Oberwolfach Report: The algebraic geometry of KLS-polynomials

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This report is based on the paper [Pro18]. Let P be a finite poset. Let

$$I(P) := \prod_{x \leq y} \mathbb{Z}[t].$$

For any $f \in I(P)$ and $x \leq y \in P$, let $f_{xy}(t) \in \mathbb{Z}[t]$ denote the corresponding component of f. The group $I(P)$ admits a ring structure with product given by convolution:

$$(fg)_{xz}(t) := \sum_{x \leq y \leq z} f_{xy}(t) g_{yz}(t).$$

Let $r : P \to \mathbb{Z}$ be a function with the property that, if $x < y$, then $r_{xy} := r(y) - r(x) > 0$. Let $\mathscr{I}(P) \subset I(P)$ denote the subring of functions f with the property that the degree of $f_{xy}(t)$ is less than or equal to r_{xy} for all $x \leq y$. The ring $\mathscr{I}(P)$ admits an involution $f \mapsto \bar{f}$ defined by the formula

$$\bar{f}_{xy}(t) := t^{r_{xy}} f_{xy}(t^{-1}).$$

An element $\kappa \in \mathscr{I}(P)$ is called a ***P*-kernel** if $\kappa_{xx}(t) = 1$ for all $x \in P$ and $\kappa^{-1} = \bar{\kappa}$. Let

$$\mathscr{I}_{1/2}(P) := \Big\{ f \in \mathscr{I}(P) \;\Big|\; f_{xx}(t) = 1 \text{ for all } x \in P \text{ and } \deg f_{xy}(t) < r_{xy}/2 \text{ for all } x < y \in P \Big\}.$$

Various versions of the following theorem appear in [Sta92, Corollary 6.7], [Dye93, Proposition 1.2], and [Bre99, Theorem 6.2]; see [Pro18, Theorem 2.2] for this precise statement.

Theorem 1. *If $\kappa \in \mathscr{I}(P)$ is a P-kernel, there exists a unique pair of functions $f, g \in \mathscr{I}_{1/2}(P)$ such that $\bar{f} = \kappa f$ and $\bar{g} = g\kappa$.*

The polynomials $f_{xy}(t)$ and $g_{xy}(t)$ are called right and left Kazhdan-Lusztig-Stanley polynomials, or **KLS-polynomials** for short. There are a number of special cases in which these polynomials have been studied.

- Let W be a Coxeter group, equipped with the Bruhat order and the rank function given by the length of an element of W. The classical R-polynomials $\{R_{vw}(t) \mid v \leq w \in W\}$ form a W-kernel, and the classical Kazhdan-Lusztig polynomials $\{f_{xy}(t) \mid v \leq w \in W\}$ are the associated right KLS-polynomials. If W is finite, then there is a maximal element $w_0 \in W$, and $g_{vw}(t) = f_{(w_0 w)(w_0 v)}(t)$.
- Let P be the poset of faces of a polytope Δ, with weak rank function given by relative dimension (where $\dim \emptyset = -1$). Then the function $\kappa_{xy}(t) = (t-1)^{r_{xy}}$ is a P-kernel, and $g_{\emptyset\Delta}(t)$

is called the g-polynomial of Δ [Sta92, Example 7.2]. The dual polytope Δ^* has the property that its face poset is opposite to P, and this implies that $f_{\emptyset\Delta}(t)$ is equal to the g-polynomial of Δ^*.

- For any P, define $\zeta \in \mathscr{I}(P)$ by the formula $\zeta_{xy}(t) = 1$ for all $x \leq y \in P$. Then the characteristic polynomial $\chi := \zeta^{-1}\bar{\zeta}$ is a P-kernel. The associated left KLS-polynomials are identically 1, but the right KLS-polynomials can be very interesting! In particular, each coefficient of $f_{xy}(t)$ can be expressed as alternating sums of multi-indexed Whitney numbers for the interval $[x, y] \subset P$ [PXY18, Theorem 3.3]. If P is the lattice of flats of a matroid M with the usual rank function, with minimum element 0 and maximum element 1, then $f_{01}(t)$ is called the Kazhdan-Lusztig polynomial of M [EPW16].

Each of these families of examples has a subfamily in which the KLS-polynomials have a cohomological interpretation.

- Let G be a split reductive algebraic group. Let $B, B^* \subset G$ be Borel subgroups with the property that $T := B \cap B^*$ is a maximal torus. Let $W := N(T)/T$ be the Weyl group. For all $w \in W$, let
$$V_w := \{gB \mid g \in BwB\}$$
be the corresponding Schubert cell in the flag variety G/B. For any $v \leq w$, the Kazhdan-Lusztig polynomial $f_{v,w}(t)$ is equal to the Poincaré polynomial for the cohomology of the stalk of the intersection cohomology sheaf IC_{V_w} at a point of V_v [KL80, Corollary 4.8].

- Let Δ be a rational polytope with associated projective toric variety $X(\Delta)$, and let $Y(\Delta)$ denote the affine cone over $X(\Delta)$. Then the g-polynomial $g_{\emptyset\Delta^*}(t) = f_{\emptyset\Delta}(t)$ is equal to the Poincaré polynomial for the intersection cohomology of $Y(\Delta)$ [DL91, Theorem 6.2], [Fie91, Theorem 1.2], or equivalently the Poincaré polynomial for the stalk of $\mathrm{IC}_{Y(\Delta)}$ at the cone point.

- Let $\mathcal{A}$ be a collection of nonzero linear forms on a vector space V, and let M be the associated matroid. Let $R_{\mathcal{A}}$ be the Orlik-Terao algebra, which is the subalgebra of rational functions on V generated by the reciprocals of the linear forms. Then the Kazhdan-Lusztig polynomial of M is equal to the Poincaré polynomial for the intersection cohomology of $\operatorname{Spec} R_{\mathcal{A}}$ [EPW16, Theorem 3.10], or equivalently the Poincaré polynomial for the stalk of $\mathrm{IC}_{R_{\mathcal{A}}}$ at the cone point.

Each of these statements was proved independently, but it is in fact possible to prove all three in a uniform way. Suppose that we have a variety Y over $\mathbb{F}_q$ and a stratification

$$Y = \bigsqcup_{x \in P} V_x.$$

We define a partial order on P by putting $x \leq y \iff V_x \subset \bar{V}_y$ and a rank function $r(x) = \dim V_x$. Suppose that, for each $x \in P$, we have a **conical slice** $C_x \subset Y$ to the stratum V_x (see [Pro18, Section 3.1] for a precise definition of a conical slice). Finally, suppose that there exists an element $\kappa \in \mathscr{I}(P)$ such that $|C_x(\mathbb{F}_{q^s}) \cap V_y(\mathbb{F}_{q^s})| = \kappa(q^s)$ for all $s > 0$.

Theorem 2. [Pro18, Theorem 3.6] *The element $\kappa \in \mathscr{I}(P)$ is a P-kernel, and for any $x \leq y$, the associated right KLS-polynomial $f_{xy}(t)$ is equal to the Poincaré polynomial for the ℓ-adic étale cohomology of the stalk of $\mathrm{IC}_{\bar{V}_y}$ at a point of V_x.*

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