

The equivariant Kazhdan-Lusztig polynomial of a matroid

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Abstract. We define the equivariant Kazhdan-Lusztig polynomial of a matroid equipped with a group of symmetries, generalizing the nonequivariant case. We compute this invariant for arbitrary uniform matroids and for braid matroids of small rank.

1 Introduction

The Kazhdan-Lusztig polynomial $P_M(t) \in \mathbb{Z}[t]$ of a matroid M was introduced in [EPW16]. In the case where M is realizable by a linear space $V \subset \mathbb{C}^n$, the coefficient of t^i in $P_M(t)$ is equal to the dimension of the intersection cohomology group $IH^{2i}(X_V; \mathbb{C})$, where X_V is the “reciprocal plane” of V [EPW16, Proposition 3.12]. In particular, this implies that $P_M(t) \in \mathbb{N}[t]$ whenever M is realizable. We conjectured [EPW16, Conjecture 2.3] that $P_M(t) \in \mathbb{N}[t]$ for every matroid M . We also gave some computations of $P_M(t)$ for uniform matroids and braid matroids of small rank.

The purpose of this paper is to define a more refined invariant. Given a matroid M equipped with an action of a finite group W , we define the equivariant Kazhdan-Lusztig polynomial $P_M^W(t)$. The coefficients of this polynomial are not integers, but rather virtual representations of the group W . If W is the trivial group, the ring of virtual representations of W is \mathbb{Z} , and $P_M^W(t)$ is equal to the ordinary polynomial $P_M(t)$. More generally, the polynomial $P_M(t)$ may be obtained from $P_M^W(t)$ by sending a virtual representation to its dimension. If M is equivariantly realizable by a linear space $V \subset \mathbb{C}^n$, the coefficient of t^i in $P_M^W(t)$ is equal to the intersection cohomology group $IH^{2i}(X_V; \mathbb{C})$, regarded as a representation of W (Corollary 2.12). In particular, this implies that the coefficients of $P_M^W(t)$ are honest (rather than virtual) representations of W whenever M is equivariantly realizable. We conjecture that this is the case even in the non-realizable case (Conjecture 2.13). We compute the coefficients of $P_M^W(t)$ for arbitrary uniform matroids (Theorem 3.1) and for braid matroids of small rank (Section 4.3).

It is reasonable to ask why bother with an equivariant version of this invariant, especially since there are still many things that we do not understand about the nonequivariant version. We have four answers to this question, all of which are illustrated by the case of uniform matroids. To set notation, let $U_{m,d}$ be the uniform matroid of rank d on a set of $m+d$ elements, which is equipped with a natural action of the symmetric group S_{m+d} . Let $C_{i,m,d}$ be the coefficient of t^i in the equivariant Kazhdan-Lusztig polynomial of $U_{m,d}$, and let $c_{i,m,d} = \dim C_{i,m,d}$ be the coefficient of t^i in the nonequivariant Kazhdan-Lusztig polynomial.

- **Nicer formulas:** Our formula for $C_{i,m,d}$ (Theorem 3.1) is very simple; it is a multiplicity-free sum of irreducible representations that are easy to describe. We could of course use the hook-

length formula for the dimension of an irreducible representation of S_{m+d} to derive a formula for $c_{i,m,d}$, but the resulting formula is messy and unenlightening. Indeed, we computed a table in the appendix of [EPW16] consisting of the numbers $c_{i,m,d}$ for small values of i , m , and d , and at that time we were unable even to guess the general formula. It was only by keeping track of the extra structure that we were able to see the essential pattern.

- **More powerful tools:** After we figured out the correct statement of Theorem 3.1, we attempted to prove the formula for $c_{i,m,d}$ directly (without going through Theorem 3.1), and we failed. The Schubert calculus techniques that we employ in the proof of Theorem 3.1 are considerably more powerful than the tools to which we have access in the nonequivariant setting.
- **Representation stability:** The sequence of representations $C_{i,m,d}$ is uniformly representation stable in the sense of Church and Farb [CF13], which essentially means that it admits a description that is independent of d , provided that $d \geq m + 2i$ (Remark 3.6). This phenomenon cannot be seen by looking at the numbers $c_{i,m,d}$.
- **Non-realizable examples:** It is difficult to write down examples of non-realizable irreducible matroids for which we can compute the Kazhdan-Lusztig polynomial, and therefore we had no nontrivial checks of our non-negativity conjecture in the non-equivariant setting. On the other hand, the uniform matroid $U_{m,d}$ is equivariantly non-realizable provided that both d and m are greater than 1. This means that Theorem 3.1 provides good evidence for Conjecture 2.13, and therefore by extension for [EPW16, Conjecture 2.3].

The paper is structured as follows. In Section 2, we define the equivariant characteristic polynomial and use it to define the equivariant Kazhdan-Lusztig polynomial. This section closely mirrors Section 2 of [EPW16], but some of the basic lemmas are much more technical in the equivariant setting. In particular, Lemma 2.4 is an equivariant version of a well-known statement that is usually proved via Möbius inversion. This proof does not work in the equivariant context (due essentially to the fact that the equivariant analogue of the Möbius algebra is not associative), so we needed to find a different approach.

Section 3 is devoted to the study of uniform matroids, and in particular the statement and proof of Theorem 3.1. Our main technique is to express everything in terms of generating functions that encode all three parameters i , m , and d , and then to manipulate our functional equations until they can be solved using repeated applications of the Pieri rule. Section 4 treats the case of braid matroids. In this case we are not able to give a general formula for the equivariant Kazhdan-Lusztig polynomial, but we do derive generating function identities that allow us to compute the polynomial explicitly in small rank.

Finally, in Section 5 we introduce the notion of equivariant log concavity, which is a generalization of the usual notion of log concavity to the equivariant setting. The statement that the coefficients of the characteristic polynomial of a matroid form a log concave sequence goes back to the 1960s, and was only recently proved by Adiprasito, Huh, and Katz [AHK]. The statement

that the coefficients of the Kazhdan-Lusztig polynomial of a matroid form a log concave sequence was conjectured in [EPW16, Conjecture 2.5]. Here we make the two analogous conjectures in the equivariant setting (Conjecture 5.3), and we prove equivariant log concavity of the characteristic polynomial of a uniform matroid (Proposition 5.7). The notion of equivariant log concavity will be further developed in a future paper.

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2 Definition

Let M be a matroid on the ground set \mathcal{I} , and let W be a finite group acting on \mathcal{I} and preserving M . We will refer to this collection of data as an **equivariant matroid** $W \curvearrowright M$. Let

$$\text{grVRep}(W) := \text{VRep}(W) \otimes_{\mathbb{Z}} \mathbb{Z}[t] \quad \text{and} \quad \text{grRep}(W) := \text{Rep}(W) \otimes_{\mathbb{N}} \mathbb{N}[t].$$

Note that, for any group homomorphism $\varphi : W' \rightarrow W$, we obtain ring maps

$$\varphi^* : \text{VRep}(W) \rightarrow \text{VRep}(W') \quad \text{and} \quad \varphi^* : \text{grVRep}(W) \rightarrow \text{grVRep}(W')$$

taking honest representations to honest representations.

2.1 The equivariant characteristic polynomial

Let $OS_{M,i}^W \in \text{Rep}(W)$ be the degree i part of the Orlik-Solomon algebra of M . We define the **equivariant characteristic polynomial**

$$H_M^W(t) := \sum_{p=0}^{\text{rk } M} (-1)^p t^{\text{rk } M - p} OS_{M,p}^W \in \text{grVRep}(W).$$

Note that the graded dimension of $H_M^W(t)$ is just the usual characteristic polynomial $\chi_M(t) \in \mathbb{Z}[t]$. The following lemma is an equivariant version of the statement that $\chi_M(1) = 0$ for any matroid M of positive rank.

Lemma 2.1. *For any equivariant matroid $W \curvearrowright M$ of positive rank, $H_M^W(1) = 0$.*

Proof. Let $e = \sum_{i \in \mathcal{I}} e_i \in OS_{M,1}$, and consider the complex of W -representations with i^{th} term $OS_{M,i}^W$ and with differential given by multiplication by e . Then $H_M^W(1)$ is equal to the Euler characteristic of this complex, which is equal to the Euler characteristic of its homology. But the homology is zero provided that M has positive rank [Yuz95, 2.1]. \square

Let L be the lattice of flats of M . Given a flat $F \in L$, let $W_F \subset W$ be the stabilizer of F . For any pair of flats $F, G \in L$, let $W_{FG} := W_F \cap W_G$. Let M_F be the **localization** of M at F ; this is

the matroid on the ground set F whose lattice of flats is isomorphic to $L_F := \{G \in L \mid G \leq F\}$. Dually, let M^F be the **contraction** of M at F ; this is the matroid on the ground set $\mathcal{I} \setminus F$ whose lattice of flats is isomorphic to $L^F := \{G \in L \mid G \geq F\}$.¹ The action of W on M induces an action of W_F on both M_F and M^F .

Lemma 2.2. *For any equivariant matroid $W \curvearrowright M$,*

$$\begin{aligned} H_M^W(t) &= \sum_{[F] \in L/W} (-1)^{\text{rk } F} t^{\text{crk } F} \text{Ind}_{W_F}^W \left(OS_{M_F, \text{rk } M_F}^{W_F} \right) \\ &= \sum_{F \in L} \frac{|W_F|}{|W|} (-1)^{\text{rk } F} t^{\text{crk } F} \text{Ind}_{W_F}^W \left(OS_{M_F, \text{rk } M_F}^{W_F} \right). \end{aligned}$$

Proof. Brieskorn's lemma says that the natural map

$$\bigoplus_{\substack{F \in L \\ \text{rk } F = p}} OS_{M_F, p} \rightarrow OS_{M, p}$$

is an isomorphism. When we incorporate the action of W , this map gives us the equation

$$OS_{M, p}^W = \sum_{\substack{[F] \in L/W \\ \text{rk } F = p}} \text{Ind}_{W_F}^W \left(OS_{M_F, p}^{W_F} \right) = \sum_{\substack{F \in L \\ \text{rk } F = p}} \frac{|W_F|}{|W|} \text{Ind}_{W_F}^W \left(OS_{M_F, p}^{W_F} \right).$$

Our statement follows immediately from this. □

Lemma 2.3. *For any equivariant matroid $W \curvearrowright M$ of positive rank,*

$$\sum_{[F] \in L/W} (-1)^{\text{rk } F} \text{Ind}_{W_F}^W \left(OS_{M_F, \text{rk } M_F}^{W_F} \right) = \sum_{F \in L} \frac{|W_F|}{|W|} (-1)^{\text{rk } F} t^{\text{crk } F} \text{Ind}_{W_F}^W \left(OS_{M_F, \text{rk } M_F}^{W_F} \right) = 0.$$

Proof. This follows immediately from Lemmas 2.1 and 2.2. □

Lemma 2.3 is an equivariant version of the statement that

$$\sum_{F \in L} \mu(\emptyset, F) = 0$$

when M has positive rank. There is also a dual statement, which says that

$$\sum_{F \in L} \mu(F, \mathcal{I}) = 0$$

¹In the matroid literature, M_F is often called a restriction instead of a localization. On the other hand, in the hyperplane arrangement literature, M^F is often called a restriction instead of a contraction (this was the terminology used in [EPW16]). In this paper we have used the words localization and contraction and eliminated the word restriction to avoid any possible confusion.

when M has positive rank. Lemma 2.4 is an equivariant version of this dual equation. Surprisingly, the proof of Lemma 2.4 is much more difficult than the proof of Lemma 2.3.

Lemma 2.4. *For any equivariant matroid $W \curvearrowright M$ of positive rank,*

$$\sum_{[F] \in L/W} (-1)^{\text{crk } F} \text{Ind}_{W_F}^W \left(OS_{M^F, \text{crk } F}^{W_F} \right) = \sum_{F \in L} \frac{|W_F|}{|W|} (-1)^{\text{crk } F} \text{Ind}_{W_F}^W \left(OS_{M^F, \text{crk } F}^{W_F} \right) = 0.$$

Proof. Fix an ordering of the ground set \mathcal{I} . For any subset $S \subset \mathcal{I}$, let e_S be the corresponding square-free monomial, with the product taken in the order induced from that on \mathcal{I} . If $S = \{i_1, \dots, i_k\}$ in order, let $\partial e_S := \sum_{j=1}^k (-1)^j e_{S \setminus \{i_j\}}$. For any flat F , we have

$$OS_{M^F, \text{crk } F} = \mathbb{C}\{e_S \mid S \text{ a basis for } M^F\} \Big/ \mathbb{C}\{\partial e_C \mid C \text{ a circuit for } M^F \text{ of rank } \text{crk } F\}.$$

If $F \leq G$ with $\text{crk } F = p$ and $\text{crk } G = p - 1$, we define a map

$$\varphi_G^F : OS_{M^F, p} \rightarrow OS_{M^G, p-1}$$

by the formula $\varphi_G^F(e_S) := \partial e_S$ for any basis S of M^F , where we implicitly set $e_i = 0$ for all $i \in G$. More precisely, we note that S can contain at most one element of G . If S contains no elements of G , then $\varphi_G^F(e_S) := \partial e_S = 0 \in OS_{M^G, p-1}$. If $S = \{i_1, \dots, i_r\}$ and $i_k \in G$, then we put $S_k := S \setminus \{i_k\}$ and $\varphi_G^F(e_S) := (-1)^k e_{S_k}$. This is well defined because $\partial^2 = 0$.

Let

$$C_p(M) := \bigoplus_{\text{crk } F=p} OS_{M^F, p},$$

and combine the various maps φ_G^F to obtain a map $\varphi_p : C_p(M) \rightarrow C_{p-1}(M)$. We claim that $(C_\bullet(M), \varphi_\bullet)$ is an exact sequence.

To show that $\varphi_p \circ \varphi_{p+1} = 0$, we need to show that, for all $E \leq G$ with $\text{crk } E = p + 1$ and $\text{crk } G = p - 1$, we have

$$\sum_{E < F < G} \varphi_G^F \circ \varphi_F^E = 0.$$

Let S be a basis for M^E . Then $\varphi_G^F \circ \varphi_F^E(e_S) = 0$ unless F contains exactly one element of S and G contains exactly two elements of S . Thus we can reduce to the situation where $S = \{i_1, \dots, i_r\}$, F_k is the flat spanned by S and i_k , F_ℓ is the flat spanned by S and i_ℓ , and G is the flat spanned by S , i_k , and i_ℓ , and we need to show that $\varphi_G^{F_k} \circ \varphi_{F_k}^E(e_S) + \varphi_G^{F_\ell} \circ \varphi_{F_\ell}^E(e_S) = 0$. This is easily checked by hand. Thus $(C_\bullet(M), \varphi_\bullet)$ is a complex.

To prove that our complex is exact, we proceed by induction on $\text{rk } M$. The case $\text{rk } M = 1$ is trivial. Fix an M of rank strictly greater than 1, and assume that the statement is proved for all smaller ranks. Choose an index $i \in \mathcal{I}$, and consider the sum

$$C'_\bullet(M) := \bigoplus_{i \in F} OS_{M^F, \text{crk } F} \subset C_\bullet(M)$$

ranging over all flats F that contain the index i . It is clear that this is a subcomplex, and that

$$C'_\bullet(M) \cong C_\bullet(M'),$$

where M' is the contraction of F at the unique flat of rank 1 containing i . Let

$$C''_\bullet(M) := C_\bullet(M)/C'_\bullet(M)$$

be the quotient complex. As a vector space, we have

$$C''_p(M) \cong \bigoplus_{\substack{\text{crk } F=p \\ i \notin F}} OS_{M^F, p}.$$

Furthermore, for each flat F of corank p that does not contain i , we have an isomorphism

$$\bigoplus_{\substack{i \notin G \geq F \\ \text{crk } G=1}} OS_{M_G^F, p-1} \rightarrow OS_{M^F, p}$$

given by multiplication by e_i . (Indeed, if we choose an order on \mathcal{I} such that i is the maximal element, then multiplication by e_i gives a bijection from the nbc monomial basis for the left-hand side to the nbc monomial basis for the right-hand side.²) These isomorphisms fit together into an isomorphism of complexes

$$C''_\bullet(M) \cong \bigoplus_{\substack{i \notin G \\ \text{crk } G=1}} C_{\bullet-1}(M_G).$$

Now consider the short exact sequence of complexes

$$0 \rightarrow C'_\bullet(M) \rightarrow C_\bullet(M) \rightarrow C''_\bullet(M) \rightarrow 0.$$

Since $\text{rk } M' = \text{rk } M_G = \text{rk } M - 1 > 0$, our inductive hypotheses imply that $C'_\bullet(M)$ and $C''_\bullet(M)$ both have trivial homology. Then the long exact sequence in homology tells us that homology of $C_\bullet(M)$ vanishes, as well.

Finally, we note that the complex $C_\bullet(M)$ admits an action of W with

$$C_p(M) = \sum_{\substack{[F] \in L/W \\ \text{crk } F=p}} \text{Ind}_{W_F}^W \left(OS_{M^F, p}^{W_F} \right) \in \text{Rep}(W).$$

Since $C_\bullet(M)$ has trivial homology, its Euler characteristic is zero. This proves the lemma. \square

The following lemma is an equivariant version of the statement that $\sum_{F \in L} \chi_{M^F}(t) = t^{\text{rk } M}$.

²The abbreviation nbc stands for “no broken circuit”; see [Yuz01, Theorem 2.8] for a discussion of this basis.

Lemma 2.5. For any equivariant matroid $W \curvearrowright M$,

$$\sum_{[F] \in L/W} \text{Ind}_{W_F}^W \left(H_{M^F}^{W_F}(t) \right) = \sum_{F \in L} \frac{|W_F|}{|W|} \text{Ind}_{W_F}^W \left(H_{M^F}^{W_F}(t) \right) = t^{\text{rk } M} \tau_W,$$

where τ_W is the trivial representation of W .

Proof. Applying Lemma 2.2 to $W_F \curvearrowright M^F$, we have

$$\begin{aligned} & \sum_{F \in L} \frac{|W_F|}{|W|} \text{Ind}_{W_F}^W \left(H_{M^F}^{W_F}(t) \right) \\ &= \sum_{F \leq G} \frac{|W_{FG}|}{|W|} (-1)^{\text{rk } G - \text{rk } F} t^{\text{crk } G} \text{Ind}_{W_{FG}}^W \left(OS_{M_G^F, \text{rk } G - \text{rk } F}^{W_{FG}} \right) \\ &= \sum_{G \in L} \frac{|W_G|}{|W|} (-1)^{\text{rk } G} t^{\text{crk } G} \text{Ind}_{W_G}^W \left(\sum_{F \leq G} \frac{|W_{FG}|}{|W_G|} (-1)^{\text{rk } F} \text{Ind}_{W_{FG}}^{W_G} \left(OS_{M_G^F, \text{rk } G - \text{rk } F}^{W_{FG}} \right) \right). \end{aligned}$$

Applying Lemma 2.4 to $W_G \curvearrowright M_G$, we have

$$\sum_{F \leq G} \frac{|W_{FG}|}{|W_G|} (-1)^{\text{rk } F} \text{Ind}_{W_{FG}}^{W_G} \left(OS_{M_G^F, \text{rk } G - \text{rk } F}^{W_{FG}} \right) = 0$$

unless G is equal to the unique flat of rank 0, in which case it is equal to τ_W . \square

Remark 2.6. Suppose that M is W -equivariantly realizable by a complex linear space V . More precisely, suppose that we are given a linear subspace $V \subset \mathbb{C}^{\mathcal{I}}$, preserved by the action of W , such that a subset $B \subset \mathcal{I}$ is a basis for M if and only if the projection of V onto \mathbb{C}^B is an isomorphism. In this case, Lemma 2.5 has a nice geometric interpretation. The right-hand side of the equation is clearly isomorphic to the compactly supported cohomology of V . It is possible to compute this cohomology via a spectral sequence whose E_1 page consists of the compactly supported cohomology groups of the various strata. By comparing the mixed Hodge structures on the various groups, we can conclude that this spectral sequence degenerates at the E_2 page, which is given by the left-hand side of the equation.

The following lemma is an equivariant version of [EPW16, Lemma 2.1]. In the proof of this lemma, and elsewhere in this paper, we make frequent use of the following standard fact. Suppose that $W'' \subset W' \subset W$ are groups, V' is a representation of W' , and V'' is a representation of W'' . Then

$$\text{Ind}_{W'}^W \left(\text{Ind}_{W''}^{W'}(V'') \otimes V' \right) = \text{Ind}_{W''}^W \left(V'' \otimes \text{Res}_{W''}^{W'}(V') \right).$$

Lemma 2.7. For any equivariant matroid $W \curvearrowright M$ of positive rank,

$$\sum_{[F] \in L/W} \text{Ind}_{W_F}^W \left(t^{\text{rk } F} H_{M^F}^{W_F}(t^{-1}) \otimes H_{M^F}^{W_F}(t) \right) = \sum_{F \in L} \frac{|W_F|}{|W|} \text{Ind}_{W_F}^W \left(t^{\text{rk } F} H_{M^F}^{W_F}(t^{-1}) \otimes H_{M^F}^{W_F}(t) \right) = 0.$$

Proof. Applying Lemma 2.2 to $W_F \curvearrowright M_F$, we have

$$\begin{aligned}
& \sum_{F \in L} \frac{|W_F|}{|W|} \operatorname{Ind}_{W_F}^W \left(t^{\operatorname{rk} F} H_{M_F}^{W_F}(t^{-1}) \otimes H_{M_F}^{W_F}(t) \right) \\
&= \sum_{E \leq F} \frac{|W_{EF}|}{|W|} (-1)^{\operatorname{rk} E} t^{\operatorname{rk} E} \operatorname{Ind}_{W_F}^W \left(\operatorname{Ind}_{W_{EF}}^{W_F} \left(OS_{M_E, \operatorname{rk} E}^{W_{EF}} \right) \otimes H_{M_F}^{W_F}(t) \right) \\
&= \sum_{E \leq F} \frac{|W_{EF}|}{|W|} (-1)^{\operatorname{rk} E} t^{\operatorname{rk} E} \operatorname{Ind}_{W_{EF}}^W \left(OS_{M_E, \operatorname{rk} E}^{W_{EF}} \otimes H_{M_F}^{W_{EF}}(t) \right) \\
&= \sum_{E \leq F} \frac{|W_{EF}|}{|W|} (-1)^{\operatorname{rk} E} t^{\operatorname{rk} E} \operatorname{Ind}_{W_E}^W \left(OS_{M_E, \operatorname{rk} E}^{W_E} \otimes \operatorname{Ind}_{W_{EF}}^{W_E} \left(H_{M_F}^{W_{EF}}(t) \right) \right) \\
&= \sum_{E \in L} \frac{|W_E|}{|W|} (-1)^{\operatorname{rk} E} t^{\operatorname{rk} E} \operatorname{Ind}_{W_E}^W \left(OS_{M_E, \operatorname{rk} E}^{W_E} \otimes \sum_{E \leq F} \frac{|W_{EF}|}{|W_E|} \operatorname{Ind}_{W_{EF}}^{W_E} \left(H_{M_F}^{W_{EF}}(t) \right) \right).
\end{aligned}$$

Applying Lemma 2.5 to $W_E \curvearrowright M^E$, this becomes

$$\begin{aligned}
& \sum_{E \in L} \frac{|W_E|}{|W|} (-1)^{\operatorname{rk} E} t^{\operatorname{rk} E} \operatorname{Ind}_{W_E}^W \left(OS_{M_E, \operatorname{rk} E}^{W_E} \otimes t^{\operatorname{crk} E} \tau_{W_E} \right) \\
&= t^{\operatorname{rk} M} \sum_{E \in L} \frac{|W_E|}{|W|} (-1)^{\operatorname{rk} E} \operatorname{Ind}_{W_E}^W \left(OS_{M_E, \operatorname{rk} E}^{W_E} \right) \\
&= t^{\operatorname{rk} M} H_M^W(1),
\end{aligned}$$

which vanishes by Lemma 2.1. □

2.2 The equivariant Kazhdan-Lusztig polynomial

Now that we have established some basic properties of the equivariant characteristic polynomial, we are ready to define the equivariant Kazhdan-Lusztig polynomial. In the non-equivariant case, this polynomial is defined in [EPW16, Theorem 2.2]. The following theorem is a categorical version of that result.

Theorem 2.8. *There is a unique way to assign to each equivariant matroid $W \curvearrowright M$ an element $P_M^W(t) \in \operatorname{grVRep}(W)$, called the **equivariant Kazhdan-Lusztig polynomial**, such that the following conditions are satisfied:*

1. If $\operatorname{rk} M = 0$, then $P_M^W(t)$ is equal to the trivial representation in degree 0.
2. If $\operatorname{rk} M > 0$, then $\deg P_M^W(t) < \frac{1}{2} \operatorname{rk} M$.
3. For every M , $t^{\operatorname{rk} M} P_M^W(t^{-1}) = \sum_{[F] \in L/W} \operatorname{Ind}_{W_F}^W \left(H_{M_F}^{W_F}(t) \otimes P_{M_F}^{W_F}(t) \right)$.
4. Given a homomorphism $\varphi : W' \rightarrow W$, $P_M^{W'}(t) = \varphi^* P_M^W(t)$.

Proof. Let M be a matroid of positive rank. We may assume inductively that $P_{M'}^{W'}(t)$ has been defined for every matroid M' of rank strictly smaller than $\text{rk } M$ and every group W' acting on M' . In particular, $P_{M^F}^{W^F}(t)$ has been defined for all $\emptyset \neq F \in L(M)$. Let

$$R_M^W(t) := \sum_{\emptyset \neq [F] \in L/W} \text{Ind}_{W_F}^W \left(H_{M_F}^{W_F}(t) \otimes P_{M^F}^{W^F}(t) \right);$$

then item 3 says that we want

$$t^{\text{rk } M} P_M^W(t^{-1}) - P_M^W(t) = R_M^W(t).$$

It is clear that there can be at most one element $P_M^W(t) \in \text{grVRep}(W)$ of degree strictly less than $\frac{1}{2} \text{rk } M$ satisfying this condition. The existence of such a polynomial is equivalent to the statement that

$$t^{\text{rk } M} R_M^W(t^{-1}) = -R_M^W(t),$$

so this is what we need to prove. We have

$$\begin{aligned} t^{\text{rk } M} R_M^W(t^{-1}) &= t^{\text{rk } M} \sum_{\emptyset \neq [F] \in L/W} \text{Ind}_{W_F}^W \left(H_{M_F}^{W_F}(t^{-1}) \otimes P_{M^F}^{W^F}(t^{-1}) \right) \\ &= \sum_{\emptyset \neq [F] \in L/W} \text{Ind}_{W_F}^W \left(t^{\text{rk } F} H_{M_F}^{W_F}(t^{-1}) \otimes t^{\text{rk } M^F} P_{M^F}^{W^F}(t^{-1}) \right) \\ &= \sum_{\emptyset \neq [F] \in L/W} \text{Ind}_{W_F}^W \left(t^{\text{rk } F} H_{M_F}^{W_F}(t^{-1}) \otimes \sum_{[G] \in L^F/W_F} \text{Ind}_{W_{FG}}^{W_F} \left(H_{M_G^F}^{W_{FG}}(t) \otimes P_{M^G}^{W_{FG}}(t) \right) \right) \\ &= \sum_{\emptyset \neq F \in L} \frac{|W_F|}{|W|} \text{Ind}_{W_F}^W \left(t^{\text{rk } F} H_{M_F}^{W_F}(t^{-1}) \otimes \sum_{G \in L^F} \frac{|W_{FG}|}{|W_F|} \text{Ind}_{W_{FG}}^{W_F} \left(H_{M_G^F}^{W_{FG}}(t) \otimes P_{M^G}^{W_{FG}}(t) \right) \right) \\ &= \sum_{\emptyset \neq F \leq G} \frac{|W_{FG}|}{|W|} \text{Ind}_{W_F}^W \left(t^{\text{rk } F} H_{M_F}^{W_F}(t^{-1}) \otimes \text{Ind}_{W_{FG}}^{W_F} \left(H_{M_G^F}^{W_{FG}}(t) \otimes P_{M^G}^{W_{FG}}(t) \right) \right) \\ &= \sum_{\emptyset \neq F \leq G} \frac{|W_{FG}|}{|W|} \text{Ind}_{W_{FG}}^W \left(t^{\text{rk } F} H_{M_F}^{W_{FG}}(t^{-1}) \otimes H_{M_G^F}^{W_{FG}}(t) \otimes P_{M^G}^{W_{FG}}(t) \right) \\ &= \sum_{G \neq \emptyset} \sum_{F \leq G} \frac{|W_{FG}|}{|W|} \text{Ind}_{W_{FG}}^W \left(t^{\text{rk } F} H_{M_F}^{W_{FG}}(t^{-1}) \otimes H_{M_G^F}^{W_{FG}}(t) \otimes P_{M^G}^{W_{FG}}(t) \right) - R_M^W(t). \end{aligned}$$

Thus it will suffice to show that, for any flat $G \neq \emptyset$,

$$\sum_{F \leq G} \frac{|W_{FG}|}{|W|} \text{Ind}_{W_{FG}}^W \left(t^{\text{rk } F} H_{M_F}^{W_{FG}}(t^{-1}) \otimes H_{M_G^F}^{W_{FG}}(t) \otimes P_{M^G}^{W_{FG}}(t) \right) = 0.$$

Indeed, fixing a flat $G \neq \emptyset$, we have

$$\begin{aligned}
& \sum_{F \leq G} \frac{|W_{FG}|}{|W|} \operatorname{Ind}_{W_{FG}}^W \left(t^{\operatorname{rk} F} H_{M_F}^{W_{FG}}(t^{-1}) \otimes H_{M_F}^{W_{FG}}(t) \otimes P_{M^G}^{W_{FG}}(t) \right) \\
&= \sum_{F \leq G} \frac{|W_{FG}|}{|W|} \operatorname{Ind}_{W_G}^W \left(P_{M^G}^{W_G}(t) \otimes \operatorname{Ind}_{W_{FG}}^{W_G} \left(t^{\operatorname{rk} F} H_{M_F}^{W_{FG}}(t^{-1}) \otimes H_{M_F}^{W_{FG}}(t) \right) \right) \\
&= \frac{|W_G|}{|W|} \operatorname{Ind}_{W_G}^W \left(P_{M^G}^{W_G}(t) \otimes \sum_{F \in L_G} \frac{|W_{FG}|}{|W_G|} \operatorname{Ind}_{W_{FG}}^{W_G} \left(t^{\operatorname{rk} F} H_{M_F}^{W_{FG}}(t^{-1}) \otimes H_{M_F}^{W_{FG}}(t) \right) \right).
\end{aligned}$$

Lemma 2.7, applied to $W_G \curvearrowright M_G$, says that the internal sum is zero, as desired. \square

Proposition 2.9. *Let $C_{M,i}^W \in \operatorname{VRep}(W)$ be the coefficient of t^i in $P_M^W(t)$. If $i < \frac{1}{2} \operatorname{rk} M$, then*

$$C_{M,i}^W = \sum_{\substack{[F] \in L/W \\ 0 \leq j \leq \operatorname{rk} F}} (-1)^j \operatorname{Ind}_{W_F}^W \left(OS_{M^F, j}^{W_F} \otimes C_{M^F, \operatorname{crk} F - i + j}^{W_F} \right).$$

Proof. This follows immediately by looking at the coefficient of $t^{\operatorname{rk} M - i}$ on both sides of the equation in Theorem 2.8(3). \square

Corollary 2.10. *For any equivariant matroid $W \curvearrowright M$,*

$$C_{M,0}^W = \tau_W \quad \text{and} \quad C_{M,1}^W = \sum_{\substack{[F] \in L/W \\ \operatorname{crk} F = 1}} \operatorname{Ind}_{W_F}^W (\tau_{W_F}) - OS_{M,1}^W.$$

Proof. We apply Proposition 2.9. When $i = 0$, $C_{M^F, \operatorname{crk} F - i + j}^{W_F} \neq 0$ only if $j = 0 = \operatorname{crk} F$. The proposition then says that $C_{M,0}^W$ is equal to $C_{M^F,0}^W$, where F is the unique flat of corank 0. By part 1 of Theorem 2.8, this is equal to τ_W .

When $i = 1$, we have $C_{M^F, \operatorname{crk} F - i + j}^{W_F} \neq 0$ only if $j = 0$ and $\operatorname{crk} F = 1$ or $j = 1$ and $\operatorname{crk} F = 0$. The first case gives us a contribution of

$$\sum_{\substack{[F] \in L/W \\ \operatorname{crk} F = 1}} \operatorname{Ind}_{W_F}^W (\tau_{W_F})$$

(the permutation representation given by the action of W on the set of corank 1 flats) and the second case gives us a contribution of $-OS_{M,1}^W$. \square

Remark 2.11. By taking dimensions of the representations in Corollary 2.10, we recover Propositions 2.11 and 2.12 of [EPW16].

Suppose that M is W -equivariantly realizable by a complex linear space $V \subset \mathbb{C}^{\mathcal{I}}$. Let X_V be the **reciprocal plane**, which is defined as follows:

$$X_V := \overline{\{z \in (\mathbb{C}^\times)^{\mathcal{I}} \mid z^{-1} \in V\}} \subset \mathbb{C}^{\mathcal{I}}.$$

The action of W on \mathcal{I} induces an action on X_V . The following corollary is an equivariant version of [EPW16, Proposition 3.12].

Corollary 2.12. *If M is W -equivariantly realizable by a linear subspace $V \subset \mathbb{C}^{\mathcal{I}}$, then we have $C_{M,i}^W = IH^{2i}(X_V; \mathbb{C}) \in \text{grRep}(W)$.*

Proof. This follows from Proposition 2.9 and [PWY16, Remark 3.6]. \square

By definition, the coefficients $C_{M,i}^W$ are virtual representations of W . When M is W -equivariantly realizable, however, Corollary 2.12 implies that they are honest representations. We conjecture that this is always the case, even if M is not equivariantly realizable.

Conjecture 2.13. *For any equivariant matroid $W \curvearrowright M$, $P_M^W(t) \in \text{grRep}(W)$.*

Remark 2.14. When W is the trivial group, Conjecture 2.13 says that the coefficients of the ordinary Kazhdan-Lusztig polynomial are natural numbers rather than just integers. This conjecture appeared in [EPW16, Conjecture 2.3]. We note, however, that it is much easier to construct non-realizable examples of equivariant matroids than it is to construct non-realizable examples of ordinary matroids. For example, let $U_{m,d}$ be the the uniform matroid of rank d on $m+d$ elements. This matroid is always realizable. However, it has an action of the symmetric group S_{m+d} , and it is equivariantly realizable if and only if $d \in \{0, 1\}$ or $m \in \{0, 1\}$. In the following section, we will prove Conjecture 2.13 for arbitrary uniform matroids.

3 Uniform matroids

Let $U_{m,d}$ be the the uniform matroid of rank d on $m+d$ elements, which admits an action of the symmetric group S_{m+d} . Let

$$H_{m,d}(t) := H_{U_{m,d}}^{S_{m+d}}(t), \quad P_{m,d}(t) := P_{U_{m,d}}^{S_{m+d}}(t), \quad \text{and} \quad C_{m,d,i} := C_{U_{m,d},i}^{S_{m+d}}.$$

For any partition λ of $m+d$, let $V[\lambda]$ be the irreducible representation of S_{m+d} indexed by λ . The purpose of this section is to prove the following result.

Theorem 3.1. *For all $i > 0$,*

$$C_{m,d,i} = \sum_{b=1}^{\min(m,d-2i)} V[d+m-2i-b+1, b+1, 2^{i-1}] \in \text{Rep}(S_{m+d}).$$

Corollary 3.2. *Conjecture 2.13 holds for all uniform matroids.*

Remark 3.3. When $m = 1$, Theorem 3.1 specializes to the main result of [PWY16].

Remark 3.4. One can use Theorem 3.1, along with the hook length formula for the dimension of V_λ , to compute the coefficients of the ordinary Kazhdan-Lusztig polynomial of $U_{m,d}$. (The formula is unenlightening, so we will not reproduce it here.) This is a computation that we were unable to

do in [EPW16]; see Section 2.4 and the appendix of that paper. Indeed, we still know of no way to compute these numbers that does not go through Theorem 3.1.

Remark 3.5. One immediate consequence of Theorem 3.1 is that, for any triple (m, d, i) , we have $C_{m,d,i} = C_{d-2i,m+2i,i}$. This was first observed empirically in the non-equivariant setting by Max Wakefield, based on computer calculations. We still have no philosophical explanation for which this symmetry should exist.

Remark 3.6. Another consequence of Theorem 3.1 is that the sequence $(C_{m,d,i})_{d \in \mathbb{N}}$ is uniformly representation stable in the sense of Church and Farb [CF13, Definition 2.3], with the stable range beginning at $d = m + 2i$.

3.1 Translating to symmetric functions

The **Frobenius characteristic** is an isomorphism of vector spaces

$$\text{ch} : \text{grVRep}(S_n) \rightarrow \Lambda_n[t],$$

where Λ_n is the space of symmetric functions of degree n [Mac95, Section I.7]. It has the property that, given two graded virtual representations $V_1 \in \text{grVRep}(S_{n_1})$ and $V_2 \in \text{grVRep}(S_{n_2})$, we have

$$\text{ch} \text{Ind}_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}} (V_1 \boxtimes V_2) = \text{ch}(V_1) \text{ch}(V_2).$$

Let

$$\mathcal{H}_{m,d}(t) := \text{ch } H_{m,d}(t), \quad \mathcal{P}_{m,d}(t) := \text{ch } P_{m,d}(t), \quad \text{and} \quad \mathcal{C}_{m,d,i} := \text{ch } C_{m,d,i}.$$

Applying the Frobenius characteristic to the equation in Theorem 2.8(3) (and applying Corollary 2.10), we obtain the statement

$$t^d \mathcal{P}_{m,d}(t^{-1}) = \mathcal{H}_{m,d}(t) + \sum_{k=1}^d \mathcal{H}_{0,d-k}(t) \mathcal{P}_{m,k}(t). \quad (1)$$

3.2 Generating functions

In this section we will work in the ring $\Lambda[[t, u, x]]$ of completed symmetric functions with coefficients in the ring of formal power series in t, u , and x . Let

$$\mathcal{H}(t, u, x) := \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} \mathcal{H}_{m,d}(t) u^d x^m \quad \text{and} \quad \mathcal{P}(t, u, x) := \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} \mathcal{P}_{m,d}(t) u^d x^m.$$

Then Equation (1) for all values of m and d is equivalent to the generating function equation

$$\mathcal{P}(t^{-1}, tu, x) = \mathcal{H}(t, u, x) + (1 + \mathcal{H}(t, u, 0)) \mathcal{P}(t, u, x). \quad (2)$$

Remark 3.7. Once we have Equation (2), we obtain for free the corresponding functional equation involving the (non-equivariant) exponential generating functions. Let

$$H(t, u, x) := \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} \dim H_{m,d}(t) \frac{u^d x^m}{(d+m)!} = \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} \chi_{U_{m,d}}(t) \frac{u^d x^m}{(d+m)!}$$

and

$$P(t, u, x) := \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} \dim P_{m,d}(t) \frac{u^d x^m}{(d+m)!} = \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} P_{U_{m,d}}(t) \frac{u^d x^m}{(d+m)!}.$$

Then we have

$$P(t^{-1}, tu, x) = H(t, u, x) + (1 + H(t, u, 0))P(t, u, x). \quad (3)$$

Equation (3) follows from Equation (2) using the following easy observation: Let V_i be a representation of S_{n_i} for $i \in \{1, 2\}$, and let $V = \text{Ind}_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}}(V_1 \boxtimes V_2)$. Then $\frac{\dim V}{(n_1+n_2)!} = \frac{\dim V_1}{n_1!} \cdot \frac{\dim V_2}{n_2!}$.

The remainder of this section is devoted to deriving explicit expressions for $\mathcal{H}(t, u, x)$ and $H(t, u, x)$. Given a partition λ of n , let $s[\lambda] := \text{ch } V[\lambda]$ be the Schur function associated with λ . Let

$$s(t) := \sum_{n=0}^{\infty} t^n s[n],$$

and recall the well-known identity

$$s(-t)^{-1} = \sum_{n=0}^{\infty} t^n s[1^n].$$

Lemma 3.8. $\sum_{e=0}^{\infty} \sum_{m=0}^{\infty} t^e u^m s[m+1, 1^e] = \frac{1}{t+u} \left(-1 + \frac{s(u)}{s(-t)} \right).$

Proof. We have

$$\begin{aligned} -1 + \frac{s(u)}{s(-t)} &= -1 + \left(\sum_{j=0}^{\infty} t^j s[1^j] \right) \left(\sum_{k=0}^{\infty} u^k s[k] \right) \\ &= -1 + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} t^j u^k s[1^j] s[k] \\ &= \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} t^j u^k s[k+1, 1^{j-1}] + \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} t^j u^k s[k, 1^j], \end{aligned}$$

where the last equality follows from the Pieri rule. Next, observe that

$$\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} t^j u^k s[k+1, 1^{j-1}] = t \sum_{e=0}^{\infty} \sum_{m=0}^{\infty} t^e u^m s[m+1, 1^e]$$

and

$$\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} t^j u^k s[k, 1^j] = u \sum_{e=0}^{\infty} \sum_{m=0}^{\infty} t^e u^m s[m+1, 1^e].$$

Adding them together and dividing by $t+u$, we obtain the desired equation. \square

Proposition 3.9. *We have*

$$\mathcal{H}(t, u, x) = \frac{u}{u-x} \left(-1 + \frac{s(x)}{s(u)} \right) + \frac{tu}{tu-x} \left(\frac{s(tu)}{s(u)} - \frac{s(x)}{s(u)} \right)$$

and

$$1 + \mathcal{H}(t, u, 0) = \frac{s(tu)}{s(u)}.$$

Proof. If $i < d$, then

$$OS_{U_{m,d},i}^{S_{m+d}} = \wedge^i \mathbb{C}^{m+d} = V[m+d-i, 1^i] + V[m+d-i+1, 1^{i-1}].$$

Applying the Frobenius characteristic, we have

$$\text{ch } OS_{U_{m,d},i}^{S_{m+d}} = s[m+d-i, 1^i] + s[m+d-i+1, 1^{i-1}] = s[1^i]s[m+d-i],$$

where the second equation follows from the Pieri rule. We also have

$$OS_{U_{m,d},d}^{S_{m+d}} = V[m+1, 1^{d-1}],$$

and therefore

$$\text{ch } OS_{U_{m,d},d}^{S_{m+d}} = s[m+1, 1^{d-1}].$$

By definition,

$$\mathcal{H}(t, u, x) = \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} (-t^{-1})^i (tu)^d x^m \text{ch } OS_{U_{m,d},i}^{S_{m+d}}.$$

Letting $k = d - i$, this tells us that

$$\begin{aligned} \mathcal{H}(t, u, x) &= \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} (-u)^d x^m \text{ch } OS_{U_{m,d},d}^{S_{m+d}} + \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} (-t^{-1})^i (tu)^{i+k} x^m \text{ch } OS_{U_{m,i+k},i}^{S_{m+i+k}} \\ &= \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} (-u)^d x^m s[m+1, 1^{d-1}] + \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} (-u)^i (tu)^k x^m s[1^i] s[m+k] \\ &= \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} (-u)^d x^m s[m+1, 1^{d-1}] + \left(\sum_{i=0}^{\infty} (-u)^i s[1^i] \right) \left(\sum_{k=1}^{\infty} \sum_{m=0}^{\infty} (tu)^k x^m s[m+k] \right) \\ &= \frac{u}{u-x} \left(-1 + \frac{s(x)}{s(u)} \right) + \frac{tu}{tu-x} \cdot \frac{s(tu) - s(x)}{s(u)}, \end{aligned}$$

where the last equation follows from Lemma 3.8. The second statement is obtained from the first

by setting x equal to zero. □

The following Corollary follows immediately from Proposition 3.9 as in Remark 3.7. We use the fact that $s(t)$ is the Frobenius characteristic of the trivial representation in every degree, so the exponential generating function for its dimensions is e^t .

Corollary 3.10. *We have*

$$H(t, u, x) = \frac{u}{u-x} (-1 + e^{x-u}) + \frac{tu}{tu-x} (e^{tu-u} - e^{x-u})$$

and

$$1 + H(t, u, 0) = e^{tu-u}.$$

Proposition 3.9 combines with Equation (2) to tell us that

$$\mathcal{P}(t^{-1}, tu, x) = \frac{u}{u-x} \left(-1 + \frac{s(x)}{s(u)} \right) + \frac{tu}{tu-x} \left(\frac{s(tu)}{s(u)} - \frac{s(x)}{s(u)} \right) + \mathcal{P}(t, u, x) \frac{s(tu)}{s(u)}.$$

Rearranging terms, this is equivalent to the equation

$$\left(\frac{u}{u-x} + \mathcal{P}(t^{-1}, tu, x) \right) s(u) - \frac{u}{u-x} s(x) = \left(\frac{tu}{tu-x} + \mathcal{P}(t, u, x) \right) s(tu) - \frac{tu}{tu-x} s(x). \quad (4)$$

Let

$$\begin{aligned} \mathcal{R}(t, u, x) &:= \left(\frac{tu}{tu-x} + \mathcal{P}(t, u, x) \right) s(tu) - \frac{tu}{tu-x} s(x) \\ &= \left(\frac{tu}{tu-x} + \mathcal{P}(t, u, x) \right) \sum_{n=0}^{\infty} (tu)^n s[n] - \frac{tu}{tu-x} \sum_{n=0}^{\infty} x^n s[n] \end{aligned}$$

be the expression on the right-hand side of Equation (4). Then Equation (4) becomes

$$\mathcal{R}(t^{-1}, tu, x) = \mathcal{R}(t, u, x).$$

The results of this section can be summarized as follows.

Proposition 3.11. *The element $\mathcal{P}(t, u, x) \in \Lambda[[t, u, x]]$ is uniquely characterized by the following properties:*

- $\mathcal{P}(t, 0, x) = 0$,
- the coefficient of $t^i u^d$ in $\mathcal{P}(t, u, x)$ is zero if $2i \geq d$,
- $\mathcal{R}(t^{-1}, tu, x) = \mathcal{R}(t, u, x)$, where $\mathcal{R}(t, u, x)$ is defined above.

Remark 3.12. It is interesting to observe exactly what our manipulations of generating functions has bought us. The straightforward approach to proving Theorem 3.1 would have been to apply Proposition 2.9 and proceed by induction on d . This works in theory, but it involves repeated

applications of the Littlewood-Richardson rule for hooks, and the combinatorics very quickly gets out of hand. Instead, we will prove Theorem 3.1 by taking our “guess” for $\mathcal{P}(t, u, x)$ and verifying the equation $\mathcal{R}(t^{-1}, tu, x) = \mathcal{R}(t, u, x)$. From the definition of $\mathcal{R}(t, u, x)$, we see that this will involve repeated applications of the Pieri rule, which is much simpler than the general Littlewood-Richardson rule. This simplification is exactly what makes our proof possible.

3.3 Proving the theorem

We are now ready to prove Theorem 3.1. Let

$$\mathcal{P}'(t, u, x) := \sum_{d=1}^{\infty} u^d s[d] + \sum_{i=0}^{\infty} \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} t^i u^d x^m \sum_{b=1}^{\min(m, d-2i)} s[d+m-2i-b+1, b+1, 2^{i-1}].$$

Here and throughout this section we adopt the notational convention that

$$s[a, 2, 2^{-1}] = s[a] \quad \text{and} \quad s[a, b+1, 2^{-1}] = 0 \text{ if } b > 1;$$

in particular, the coefficient of $t^0 u^d x^m$ in $\mathcal{P}'(t, u, x)$ is equal to $s[d+m]$ for any $d \geq 1$ and $m \geq 0$.

Let

$$\mathcal{R}'(t, u, x) := \left(\frac{tu}{tu-x} + \mathcal{P}'(t, u, x) \right) \sum_{n=0}^{\infty} (tu)^n s[n] - \frac{tu}{tu-x} \sum_{n=0}^{\infty} x^n s[n].$$

By Proposition 3.11, Theorem 3.1 is equivalent to the statement that $\mathcal{R}'(t^{-1}, tu, x) = \mathcal{R}'(t, u, x)$.

The coefficient of $t^i u^d x^m$ in $\mathcal{R}'(t, u, x)$ is equal to

$$\sum_{b=1}^m \sum_{k=b-d+2i}^i s[k] s[d+k+m-2i-b+1, b+1, 2^{i-k-1}] + \begin{cases} s[i] s[d-i] & \text{if } m=0 < d-i \\ 0 & \text{otherwise} \end{cases} + \begin{cases} s[m+d] & \text{if } i=d > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we need to show that this expression is invariant under the substitution $i \leftrightarrow d-i$. If $d=0$ or $i > d$, the expression is equal to zero. If $d > 0$ and $i=0$ or $i=d$, then the expression is equal to $s[d+m]$. Thus, we may assume that $0 < i < d$. If $m=0$, the expression is equal to $s[i] s[d-i]$, which is clearly invariant. Thus, we may further assume that $m \neq 0$, which means that we can restrict our attention to the expression

$$\sum_{b=1}^m \sum_{k=b-d+2i}^i s[k] s[d+k+m-2i-b+1, b+1, 2^{i-k-1}].$$

Letting $r = d - 2i$, we can rewrite this expression as

$$\Psi(i, r, m) := \sum_{b=1}^m \sum_{k=b-r}^i s[k]s[r+k+m-b+1, b+1, 2^{i-k-1}],$$

and we want to show that it is equal to

$$\Phi(i, r, m) := \Psi(i+r, -r, m) = \sum_{b=1}^m \sum_{j=b}^i s[j+r]s[j+m-b+1, b+1, 2^{i-j-1}].$$

The Pieri rule tells us that any Schur function appearing $\Psi(i, r, m)$ must be of the form $s[A, B, C, 2^D]$ or $s[A, B, C, 2^D, 1]$, where we continue to adhere to our notational convention:

$$s[A, B, 2, 2^{-1}] = s[A, B], \quad s[A, 2, 2, 2^{-2}] = s[A],$$

and so on.

Let us focus on the case of $s[A, B, C, 2^D]$ with $D \geq 0$ and $A + B + C + 2D = 2i + r + m$. The Schur function $s[A, B, C, 2^D]$ can appear in the k summand of $\Psi(i, r, m)$ when $k = i - D - 2$ or $k = i - D - 1$. In each of these summands, the number of times that $s[A, B, C, 2^D]$ appears is equal to the number of values of $b \leq \min(m, k+r)$ satisfying the inequalities

$$A \geq r + k + m - b + 1 \geq B \geq b + 1 \geq C \geq 2,$$

which ensure that the partitions $[r+k+m-b+1, b+1, 2^{i-k-1}]$ and $[A, B, C, 2^D]$ interlace. More precisely, let

$$\varepsilon_1 := \min(m, B-1) + \min(0, r+i-D-1-B), \quad \varepsilon_2 := \min(m, B-1) + \min(0, r+i-D-B),$$

$$\Upsilon_1 := \max(C-1, r+i-D-1+m-A), \quad \text{and} \quad \Upsilon_2 := \max(C-1, r+i-D+m-A).$$

A priori, we would want to let

$$\varepsilon_1 = \min(m, B-1, r+i-D-1+m-B, i-D-2+r).$$

The fact that $m \leq B-1$ if and only if $r+i-D+m-1-B \leq r+i-D-2$ tells us that the two expressions for ε_1 are the same. A similar argument holds for ε_2 .

The coefficient of $s[A, B, C, 2^D]$ in $\Psi(i, r, m)$ is then equal to

$$\max(0, \varepsilon_1 - \Upsilon_1 + 1) + \max(0, \varepsilon_2 - \Upsilon_2 + 1),$$

where the first summand represents the number of possible values for b when $k = i - D - 2$, and the second summand represents the number of possible values for b when $k = i - D - 1$.

Similarly, let

$$\varepsilon'_1 := \min(m, B - 1) + \min(0, i - D - 1 - B), \quad \varepsilon'_2 := \min(m, B - 1) + \min(0, i - D - B),$$

$$\Upsilon'_1 := \max(C - 1, i - D - 1 + m - A), \quad \text{and} \quad \Upsilon'_2 := \max(C - 1, i - D + m - A).$$

Then the coefficient of $s[A, B, C, 2^D]$ in $\Phi(i, r, m)$ is equal to

$$\max(0, \varepsilon'_1 - \Upsilon'_1 + 1) + \max(0, \varepsilon'_2 - \Upsilon'_2 + 1),$$

where the first summand represents the number of possible values for b when $j = i - D - 2$, and the second summand represents the number of possible values for b when $j = i - D - 1$. Our plan is to show that

$$\varepsilon_1 - \Upsilon_1 = \varepsilon'_2 - \Upsilon'_2 \quad \text{and} \quad \varepsilon_2 - \Upsilon_2 = \varepsilon'_1 - \Upsilon'_1,$$

which will tell us that $s[A, B, C, 2^D]$ appears with the same coefficient in $\Psi(i, r, m)$ and $\Phi(i, r, m)$.

Lemma 3.13. *We have*

1. $\Upsilon_1 = C - 1$ if and only if $\varepsilon'_2 = \min(m, B - 1)$,
2. $\Upsilon_2 = C - 1$ if and only if $\varepsilon'_1 = \min(m, B - 1)$,
3. $\Upsilon'_1 = C - 1$ if and only if $\varepsilon_2 = \min(m, B - 1)$,
4. $\Upsilon'_2 = C - 1$ if and only if $\varepsilon_1 = \min(m, B - 1)$.

Proof. We prove the forward direction of Lemma 3.13(1) and note that the other cases are identical. Since $\Upsilon_1 = C - 1$, we have

$$C - 1 \geq r + i - D - 1 + m - A = B + C + D - i - 1,$$

which implies that $i - D - 1 \geq B - 1$. Adding $m + 1$ to both sides and subtracting B tells us that $i - D + m - B \geq m$, hence $\varepsilon'_2 = m$ or $B - 1$. \square

Now consider the expression

$$\varepsilon_1 - \Upsilon_1 - \varepsilon'_2 + \Upsilon'_2; \tag{5}$$

we will use a case-by-case analysis to prove that this expression is equal to zero.

Case 1: $\varepsilon_1 - \varepsilon'_2 = 0$. Then $\Upsilon_1 = C - 1 = \Upsilon'_2$, and (5) vanishes.

Case 2: $\varepsilon_1 - \varepsilon'_2 = r - 1$. We have $\Upsilon_1 \neq C - 1 \neq \Upsilon'_2$ and hence $\Upsilon'_2 - \Upsilon_1 = 1 - r$, which tells us that the expression (5) vanishes.

Case 3: $\varepsilon_1 - \varepsilon'_2 = -i + D + B$. Then $\Upsilon'_2 = C - 1$, and $\Upsilon_1 = r + i - D - 1 + m - A$, and

therefore that the expression (5) is equal to $r + 2i + m - A - B - C - 2D = 0$.

We have now shown that the expression (5) is equal to zero, and therefore that $\varepsilon_1 - \Upsilon_1 = \varepsilon'_2 - \Upsilon'_2$. A similar argument allows us to conclude that $\varepsilon_2 - \Upsilon_2 = \varepsilon'_1 - \Upsilon'_1$. This completes the proof that $s[A, B, C, 2^D]$ appears with the same coefficient in $\Psi(i, r, m)$ and $\Phi(i, r, m)$. The other cases, namely Schur functions of the form $s[A, B, C, 2^D, 1]$ and Schur functions of the form $s[A, B, C, 2^D]$ with $D < 0$, can be analyzed in a similar fashion; we leave the details of these cases to the reader. This completes the proof of Theorem 3.1. \square

4 Braid matroids

Let B_n be the braid matroid of rank $n - 1$. Equivalently, B_n is the matroid associated with the complete graph on n vertices. The ground set is equal to the set of all 2-element subsets of $[n]$, and the lattice of flats is equal to the lattice of set-theoretic partitions of n ; the rank of a flat is equal to n minus the number of parts of the partition. The group S_n acts on B_n in the obvious manner.

Let

$$K_n(t) := H_{B_n}^{S_n}(t), \quad Q_n(t) := P_{B_n}^{S_n}(t), \quad \text{and} \quad D_{n,i} := C_{B_n,i}^{S_n}.$$

For any partition $\lambda \vdash n$, let $S_\lambda \subset S_n$ be the stabilizer of a set-theoretic partition of type λ . Then Theorem 2.8(3) says

$$t^{n-1}Q_n(t^{-1}) = \sum_{\lambda \vdash n} \text{Ind}_{S_\lambda}^{S_n} \left(K_{\lambda_1}(t) \otimes \cdots \otimes K_{\lambda_{\ell(\lambda)}}(t) \otimes Q_{\ell(\lambda)}(t) \right). \quad (6)$$

The polynomial $K_n(t)$ is well understood, going back to Lehrer and Solomon [LS86]. An explicit formula for $\text{ch } K_n(t)$ appears in [HR, Theorem 2.7]. We cannot give an explicit formula for $Q_n(t)$ in the same way that we did for uniform matroids, but we will describe the recursion that can be used to compute it and calculate some examples for small n .

Remark 4.1. As defined above, B_n is not equivariantly realizable. However, let B'_n be the analogously defined matroid with ground set \mathcal{I}_n equal the set of **ordered** pairs of distinct elements of $[n]$, with $\{(i, j), (j, i)\}$ a dependent set. Then B_n is the underlying simple matroid of B'_n , so they have isomorphic lattices of flats, the same equivariant characteristic polynomial, and the same equivariant Kazhdan-Lusztig polynomial. Let $V_n = \mathbb{C}^n / \mathbb{C}_\Delta$, and consider the embedding of V_n into $\mathbb{C}^{\mathcal{I}_n}$ given by $x_i - x_j$ in the (i, j) -coordinate. This embedding is an S_n -equivariant realization of B'_n . It follows from Corollary 2.12 that

$$Q_n(t) = P_{B_n}^{S_n}(t) = P_{B'_n}^{S_n}(t) \in \text{grRep}(S_n).$$

4.1 Generating functions

Let K be the (virtual, graded) linear species that assigns to the set $[n]$ the (virtual, graded) representation $K_n(t)$ of S_n . Similarly, let Q be the (graded) linear species that assigns to the set

$[n]$ the (graded) representation $Q_n(t)$ of S_n . Finally, motivated by Equation (6), let \bar{Q} be the (graded) linear species that assigns to the set $[n]$ the (graded) representation $t^{n-1}Q_n(t^{-1})$ of S_n . In the language of species, Equation (6) says that $\bar{Q} = Q \circ K$.

Consider the following six generating functions:

$$K(t, z) := \sum_{n=1}^{\infty} \dim K_n(t) \frac{z^n}{n!}, \quad Q(t, z) := \sum_{n=1}^{\infty} \dim Q_n(t) \frac{z^n}{n!}, \quad \bar{Q}(t) := \sum_{n=1}^{\infty} \dim \bar{Q}_n(t) \frac{z^n}{n!} \in \mathbb{Q}[[t, z]],$$

$$\mathcal{K}(t) := \sum_{n=1}^{\infty} \text{ch } K_n(t), \quad \mathcal{Q}(t) := \sum_{n=1}^{\infty} \text{ch } Q_n(t), \quad \bar{\mathcal{Q}}(t) := \sum_{n=1}^{\infty} \text{ch } \bar{Q}_n(t) \in \Lambda[[t]].$$

Note that $\dim K_n(t) = \chi_{B_n}(t) = (t-1) \cdots (t-n+1)$, $\dim Q_n(t) = P_{B_n}(t)$, and $\dim \bar{Q}_n(t) = t^{n-1}P_{B_n}(t^{-1})$. Since we have $\bar{Q} = Q \circ K$, the theory of species tells us that

$$\frac{1}{t}Q(t^{-1}, tz) = \bar{Q}(t, z) = Q(t, K(t, z)) \quad \text{and} \quad \bar{\mathcal{Q}}(t) = \mathcal{Q}(t)[\mathcal{K}(t)], \quad (7)$$

where square brackets denote plethysm. See, for example, [Mén91, Proposition 2.1], which can be extended to virtual species as in [Joy86].

Remark 4.2. Equation (7) can be derived directly from Equation (6). The second half of Equation (7) follows from the fact that, for any representation V_i of S_i ($i \in \{1, 2\}$), $\text{ch } \text{Ind}_{S_i S_j}^{S_{ij}}(V_j \otimes V_i^{\otimes j})$ is equal to $\text{ch } V_j[\text{ch } V_i]$. The first half follows then from the second half by computing the effect of this induction on the dimension of a (virtual, graded) representation. The language of species simply provides a tidy formalism for these observations.

Remark 4.3. The generating functions $K(t, z)$ and $\mathcal{K}(t)$ can be understood very explicitly. We have

$$(z+1)^t = \sum_{n=0}^{\infty} \binom{t}{n} z^n = 1 + t \cdot \sum_{n=1}^{\infty} \chi_{B_n}(t) \frac{z^n}{n} = 1 + tK(t, z).$$

Based on the work of Lehrer and Solomon, Getzler gives the following analogous formula for $\mathcal{K}(t)$:

$$1 + t\mathcal{K}(t) = \prod_{k=1}^{\infty} (1 + p_k)^{\frac{1}{k} \sum_{d|k} \mu(k/d) t^d},$$

where p_k is the k^{th} power sum symmetric function. See the second displayed equation in the proof of Proposition 2.4 of [Get].

4.2 Linear term

Though we have no general formula for $Q_n(t)$, we can use Corollary 2.10 to calculate $D_{n,1}$.

Proposition 4.4. *When $n \leq 3$, $D_{n,1} = 0$. When $n \geq 4$, we have*

$$D_{n,1} = V[n]^{\oplus \lfloor \frac{n-2}{2} \rfloor} \oplus V[n-1, 1]^{\oplus \lfloor \frac{n-3}{2} \rfloor} \oplus V[n-2, 2]^{\oplus \lfloor \frac{n-4}{2} \rfloor} \oplus \bigoplus_{3 \leq i \leq \lfloor n/2 \rfloor} V[n-i, i]^{d(n,i)},$$

where

$$d(n, i) = \begin{cases} n/2 - i & \text{if } n \text{ is even and } i \text{ is odd,} \\ (n+1)/2 - i & \text{if } n \text{ is odd,} \\ (n+2)/2 - i & \text{if } n \text{ and } i \text{ are both even.} \end{cases}$$

Proof. Let $W = S_n$. By Proposition 2.10,

$$D_{n,1} = \sum_{\substack{[F] \in L/W \\ \text{crk } F=1}} \text{Ind}_{W_F}^W(\tau_{W_F}) - OS_{B_{n,1}}^{S_n}.$$

We have

$$OS_{B_{n,1}}^{S_n} = \text{Sym}^2(\mathbb{C}^n) - \mathbb{C}^n = \text{Sym}^2 V[n-1, 1] = V[n-2, 2] \oplus V[n-1, 1] \oplus V[n].$$

Let F_k be the partition of $[n]$ into $[k]$ and its complement; then $\{F_1, \dots, F_{\lfloor n/2 \rfloor}\}$ is a complete set of representatives of S_n -orbits of corank 1 flats.

Suppose that n is odd. Then $W_{F_k} = S_k \times S_{n-k}$, so

$$\begin{aligned} \text{ch } D_{n,1} &= \sum_{k=1}^{\frac{n-1}{2}} s[k]s[n-k] - (s[n-2, 2] + s[n-1, 1] + s[n]) \\ &= \sum_{k=1}^{\frac{n-1}{2}} \sum_{i=0}^k s[n-i, i] - (s[n-2, 2] + s[n-1, 1] + s[n]) \\ &= \frac{n-3}{2}s[n] + \frac{n-3}{2}s[n-1, 1] + \frac{n-5}{2}s[n-2, 2] + \sum_{i=3}^{\frac{n-1}{2}} \left(\frac{n+1}{2} - i \right) s[n-i, i]. \end{aligned}$$

If n is even, then $W_{F_k} = S_k \times S_{n-k}$ for all $k < n/2$, but $W_{F_{n/2}} = S_{n/2} \wr S_2$. We therefore have

$$\begin{aligned} \text{ch } D_{n,1} &= \sum_{k=1}^{\frac{n-2}{2}} s[k]s[n-k] + s[2] \left[s[n/2] \right] - (s[n-2, 2] + s[n-1, 1] + s[n]) \\ &= \frac{n-4}{2}s[n] + \frac{n-4}{2}s[n-1, 1] + \frac{n-6}{2}s[n-2, 2] + \sum_{i=3}^{\frac{n-2}{2}} \left(\frac{n}{2} - i \right) s[n-i, i] + s[2] \left[s[n/2] \right]. \end{aligned}$$

We also have

$$s[2] \left[s[n/2] \right] = \sum_{j=0}^{\lfloor n/4 \rfloor} s[n-2j, 2j],$$

and the proposition follows. \square

Remark 4.5. Proposition 4.4 implies that the sequence $\{D_{n,1}\}$ of S_n -representations is **not** representation stable in the sense of Church and Farb.

4.3 Calculations

We conclude our discussion of braid matroids with a calculation of $D_{n,i}$ for $n \leq 9$ and $i \geq 2$ (since $D_{n,0} = V[n]$ by Proposition 2.10 and $D_{n,1}$ is computed in Proposition 4.4). These computations were performed in SAGE [S⁺14], using Equation (6).

$$D_{6,2} = V[2, 2, 2] \oplus V[4, 2] \oplus V[6]$$

$$D_{7,2} = V[2, 2, 2, 1] \oplus V[3, 2, 2] \oplus V[4, 2, 1]^{\oplus 2} \oplus V[4, 3]^{\oplus 2} \oplus V[5, 2]^{\oplus 2} \oplus V[6, 1]^{\oplus 2} \oplus V[7]^{\oplus 2}$$

$$D_{8,2} = V[2, 2, 2, 2] \oplus V[3, 2, 2, 1] \oplus V[4, 2, 1, 1] \oplus V[4, 2, 2]^{\oplus 4} \oplus V[4, 3, 1]^{\oplus 3} \oplus V[4, 4]^{\oplus 4} \\ \oplus V[5, 2, 1]^{\oplus 4} \oplus V[5, 3]^{\oplus 4} \oplus V[6, 1, 1] \oplus V[6, 2]^{\oplus 7} \oplus V[7, 1]^{\oplus 4} \oplus V[8]^{\oplus 4}$$

$$D_{9,2} = V[3, 2, 2, 2] \oplus V[4, 2, 2, 1]^{\oplus 3} \oplus V[4, 3, 1, 1] \oplus V[4, 3, 2]^{\oplus 4} \oplus V[4, 4, 1]^{\oplus 6} \oplus V[5, 2, 1, 1]^{\oplus 2} \oplus V[5, 2, 2]^{\oplus 7} \\ \oplus V[5, 3, 1]^{\oplus 7} \oplus V[5, 4]^{\oplus 7} \oplus V[6, 2, 1]^{\oplus 9} \oplus V[6, 3]^{\oplus 12} \oplus V[7, 1, 1]^{\oplus 3} \oplus V[7, 2]^{\oplus 12} \oplus V[8, 1]^{\oplus 8} \oplus V[9]^{\oplus 6}$$

$$D_{8,3} = V[2, 2, 2, 2] \oplus V[3, 2, 2, 1] \oplus V[3, 3, 1, 1] \oplus V[4, 1, 1, 1, 1] \oplus V[4, 2, 1, 1] \oplus V[4, 2, 2]^{\oplus 3} \\ \oplus V[4, 3, 1] \oplus V[4, 4]^{\oplus 2} \oplus V[5, 2, 1]^{\oplus 2} \oplus V[5, 3] \oplus V[6, 2]^{\oplus 2} \oplus V[7, 1] \oplus V[8]$$

$$D_{9,3} = V[2, 2, 2, 2, 1] \oplus V[3, 2, 2, 1, 1]^{\oplus 2} \oplus V[3, 2, 2, 2]^{\oplus 4} \oplus V[3, 3, 1, 1, 1]^{\oplus 2} \oplus V[3, 3, 2, 1]^{\oplus 4} \oplus V[3, 3, 3]^{\oplus 2} \\ \oplus V[4, 1, 1, 1, 1, 1] \oplus V[4, 2, 1, 1, 1]^{\oplus 3} \oplus V[4, 2, 2, 1]^{\oplus 10} \oplus V[4, 3, 1, 1]^{\oplus 7} \oplus V[4, 3, 2]^{\oplus 10} \oplus V[4, 4, 1]^{\oplus 8} \\ \oplus V[5, 1, 1, 1, 1] \oplus V[5, 2, 1, 1]^{\oplus 8} \oplus V[5, 2, 2]^{\oplus 12} \oplus V[5, 3, 1]^{\oplus 13} \oplus V[5, 4]^{\oplus 7} \oplus V[6, 1, 1, 1]^{\oplus 2} \\ \oplus V[6, 2, 1]^{\oplus 12} \oplus V[6, 3]^{\oplus 11} \oplus V[7, 1, 1]^{\oplus 4} \oplus V[7, 2]^{\oplus 9} \oplus V[8, 1]^{\oplus 5} \oplus V[9]^{\oplus 3}$$

5 Equivariant log concavity

Fix a finite group W . We define a sequence (C_0, C_1, C_2, \dots) in $\text{VRep}(W)$ to be **log concave** if, for all $i > 0$, $C_i^{\otimes 2} - C_{i-1} \otimes C_{i+1} \in \text{Rep}(W)$. We call an element of $\text{grVRep}(W)$ log concave if its sequence of coefficients is log concave.

Remark 5.1. If W is the trivial group, then $\text{VRep}(W) \cong \mathbb{Z}$ and this is the usual notion of log concavity for a sequence of integers.

Remark 5.2. More generally, we can replace $\text{VRep}(W)$ by any partially ordered ring and have a reasonable definition of a log concave sequence in that ring.

Conjecture 5.3. *Let $W \curvearrowright M$ be an equivariant matroid.*

1. *The equivariant characteristic polynomial $H_M^W(t)$ is log concave.*

2. The equivariant Kazhdan-Lusztig polynomial $P_M^W(t)$ is log concave.

Remark 5.4. When W is the trivial group, Conjecture 5.3(1) has existed in various forms since the 1960s, and was only recently proven by Adiprasito, Huh, and Katz [AHK]. Conjecture 5.3(2) for the trivial group appeared in [EPW16, Conjecture 2.5].

Remark 5.5. We have verified both parts of Conjecture 5.3 with a computer for the braid matroid B_n for all $n \leq 9$, as well as part 2 for the uniform matroid $U_{m,d}$ for all $m, d \leq 15$. Part 1 of the conjecture for uniform matroids is proved below (Proposition 5.7) for arbitrary values of m and d .

Remark 5.6. In a forthcoming paper, we will explore the notion of log concavity for W -representations in greater detail. There we will give many more conjectural examples of naturally arising log concave sequences of representations.

Proposition 5.7. *The equivariant characteristic polynomial $H_{m,d}(t)$ of the uniform matroid $U_{m,d}$ is log concave.*

Proof. First, we observe that $OS_{U_{m,d},i}^{S_{m+d}}$ is equal to $OS_{U_{0,m+d},i}^{S_{m+d}}$ when $i < d$, it is equal to a quotient of $OS_{U_{0,m+d},i}^{S_{m+d}}$ when $i = d$, and it is equal to zero when $i > d$. For this reason, it suffices to prove the proposition when $m = 0$, in which case it says that

$$\left(\bigwedge^i \mathbb{C}^d\right)^{\otimes 2} - \left(\bigwedge^{i-1} \mathbb{C}^d \otimes \bigwedge^{i+1} \mathbb{C}^d\right) \in \text{Rep}(S_d).$$

Let $V_i := V[d-i, 1^i]$. Since $\bigwedge^i \mathbb{C}^d = V_i \oplus V_{i-1}$, it is sufficient to prove that

$$V_i^{\otimes 2} - V_{i-1} \otimes V_{i+1} \in \text{Rep}(S_d) \tag{8}$$

and

$$V_{i-1} \otimes V_i - V_{i-2} \otimes V_{i+1} \in \text{Rep}(S_d). \tag{9}$$

These tensor products may be computed using a formula of Remmel [Rem89, Theorem 2.1]. We will only prove Equation (8); the proof of Equation (9) is similar.

First, Remmel observes that tensoring with the sign representation takes V_i to V_{d-i-1} , and we may use this to reduce to the case where $i < d/2$, which we will assume for the remainder of the proof. For any partition λ of d , Remmel computes

$$c(\lambda, i, j) := \dim \text{Hom}(V[\lambda], V_i \otimes V_j);$$

we need to show that $c(\lambda, i, i) \geq c(\lambda, i-1, i+1)$ for all λ .

The number $c(\lambda, i, j)$ is zero unless $\lambda = [r, 1^{d-r}]$ for some r or $\lambda = [q, p, 2^k, 1^\ell]$ for some $q \geq p \geq 2$ and $k, \ell \geq 0$. When $\lambda = [r, 1^{d-r}]$, Remmel tells us that

$$c(\lambda, i, i) = \chi(d-2i-1 \leq r \leq d) \quad \text{and} \quad c(\lambda, i-1, i+1) = \chi(d-2i-1 \leq r \leq d-2),$$

where χ of a statement is 1 if the statement is true and 0 if the statement is false. In particular, we can see that $c([r, 1^{d-r}], i, i) \geq c([r, 1^{d-r}], i-1, i+1)$.

When $\lambda = [q, p, 2^k, 1^\ell]$, we put

$$u = \max(p, d - 2i), \quad \omega = 2(d - i - k) - \ell, \quad x = \lfloor \omega/2 \rfloor,$$

$$v'_0 = \min(q, d - i - k - 2), \quad v'_1 = v_0 = \min(q, d - i - k - 1), \quad \text{and} \quad v_1 = \min(q, d - i - k).$$

Remmel tells us that

$$c(\lambda, i, i) = \begin{cases} 0 & \text{if } p + k > d - i \\ \chi(u \leq x - 1 \leq v_0) + \chi(u \leq x \leq v_1) & \text{if } p + k \leq d - i \text{ and } \ell \text{ is even} \\ \chi(u \leq x \leq v_0) + \chi(u \leq x \leq v_1) & \text{if } p + k \leq d - i \text{ and } \ell \text{ is odd} \end{cases}$$

and

$$c(\lambda, i - 1, i + 1) = \begin{cases} 0 & \text{if } p + k > d - i - 1 \\ \chi(u \leq x - 1 \leq v'_0) + \chi(u \leq x \leq v'_1) & \text{if } p + k \leq d - i - 1 \text{ and } \ell \text{ is even} \\ \chi(u \leq x \leq v'_0) + \chi(u \leq x \leq v'_1) & \text{if } p + k \leq d - i - 1 \text{ and } \ell \text{ is odd.} \end{cases}$$

Since $v'_0 \leq v_0$, $v'_1 \leq v_1$, and $d - i - 1 < d - i$, this implies that $c(\lambda, i, i) \geq c(\lambda, i - 1, i + 1)$. \square

Remark 5.8. We define a sequence (C_0, C_1, C_2, \dots) in $\text{VRep}(W)$ to be **strongly log concave** if, for all $0 \leq k \leq i \leq j$, $C_i \otimes C_j - C_{i-k} \otimes C_{j+k} \in \text{Rep}(W)$. We call an element of $\text{grVRep}(W)$ strongly log concave if its sequence of coefficients is strongly log concave.

When W is the trivial group and $C_i \geq 0$ for all i , strong log concavity is equivalent to log concavity with no internal zeros. When $W = S_2$, however, the element

$$f(t) := (1 + 2t + 2t^2 + t^3)V[2] + (3 + 2t + 2t^2 + 3t^3)V[1, 1] \in \text{grRep}(W)$$

is log concave with no internal zeros but not strongly log concave.

The notion of strong log concavity may be more natural than the notion of ordinary log concavity, since it has the property that strong log concavity of $f(t)$ and $g(t)$ implies strong log concavity of $f(t) \otimes g(t)$.³ This fails for ordinary log concavity, as we can see by taking $f(t)$ as above and $g(t) = (1+t)V[2]$. Conjecture 5.3 and Proposition 5.7 may both be generalized to the corresponding “strong” versions.

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³The proof of this statement was communicated to us by David Speyer, and will appear in a future paper.

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