

$$V^K = \mathrm{Hom}_K(\mathbb{C}_{\mathrm{triv}}, \mathrm{Res}(V)) = \mathrm{Hom}_G(\mathrm{Ind}(\mathbb{C}_{\mathrm{triv}}), V) = \mathrm{Hom}_G(\mathbb{C}[G/K], V),$$

where $\mathbb{C}_{\mathrm{triv}}$ denotes the trivial representation of K. Since there is no danger of confusion, we have abbreviated the induction and restriction functors as Ind and Res. There is a natural right action of the algebra $\mathrm{End}_G(\mathbb{C}[G/K])$ on every $V^K = \mathrm{Hom}_G(\mathbb{C}[G/K], V)$ by precomposition:

$$(\phi, \alpha) \mapsto \phi \circ \alpha$$

for all $\alpha \in \mathrm{End}_G(\mathbb{C}[G/K])$ and $\phi \in \mathrm{Hom}_G(\mathbb{C}[G/K], V)$. The algebra $\mathcal{H}(G, K) := \mathrm{End}_G(\mathbb{C}[G/K])$ is known as the **Hecke algebra** of the pair (G, K). By the above comments, there is a factorization



The Yoneda lemma implies that $\mathcal{H}(G,K)^{\mathrm{op}} \simeq \mathrm{End}((-)^K)$.

Let $\langle \mathbb{C}[G/K] \rangle$ be the full subcategory of $\mathrm{Rep}(G)$ consisting of representations V such that $\mathrm{Hom}_G(\mathbb{C}[G/K], V) \neq 0$ (equivalently, $V^K \neq 0$), together with the zero representation. This is often referred to as the **subcategory generated** by $\mathbb{C}[G/K]$.

Proposition 7. *There is an equivalence of categories* $\langle \mathbb{C}[G/K] \rangle \simeq \mathcal{H}(G,K)$-Mod*.*

Proof. The Barr-Beck theorem provides one way to see this equivalence. Since the Barr-Beck theorem will feature in a later talk, the reader may wish to read this proof after learning the Barr-Beck theorem.

A left adjoint to the exact functor $(-)^K$ is the composition $\mathrm{Vec}_{\mathbb{C}} \to \mathrm{Rep}(K) \xrightarrow{\mathrm{Ind}} \mathrm{Rep}(G)$ where the first functor is the inclusion of vector spaces as the full subcategory of trivial representations. The corresponding monad on $\mathrm{Vec}_{\mathbb{C}}$ is given by tensoring with $\mathcal{H}(G,K)$ since

$$\mathbb{C} \mapsto (\mathrm{Ind}(\mathbb{C}_{\mathrm{triv}}))^K = \mathbb{C}[G/K]^K = \mathrm{Hom}(\mathbb{C}[G/K], \mathbb{C}[G/K]) = \mathcal{H}(G,K)$$

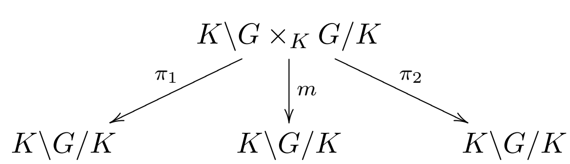
and extending additively. Since $V^K \neq 0$ for nonzero objects V of $\langle \mathbb{C}[G/K] \rangle$, the Barr-Beck theorem immediately implies the result. □

Therefore, the Hecke algebra allows us to probe into the category of representations of G. If K is small, then many representations of G will have K-invariants, so knowledge of $\mathcal{H}(G,K)$ and its category of modules is more valuable. However, in this case $\mathcal{H}(G,K)$ may be more difficult to understand. If K is large, then G/K is small and $\mathcal{H}(G,K)$ may have a simpler structure, for example it may be commutative. The disadvantage is that in this case we may acquire less information about the category $\mathrm{Rep}(G)$.

Let ${}^K\mathbb{C}[G]^K$ denote the left and right K-invariant functions in $\mathbb{C}[G]$. It is easy to see that ${}^K\mathbb{C}[G]^K$ can be identified with functions on the double cosets $\mathbb{C}[K\backslash G/K]$. In certain contexts, the Hecke algebra $\mathcal{H}(G,K)$ is defined as $\mathbb{C}[K\backslash G/K]$; this is justified by the following proposition.

Proposition 8. *The space* ${}^K\mathbb{C}[G]^K = \mathbb{C}[K\backslash G/K]$ *is a subalgebra of* $\mathbb{C}[G]$ *isomorphic to* $\mathcal{H}(G,K)$*.*

Instead of providing a detailed proof, we mention several ways to gain insight on the proposition. Recall that, in the definition of the multiplication in the group algebra, we considered a diagram with maps out of $G \times G$. The addition of appropriate quotients yields the following diagram, whose maps are well-defined:



Here '$\times_K$' again denotes the balanced product, as defined in the previous section. One can use this diagram to deduce that $\mathbb{C}[K\backslash G/K]$ is an algebra under convolution.

In order to demonstrate the isomorphism of $\mathbb{C}[K\backslash G/K]$ with the Hecke algebra, we can use the definition of the Hecke algebra and the representability of the functor $(-)^K$ to obtain isomorphisms of vector spaces

$$\mathcal{H}(G,K) = \mathrm{Hom}_G(\mathbb{C}[G/K], \mathbb{C}[G/K]) \simeq \mathbb{C}[G/K]^K \simeq \mathbb{C}[K\backslash G/K],$$

that are in fact isomorphisms of algebras.

Alternatively, consider the diagonal action of G on $\mathbb{C}[G/K \times G/K]$. In a manner similar to the above discussion of the group algebra, there are algebra isomorphisms

$$\mathcal{H}(G,K) \simeq \mathbb{C}[G/K \times G/K]^G \simeq \mathbb{C}[G\backslash(G/K \times G/K)] \simeq \mathbb{C}[K\backslash G/K].$$

Here we use the (easily verified) fact that the orbits of $G/K \times G/K$ under the diagonal action of G can be identified with the double coset space $K\backslash G/K$.

Another approach is to use the idempotents: it is a general fact that in any algebra A with an idempotent element e, the set eAe is a subalgebra isomorphic to $\mathrm{End}_A(Ae)$. In our case, take $A = \mathbb{C}[G]$ with the idempotent $e_K = \sum_{k\in K} \delta_k$. Simple computations show that $\mathbb{C}[G/K] = \mathbb{C}[G] * e_K$ and $\mathbb{C}[K\backslash G/K] = e_K * \mathbb{C}[G] * e_K$. Hence $\mathrm{End}_G(\mathbb{C}[G/K]) \simeq \mathbb{C}[K\backslash G/K]$.

To conclude this section, we describe a more general formulation of the Hecke algebra. Let W be an irreducible representation of K. Define the Hecke algebra of the triple (G,K,W) as $\mathcal{H}(G,K,W) = \mathrm{End}_G(\mathrm{Ind}(W))$. Consider the functor $\mathrm{Rep}(G) \to \mathrm{Vec}_{\mathbb{C}}$ taking a representation V to its "W-isotypic component" under K, that is, the largest subrepresentation of $\mathrm{Res}(V)$ isomorphic to some number of copies of W. Using identical arguments as above, one can see that this functor is representable by $\mathrm{Ind}(W)$ and establishes an equivalence between that category of $\mathcal{H}(G,K,W)$-modules and the full subcategory $\langle \mathrm{Ind}(W) \rangle$ of $\mathrm{Rep}(G)$.

6 Characters and the Frobenius character formula

Let V be a finite dimensional representation of a finite group G. Consider the 'matrix coefficients' map

$$\begin{aligned} \phi : \mathrm{End}(V) \simeq V^* \otimes V &\to \mathbb{C}[G] \\ v^* \otimes v &\mapsto [g \mapsto \langle v^*, g \cdot v\rangle]. \end{aligned}$$

The **character** of G on V is defined as the element $\chi_V := \phi(\mathrm{Id}_V) \in \mathbb{C}[G]$. For any fixed basis $\{e_i\}$ of V, the element $\mathrm{Id}_V \in \mathrm{End}(V)$ corresponds to $\sum e_i^* \otimes e_i \in V^* \otimes V$. So

$$\chi_V(g) = \sum_i e_i^*(g \cdot e_i) = \mathrm{tr}(e_j^*(g \cdot e_i)) = \mathrm{tr}(\rho(g))$$

where $\rho(g) = (e_j^*(g \cdot e_i))_{i,j}$ is the matrix giving the action of g on V in the basis $\{e_i\}$. Since the trace function on matricies is a class function, it follows that χ_V is also a class function, i.e. and element of $\mathbb{C}[G/G]$.

From now on, all representations of G are assumed to be finite dimensional. Let $V_1, \ldots, V_r$ be the irreducible representations of G, with characters $\chi_1, \ldots, \chi_r$. We review some basic facts about characters without proof.

1. There is a non-degenerate Hermitian inner product on the space of class functions $\mathbb{C}[G/G]$ given by
$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

2. The characters $\chi_1, \dots, \chi_r$ form an orthonormal basis for $\mathbb{C}[G/G]$. In particular, the number of irreducible representations of a finite group equals the number of conjugacy classes. Since any representation of G decomposes as a direct sum of irreducibles, we have further that a representation is determined by its character.

3. Let V and U be representations of G. Then $\chi_{V \oplus U} = \chi_V + \chi_U$ and $\chi_{V \otimes U} = \chi_V \cdot \chi_U$ (pointwise). Also,
$$\langle \chi_V, \chi_U \rangle = \dim \mathrm{Hom}_G(V, U).$$
Let K be a subgroup of G. We use the notation ψ_W for the character of a representation W of K. Therefore, Frobenius reciprocity implies that
$$\langle \chi_{\mathrm{Ind}(W)}, \chi_V \rangle = \langle \psi_W, \psi_{\mathrm{Res}(V)} \rangle \quad \text{and} \quad \langle \chi_V, \chi_{\mathrm{Ind}(W)} \rangle = \langle \psi_{\mathrm{Res}(V), \psi_W} \rangle.$$

4. As algebras, $\mathbb{C}[G] \simeq \bigoplus \mathrm{End}(V_i)$. The idempotents are
$$e_i = \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \cdot \delta_g \in \mathbb{C}[G]$$
for $1 \leq i \leq r$. On a representation U of G, the element e_i acts as a projection onto the isotypic component of U corresponding to V_i.

Let K be a subgroup of G. There is a well-defined map $\pi : K/K \to G/G$ sending the conjugacy class of k in K to its conjugacy class in G. The pushforward π_* sends a class function f on K to the class function
$$g \mapsto \frac{1}{|K|} \sum_{x \in G} \dot{f}(xgx^{-1})$$
on G, where $\dot{f} \in \mathbb{C}[G]$ conincides with f on K and is 0 otherwise. If W is a representation of K with character ψ, then we abbreviate by $\mathrm{Ind}(\psi)$ the character of the induced representation $\mathrm{Ind}_K^G(W)$. The following result is known as the **Frobenius character formula**:

Proposition 9. *Let ψ be the character of a representation W of K. Then $\mathrm{Ind}(\psi) = \pi_* \psi$, i.e.*
$$\mathrm{Ind}(\psi)(g) = \frac{1}{|K|} \sum_{x \in G} \dot{\psi}(xgx^{-1}).$$

Proof. Let $\eta \in \mathbb{C}[G/G]$ be arbitrary. Then

$$\begin{aligned}
\langle \eta, \pi_*\psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{\eta(g)} \cdot \pi_*\psi(g) = \frac{1}{|G||K|} \sum_{g \in G} \sum_{x \in G} \overline{\eta(g)} \cdot \dot{\psi}(xgx^{-1}) \\
&= \frac{1}{|G||K|} \sum_{g \in G} \sum_{k \in K} \sum_{\substack{x \in G \\ xgx^{-1} = k}} \overline{\eta(g)} \cdot \psi(k) = \frac{1}{|G||K|} \sum_{k \in K} \sum_{x \in G} \sum_{\substack{g \in G \\ g = x^{-1}kx}} \overline{\eta(g)} \cdot \psi(k) \\
&= \frac{1}{|G||K|} \sum_{k \in K} \sum_{x \in G} \overline{\eta(x^{-1}kx)} \cdot \psi(k) = \frac{1}{|G||K|} \sum_{k \in K} \sum_{x \in G} \overline{\eta(k)} \cdot \psi(k) \\
&= \frac{|G|}{|G||K|} \sum_{k \in K} \overline{\eta(k)} \cdot \psi(k) = \langle \mathrm{Res}(\eta), \psi \rangle = \langle \eta, \mathrm{Ind}(\psi) \rangle.
\end{aligned}$$

The first three equalities follow from the definitions of $\langle , \rangle$, $\pi_*\psi$, and $\dot{\psi}$. The sixth equality uses the fact that η is a class function, and the last equality invokes Frobenius reciprocity. Since η is arbitrary, the result follows from the non-degeneracy of the inner product on $\mathbb{C}[G/G]$. □

We describe another perspective on the Frobenius character formula. Recall that the **Grothendieck group** of an (essentially small) abelian category $\mathcal{C}$ is defined as the free abelian group on the set $\{[X]\}$ of isomorphism classes of objects of $\mathcal{C}$ modulo the relation $[X \oplus Y] = [X] + [Y]$. The Grothendieck group of $\mathcal{C}$ is denoted $K(\mathcal{C})$. If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between abelian categories, then F gives rise to a homomorphism $\tilde{F} : K(\mathcal{C}) \to K(\mathcal{D})$ between the Grothendieck groups defined by $\tilde{F}([X]) = [F(X)]$.

The facts listed earlier in this section imply that the complexified Grothendieck group $K(\mathrm{Rep}_f(G)) \otimes_{\mathbb{Z}} \mathbb{C}$ of the category $\mathrm{Rep}_f(G)$ of finite-dimensional complex representations of G can be identified with the vector space $\mathbb{C}[G/G]$ of class functions[1]. Passing from $\mathrm{Rep}_f(G)$ to $K(\mathrm{Rep}_f(G)) \otimes \mathbb{C}$ replaces a representation by its character. Now let K be a subgroup of G. The induction and restriction functors

$$\mathrm{Rep}_f(K) \underset{\mathrm{Res}}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathrm{Rep}_f(G)$$

give linear maps

$$\mathbb{C}[K/K] \rightleftarrows \mathbb{C}[G/G]\ .$$

The claim is that these linear maps are π_* and π^*, where $\pi : K/K \to G/G$ is the function described above. In other words, on the level of characters, $\mathrm{Ind} = \pi_*$ and $\mathrm{Res} = \pi^*$. More precisely:

Proposition 10. *Let V be a representation of G and W a representation of K. Then $\psi_{\mathrm{Res}(V)} = \pi^*(\chi_V)$ and $\chi_{\mathrm{Ind}(W)} = \pi_*(\psi_W)$.*

The first equality is easy since $\pi^*(\chi_V) = \chi_V|_K$, while second equality is just Proposition 9.

[1] In fact, $\mathrm{Rep}_f(G)$ is a tensor category, so $K(\mathrm{Rep}_f(G)) \otimes \mathbb{C}$ is an algebra, and it is isomorphic to $(\mathbb{C}[G/G], \text{ptwise})$, but this extra structure is not relevant for the present discussion.

7 Exercises

1. Complete the proof of Proposition 1 by showing that $E_{i,j}$ gives an isomorphism between V_i and V_j. Prove Proposition 2 by adopting arguments from the proof of Proposition 1.

2. Let X and Y be finite sets. Show that $\mathbb{C}[X \times Y] = \mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[Y]$. More generally, show that $\mathbb{C}[X \times_Z Y] = \mathbb{C}[X] \otimes_{\mathbb{C}[Z]} \mathbb{C}[Y]$ where Z is a finite set and with maps $X \to Z$ and $Y \to Z$.

3. Complete the proof of Proposition 3 by showing that the maps given in the text are mutual inverses.

4. Provide a proof for Propostion 4.

5. Let K be a subgroup of a finite group G and W a representation of K. Show that the representation of G on sections of the bundle $G \times_K W \to G/K$ is isomorphic to the representation $\mathrm{Coind}_K^G(W)$ (and hence also to $\mathrm{Ind}_K^G(W)$).

6. Provide a detailed proof of Proposition 8.

7. Let V be a finite dimensional representation of G. This exercise gives another way to see that χ_V is a class function. Suppose first that V is irreducible. Use Schur's lemma to prove that $\mathrm{End}(V)^G = \mathbb{C} \cdot \mathrm{Id}_V$. Prove that the map $\phi : \mathrm{End}(V) \to \mathbb{C}[G]$ dicussed in the text is G-equivariant for the action of G on $\mathbb{C}[G]$ by conjugation, and conclude that $\phi(\mathrm{Id}_V)$ is a class function. Use the complete reducibility of finite dimensional representations to prove the result for arbitrary V.

8. If G acts on a set X, show that $\mathbb{C}[X]$ carries the structure of a representation of G. Assume X is finite. Prove that $\chi_{\mathbb{C}[X]}$ counts fixed points: $\chi_{\mathbb{C}[X]}(g) = \#\{x \in X : g \cdot x = x\}$. Observe that $\mathcal{H}(G, K)$ acts on $\mathbb{C}[X]^K = \mathbb{C}[K \backslash X]$.

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