PROBLEM SET FOR DAY 1
NOTES FOR THE OREGON SUMMER SCHOOL 2013

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Topics:

- Schubert varieties on Grassmannians and flag manifolds.
- The Borel presentation.
- P-orbits on G/Q as Morse-Bott strata.
- Kleiman transversality.
- Equivariant cohomology.
- Kirwan injectivity for Hamiltonian actions on compact symplectic manifolds.
- GKM manifolds.
- The Anderson-Jantzen-Soergel/Billey restriction formula.
- Equivariant K-homology and K-cohomology.
- Double Grothendieck polynomials.
- Stanley-Reisner schemes (especially of shellable balls) and their K-classes

Consider $k \times n$ matrices of rank k. What's the relation between Gaussian elimination to reduced row-echelon form, and the Bruhat decomposition of the Grassmannian $Gr_k(\mathbb{C}^n)$?

That was too easy. What's the analogue of reduced row-echelon form, when one's thinking about 2-step flag manifolds $\{V^a < V^b \leq \mathbb{C}^n\}$?

Prove directly that the Borel presentation, in the $G = GL(n)$ case, gives a vector space of dimension $n!$.

Describe the poset of P-orbits on G/Q, when P, Q are two maximal parabolics in $GL(n)$.

How about if they're in $SO(n)$? Deal with n odd first.

Prove the statement that Omar made: If X is a T-space and $H^{odd}(X) = 0$, then X is T-equivariantly formal. (Hint: spectral sequence.)

Interpret the map $H^*_G(X) \otimes_{H^*_G(pt)} \mathbb{Q} \to H^*(X)$ in terms of the equivariant Künneth formula. Show by example that it need not be an isomorphism.

Use equivariant formality to show that $H^*(G/B) \cong H^*_T(pt) \otimes_{H^*_G(pt)} \mathbb{Q}$. Use equivariant Künneth (which works in the place you'll need it) to get $H^*_T(G/B) \cong H^*_T(pt) \otimes_{H^*_G(pt)} H^*_T(pt)$. Show that the second isomorphism implies the first.

If X is T-equivariantly formal, show that the restriction map $H^*_T(X) \to H^*_T(X^T)$ is injective, and that it is an isomorphism in high degree. (Hint: cover X with two T-invariant open sets, and use equivariant Mayer-Vietoris.)

Show that the map $H^*_T(M) \to H^*_T(M^T)$ is *not* injective if M is the union of three coordinate lines in $\mathbb{CP}^2$, or $M = \mathbb{C}^\times$ (where in each, you figure out what T should be).

Use GKM theory to understand the ring $H^*_T(\mathbb{CP}^1)$.

Use GKM theory to compute the product $S_{213}S_{132} = S_{231} + S_{312}$ in $H^*_T(\mathtt{Flags}(\mathbb{C}^3))$. Which is to say, figure out the restrictions of every class to fixed points. Then do the same with double Schubert polynomials in the Borel presentation.

Prove directly, in the simply-laced case, that the AJS/Billey formula for $S_w|_v$ is independent of the reduced word chosen for v.

Prove directly that the AJS/Billey formula for $\{S_w|_v\}$ satisfies the GKM conditions (w fixed, v varying).

Let $v = v_1 v_2 \cdots v_k$ be a length-additive factorization of v (e.g. a reduced word). Come up with a formula like AJS/Billey for computing $S_w|_v$, as a sum over some combinatorial objects related to this choice.

Compute all products in $H^*_T(\mathbb{CP}^2)$, using puzzles, without assuming commutativity.