

CHERN-SCHWARTZ-MACPHERSON CLASSES FOR SCHUBERT CELLS IN FLAG MANIFOLDS

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ABSTRACT. We obtain an algorithm describing the Chern-Schwartz-MacPherson (CSM) classes of Schubert cells in generalized flag manifolds G/B . In analogy to how the ordinary divided difference operators act on Schubert classes, each CSM class of a Schubert cell is obtained by applying certain Demazure-Lusztig type operators to the CSM class of a cell of dimension one less. By functoriality, we deduce algorithmic expressions for CSM classes of Schubert cells in any flag manifold G/P . We conjecture that the CSM classes of Schubert cells are an effective combination of (homology) Schubert classes, and prove that this is the case in several classes of examples.

1. INTRODUCTION

A classical problem in Algebraic Geometry is to define characteristic classes of singular algebraic varieties generalizing the notion of the total Chern class of the tangent bundle of a non-singular variety. The existence of a functorial theory of Chern classes for possibly singular varieties was conjectured by Grothendieck and Deligne, and established by R. MacPherson [Mac74]. This theory associates a class $c_*(\varphi) \in H_*(X)$ with every constructible function φ on X , such that $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ if X is a smooth compact complex variety. (The theory was later extended to arbitrary algebraically closed fields of characteristic 0, with values in the Chow group [Ken90], [Alu06b].) The strong functoriality properties satisfied by these classes determine them uniquely; we refer to §3 below for details. If X is a compact complex variety, then the class $c_*(\mathbb{1}_X)$ coincides with a class defined earlier by M. H. Schwartz [Sch65a, Sch65b] independently of the work mentioned above. This class is commonly known as the *Chern-Schwartz-MacPherson* (CSM) class of X . In general, we denote by $c_{\text{SM}}(W)$ the class $c_*(\mathbb{1}_W) \in H_*(X)$ for any constructible (e.g., locally closed) subset $W \subseteq X$.

Let G be a complex simple Lie group and let B be a Borel subgroup. Denote by W the Weyl group of G . The goal of this note is to provide an algorithm calculating the CSM classes of the Schubert cells $X(w)^\circ := BwB/B$ in the generalized flag manifold $X := G/B$, as w varies in W . To describe the answer, we need to recall two well known families of operators on the homology group $H_*(X)$; we refer to section 2 below for details.

Let $s_1, \dots, s_r \in W$ be the simple reflections corresponding respectively to the simple roots $\alpha_1, \dots, \alpha_r$ of G , and let $\ell : W \rightarrow \mathbb{N}$ be the length function. Denote by $X(w) := \overline{BwB/B}$ the Schubert variety corresponding to w ; it is a subvariety of X of complex dimension $\ell(w)$. For each $1 \leq k \leq r$, the classical BGG operator [BGG73] is an operator $\partial_k : H_*(X) \rightarrow H_{*+2}(X)$ which sends the Schubert class $[X(w)]$ to $[X(ws_k)]$ if $\ell(ws_k) > \ell(w)$ and to 0 otherwise. The Weyl group admits a right action on $H_*(X)$, which was originally used to *define* the BGG operator ∂_k . For $1 \leq k \leq r$ define the non-homogeneous operator

$$\mathcal{T}_k := \partial_k - s_k : H_*(X) \rightarrow H_*(X),$$

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where s_k denotes the (right) action of the simple reflection s_k . The main result of this note is the following.

Theorem 1.1. *Let $w \in W$ be a Weyl group element and $1 \leq k \leq r$. Then*

$$\mathcal{T}_k(c_{SM}(X(w)^\circ)) = c_{SM}(X(ws_k)^\circ).$$

In the case $w = \text{id}$, the Schubert cell $X(\text{id})^\circ$ is a point, and $c_{SM}([pt]) = [pt]$. More generally, if $w = s_{i_1} \dots s_{i_k}$, then the theorem implies that the CSM class $c_{SM}(X(w)^\circ)$ is obtained by composing the operators $\mathcal{T}_{i_k} \dots \mathcal{T}_{i_1}$. This is reminiscent of the classical situation in Schubert Calculus, where one generates all the Schubert classes by applying successively the BGG operators ∂_k . To further the analogy, $\partial_k^2 = 0$ and the BGG operators satisfy the braid relations. We prove that $\mathcal{T}_k^2 = 1$, and then Theorem 1.1 can be used to show that the operators \mathcal{T}_k also satisfy the braid relations (Proposition 4.1). In particular, there is a well defined operator \mathcal{T}_w associated with any $w \in W$, and this yields a new representation of the Weyl group on $H_*(X)$. The CSM classes of the Schubert cells $X(w)^\circ$ are the values obtained by applying to the class of a point $[pt]$ the operators $T_{w^{-1}}$ in this representation. Further, the action of each simple reflection \mathcal{T}_k on Schubert classes can be written explicitly using the Chevalley formula (Proposition 4.3). This gives an *explicit* algorithm to calculate the CSM class of any Schubert cell.

We also note that \mathcal{T}_k is related to a specialization of the *Demazure-Lusztig operator* defined in [Gin98, §12], in relation to degenerate Hecke algebras. We plan on investigating this connection further in a future paper.

Perhaps the most surprising feature of the CSM classes (and of the operators \mathcal{T}_k) is a positivity property. It follows from definition of CSM classes that

$$c_{SM}(X(w)^\circ) = \sum_{v \leq w} c(w; v)[X(v)],$$

where \leq denotes the Bruhat ordering, and $c(w; w) = c(w; \text{id}) = 1$. Despite the fact that \mathcal{T}_k does not preserve the positivity of a combination of Schubert classes, we conjecture that $c(w; v) > 0$ for all $v \leq w$. We have checked this by explicit computations for thousands of Schubert cells $X(w)$ in type A flag manifolds $\text{Fl}(n)$ for $n \leq 8$. We were also able to prove that this positivity holds for several families of Schubert cells, across all Lie types; see §5 below.

Let $P \subset G$ be a parabolic subgroup containing B , and let $p : G/B \rightarrow G/P$ be the natural projection. Then $p(X(w)^\circ) = X(wW_P)^\circ$, where W_P is the subgroup of W generated by the reflections in P . The functoriality of CSM classes can be used to prove that

$$p_*(c_{SM}(X(w)^\circ)) = c_{SM}(X(wW_P)^\circ)$$

(see Proposition 3.5 below). In particular, the (conjectured) positivity of CSM classes of Schubert cells in G/B implies the positivity of classes in any G/P . In the case when G/P is a Grassmann manifold this was proved in several cases by the authors of this note [AM09, Mih15], B. Jones [Jon10], J. Stryker [Str11], and it was settled for all cases by J. Huh [Huh]. Huh was able to realize the homogeneous components of CSM classes as a class of an effective cycle, but unfortunately his main technical requirements do not seem to hold for arbitrary flag manifolds G/B .

Our calculation of CSM classes is based on a construction of these classes in terms of bundles of logarithmic tangent fields. For every $W \subseteq X$, the class $c_{SM}(\mathbb{1}_W) \in H_*(X)$ may be obtained by pushing forward to X the total Chern class of $T_{\widetilde{W}}(-\log D)$, where \widetilde{W} is a resolution of \overline{W} such that $D := \widetilde{W} \setminus W$ is a simple normal crossing (SNC) divisor ([Alu99], [Alu06a]). We refer to §3 for details. This approach was used successfully in previous work

by the authors [AM09] to compute CSM classes of Schubert cells in the Grassmann manifold and prove partial positivity results. In that case one can use a resolution of a Schubert variety which is a (smooth) Schubert variety, but in a partial flag manifold. B. Jones [Jon10] gave an alternative computation of the classes, by means of a different (small) resolution, and also obtained partial positivity results. The resolution used in [AM09] has finitely many orbits of the Borel subgroup B , and this was a key fact in Huh's full proof of the positivity conjecture for CSM classes in that case [Huh].

For generalized flag manifolds G/B a resolution is given by *Bott-Samelson varieties*. These are iterated \mathbb{P}^1 -bundles, and they can be constructed from any (possibly non-reduced) word consisting of simple reflections. Section 2.3 is devoted to the definition and cohomological properties of Bott-Samelson varieties. The key technical result in the paper is Theorem 3.3, establishing the necessary cohomological formulas calculating the push-forward of the Chern class of the logarithmic tangent bundle. The operators $\hat{\partial}_k$ and s_k , and $\mathcal{T}_k = \hat{\partial}_k - s_k$, appear naturally in these push-forward formulas.

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2. PRELIMINARIES

The goal of this section is to recall some basic facts on the cohomology of flag manifolds and on Bott-Samelson resolutions. We refer to [BK05], §2.1–§2.2, for more details and references to the standard literature.

2.1. Flag manifolds and Schubert varieties. Let G be a complex simple Lie group and let $T \subseteq B \subseteq G$ be a maximal torus included in a Borel subgroup of G . Let \mathfrak{h} and \mathfrak{g} be the Lie algebras of T and G , and let $R \subseteq \mathfrak{h}^*$ be the associated root system with the set of positive roots R^+ determined by B . Denote by $\Delta := \{\alpha_1, \dots, \alpha_r\} \subseteq R^+$ the set of simple roots. Let R^\vee denote the set of coroots $\alpha^\vee \in \mathfrak{h}$ and $\langle \cdot, \cdot \rangle : R \otimes R^\vee \rightarrow \mathbb{Z}$ the evaluation pairing.

To each root $\alpha \in R$ one associates a reflection s_α in the Weyl group $W = N_G(T)/T$. The set of simple reflections $s_i := s_{\alpha_i}$ generate W , thus each $w \in W$ can be written as $w = s_{i_1} \cdots s_{i_k}$. The minimal such integer k is denoted by $\ell(w)$, the length of w . If $k = \ell(w)$, then the decomposition $w = s_{i_1} \cdots s_{i_k}$ is said to be *reduced*. There is a partial order on W called the *Bruhat ordering*, defined as follows: $u < v$ if there exists a chain

$$u_0 := u \rightarrow u_1 := us_{\beta_1} \rightarrow u_2 := u_1 s_{\beta_2} \rightarrow \cdots \rightarrow u_n := v = u_{n-1} s_{\beta_n}$$

where the β_i 's are roots in R such that $\ell(u_i) > \ell(u_{i-1})$. Let $X := G/B$ be the generalized flag variety. This is a homogeneous space for G (hence a non-singular variety) and it is stratified by *Schubert cells* $X(w)^\circ := BwB/B$, where $w \in W$, and each such cell $X(w)^\circ$ is isomorphic to $\mathbb{C}^{\ell(w)}$. The closure $X(w) := \overline{BwB/B}$ is called a *Schubert variety*. Each Schubert variety $X(w)$ has a fundamental class $[X(w)] \in H_{2\ell(w)}(X)$, and these classes form a \mathbb{Z} -basis for the (co)homology. It may be verified that $X(v) \subseteq X(w)$ if and only if $v \leq w$ in the Bruhat order. In fact, $X(w) = \bigcup_{v \leq w} X(v)^\circ$. It follows ([Ful98, Example 1.9.1]) that any class in $H_*(X(w))$ may be written uniquely as an integer linear combination $\sum_{v \leq w} c_v [X(v)]$.

Let $\mathfrak{h}_{\mathbb{Z}}^*$ be the integral weight lattice and let $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ be an integral weight. Then one constructs the G -equivariant line bundle over X

$$\mathcal{L}_\lambda := G \times^B \mathbb{C}_{-\lambda} = (G \times \mathbb{C})/B$$

where B acts on $G \times \mathbb{C}$ by $b.(g, u) = (gb^{-1}, \lambda(b)^{-1}u)$ (and the action of $B = UT$ on \mathbb{C} extends the action of T so that it is trivial over the unipotent group U).

The *Chevalley formula* states that

$$(1) \quad c_1(\mathcal{L}_\lambda) \cdot [X(w)] = \sum \langle \lambda, \beta^\vee \rangle [X(ws_\beta)],$$

where the sum is over all positive roots β such that $\ell(ws_\beta) = \ell(w) - 1$. See e.g., [FW04, Lemma 8.1].

2.2. Two operators acting $H^*(G/B)$. For each simple root $\alpha_k \in \Delta$ one can construct the BGG operator $\partial_k : H^*(X) \rightarrow H^*(X)$ defined in [BGG73] as follows. Let $P_k \subseteq G$ be the minimal parabolic subgroup corresponding to α_k . Then the natural projection $\pi : G/B \rightarrow G/P_k$ is a \mathbb{P}^1 -bundle and there is a fibre square

$$\begin{array}{ccc} G/B \times_{G/P_k} G/B & \xrightarrow{pr_1} & G/B \\ \downarrow pr_2 & & \downarrow p_k \\ G/B & \xrightarrow{p_k} & G/P_k \end{array}$$

The BGG operator is defined to be $\partial_k = p_k^*(p_k)_*$. We record next two properties of this operator; see e.g., [Knu, Proposition 2] or [Tym08] for simple proofs.

Proposition 2.1. *The operator ∂_k satisfies the following properties:*

(a) *For all Weyl group elements $w \in W$,*

$$(2) \quad \partial_k([X(w)]) = \begin{cases} 0 & \text{if } \ell(ws_k) < \ell(w) \\ [X(ws_k)] & \text{if } \ell(ws_k) > \ell(w) \end{cases}$$

In particular, $\partial_k^2 = 0$ and the BGG operators satisfy the same braid relations as the elements of the Weyl group.

(b) *For all $\gamma_1, \gamma_2 \in H^*(G/B)$,*

$$\partial_k(\gamma_1\gamma_2) = \partial_k(\gamma_1)\gamma_2 + \gamma_1\partial_k(\gamma_2) - c_1(\mathcal{L}_{\alpha_k})\partial_k(\gamma_1)\partial_k(\gamma_2)$$

where $c_1(\mathcal{L}_{\alpha_k})$ denotes the Chern class of \mathcal{L}_{α_k} .

For each $w \in W$ there is a well defined map $r_w : G/T \rightarrow G/T$ obtained by multiplying on the right with any lift of w in $N_G(T)$. This induces a ring endomorphism $r_w^* : H^*(G/T) \rightarrow H^*(G/T)$. Note that the projection $G/T \rightarrow G/B$ is a $U \simeq B/T$ -bundle and because U is contractible this implies that the cohomology rings $H^*(G/B)$ and $H^*(G/T)$ are isomorphic. This defines a right action of W on $H^*(X)$, denoted again by w ; it will be clear from the context whether we refer to the Weyl group element or to its action on $H^*(X)$. It is well known (see e.g., [BGG73, §1]) that for $w = s_k$ this operator satisfies

$$(3) \quad s_k = \text{id} - c_1(\mathcal{L}_{\alpha_k})\partial_k$$

where the Chern class $c_1(\mathcal{L}_{\alpha_k})$ acts on $H^*(X)$ by multiplication. Combining this with the Chevalley formula we obtain an identity

$$(4) \quad s_k[X(w)] = \begin{cases} [X(w)] & \text{if } \ell(ws_k) < \ell(w); \\ -[X(w)] - \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)] & \text{if } \ell(ws_k) > \ell(w). \end{cases}$$

where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(w) = \ell(ws_k s_\beta)$.

For use in §4 we also record the following commutation relation of the operators ∂_k and s_k .

Lemma 2.2. *With notation as above,*

$$s_k \hat{\partial}_k = \hat{\partial}_k \quad , \quad \hat{\partial}_k s_k = -\hat{\partial}_k.$$

In particular, $\hat{\partial}_k s_k + s_k \hat{\partial}_k = 0$.

Proof. The first equality follows immediately from the definition of s_k and the fact that $\hat{\partial}_k^2 = 0$. The second equality is a consequence of Proposition 2.1(b):

$$\begin{aligned} \hat{\partial}_k s_k &= \hat{\partial}_k - \hat{\partial}_k(c_1(\mathcal{L}_{\alpha_k})\hat{\partial}_k) \\ &= \hat{\partial}_k - \hat{\partial}_k(c_1(\mathcal{L}_{\alpha_k}))\hat{\partial}_k - c_1(\mathcal{L}_{\alpha_k})\hat{\partial}_k^2 + c_1(\mathcal{L}_{\alpha_k})\hat{\partial}_k(c_1(\mathcal{L}_{\alpha_k}))\hat{\partial}_k^2 \\ &= \hat{\partial}_k - \hat{\partial}_k(c_1(\mathcal{L}_{\alpha_k}))\hat{\partial}_k \\ &= -\hat{\partial}_k, \end{aligned}$$

where we used the fact that $\hat{\partial}_k(c_1(\mathcal{L}_{\alpha_k})) = \langle \alpha_k, \alpha_k^\vee \rangle = 2$ as may be checked using Proposition 2.1(a) and the Chevalley formula. \square

2.3. Bott-Samelson varieties. For each word $s_{i_1} s_{i_2} \dots s_{i_k}$ for an element $w \in W$ one can construct a tower of \mathbb{P}^1 bundles, the *Bott-Samelson variety* $Z := Z_{i_1, \dots, i_k}$, endowed with a map $\theta := \theta_{i_1, \dots, i_k} : Z \rightarrow X(w)$. If the word is reduced, this map is birational, giving a resolution of singularities for $X(w)$ (depending on the choice of the word for w). There are several ways to do this, but for our purpose we present an inductive construction which can be found e.g., in [BK05, §2.2].

If the word is empty, then define $Z := pt$. In general assume we have constructed $Z' := Z_{i_1, \dots, i_{k-1}}$ and the map $\theta' : Z' \rightarrow X(w')$, for $w' = s_{i_1} \dots s_{i_{k-1}}$. Define $Z = Z_{i_1, \dots, i_k}$ so that the left square in the diagram

$$(5) \quad \begin{array}{ccccc} Z & \xrightarrow{\theta_1} & G/B \times_{G/P_{i_k}} G/B & \xrightarrow{pr_1} & G/B \\ \pi \downarrow & & \downarrow pr_2 & & \downarrow p_{i_k} \\ Z' & \xrightarrow{\theta'} & G/B & \xrightarrow{p_{i_k}} & G/P_{i_k} \end{array}$$

is a fibre square; the morphisms pr_1, pr_2, p_{i_k} are the natural projections. From this construction it follows that Z_{i_1, \dots, i_k} is a smooth, projective variety of dimension k .

The Bott-Samelson variety Z is equipped with a simple normal crossing (SNC) divisor D_Z . We recall next an explicit inductive construction of this divisor, which will be needed later. If $Z = pt$, then $D_Z = \emptyset$. In general, G/B is the projectivization $\mathbb{P}(E)$ of a homogeneous rank-2 vector bundle $E \rightarrow G/P_k$, defined up to tensoring with a line bundle. Define $\mathcal{E} := E \otimes \mathcal{O}_E(1)$, a vector bundle over $G/B = \mathbb{P}(E)$. Then we have the Euler sequence of the projective bundle $\mathbb{P}(E)$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where \mathcal{Q} is the relative tangent bundle $T_{p_{i_k}}$. Note that \mathcal{E} is independent of the specific choice of E , and $pr_2 : G/B \times_{G/P_{i_k}} G/B \rightarrow G/B$, that is, the pull-back of $\mathbb{P}(E)$ via p_{i_k} , may be identified with $\mathbb{P}(\mathcal{E})$. Let $\mathcal{E}' := (\theta')^* \mathcal{E}$ and $\mathcal{Q}' := (\theta')^* \mathcal{Q}$, and pull-back the previous sequence via θ' to get an exact sequence

$$(6) \quad 0 \longrightarrow \mathcal{O}_{Z'} \longrightarrow \mathcal{E}' \longrightarrow \mathcal{Q}' \longrightarrow 0.$$

The inclusion $\mathcal{O}_{Z'} \hookrightarrow \mathcal{E}'$ gives a section $\sigma : Z' \rightarrow Z$ of π and therefore a divisor $D_k := \sigma(Z') = \mathbb{P}(\mathcal{O}_{Z'})$ in $\mathbb{P}(\mathcal{E}') = Z$. The SNC divisor on Z is defined by

$$D_Z = \pi^{-1}(D_{Z'}) \cup D_k$$

where $D_{Z'}$ is the inductively constructed SNC divisor on Z' . The following result is well known, see e.g., [BK05, §2.2]:

Proposition 2.3. *If w is a reduced word, then the image of the composition $\theta = pr_1 \circ \theta_1 : Z_{i_1, \dots, i_k} \rightarrow G/B$ is the Schubert variety $X(w)$. Moreover, $\theta^{-1}(X(w) \setminus X(w)^\circ) = D_{Z_{i_1, \dots, i_k}}$ and the restriction map*

$$\theta : Z_{i_1, \dots, i_k} \setminus D_{Z_{i_1, \dots, i_k}} \rightarrow X(w)^\circ$$

is an isomorphism.

Let $h_k := c_1(\mathcal{O}_{\mathcal{E}'}) \in H^2(Z)$. For later use we record next the class of the divisor D_k in Z , and the Chern classes of the relative tangent bundle $T_\pi = T_{\mathbb{P}(\mathcal{E}')|Z'}$ and of T_Z .

Proposition 2.4. *The following identities hold in $H^*(Z)$:*

- (a) $[D_k] = c_1(\pi^*(\mathcal{Q}') \otimes \mathcal{O}_{\mathcal{E}'}) \in H^*(Z)$;
- (b) $h_k \cdot [D_k] = 0$;
- (c) $c(T_\pi) = (1 + [D_k])(1 + h_k)$, and therefore

$$c(T_Z) = \pi^*(c(T_{Z'}))(1 + h_k)(1 + [D_k]).$$

Proof. (a) follows from the definition of D_k and [Ful98, Ex. 3.2.17], since $\mathcal{Q}' = \mathcal{E}'/\mathcal{O}_{Z'}$.

(b) holds since $h_k|_{D_k} = c_1(\mathcal{O}_{\mathcal{O}_{Z'}}) = 0$.

To prove (c), note that by (6) the Chern roots of \mathcal{E}' are 0 and $c_1(\mathcal{Q}')$; it follows from (a) that the Chern roots of $\pi^*\mathcal{E}' \otimes \mathcal{O}_{\mathcal{E}'}$ are h_k and $[D_k]$. The Euler sequence

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \pi^*\mathcal{E}' \otimes \mathcal{O}_{\mathcal{E}'} \longrightarrow T_\pi \longrightarrow 0$$

then implies $c(T_\pi) = c(\pi^*\mathcal{E}' \otimes \mathcal{O}_{\mathcal{E}'}) = (1 + h_k)(1 + [D_k])$. The last statement follows from $c(T_Z) = \pi^*(c(T_{Z'}))c(T_\pi)$. \square

3. CHERN-SCHWARTZ-MACPHERSON CLASSES OF SCHUBERT CELLS IN G/B

3.1. CSM classes. Let Y be an algebraic variety over \mathbb{C} . Denote by $\mathcal{F}(Y)$ the group of constructible functions on Y : the elements of $\mathcal{F}(Y)$ are finite sums $\sum c_i \mathbb{1}_{W_i}$ where $c_i \in \mathbb{Z}$, $W_i \subseteq Y$ are locally closed subvarieties, and $\mathbb{1}_W$ denotes the characteristic function taking value 1 on $p \in W$ and 0 otherwise. If $f : Y \rightarrow X$ is a proper morphism of varieties, one can define a push-forward $f_* : \mathcal{F}(Y) \rightarrow \mathcal{F}(X)$ by setting

$$f_*(\mathbb{1}_W)(p) = \chi(f^{-1}(p) \cap W)$$

for $W \subseteq Y$ a subvariety and $p \in X$, and extending by linearity to every $\varphi \in \mathcal{F}(Y)$; this makes \mathcal{F} into a covariant functor. Here χ denotes the topological Euler characteristic. MacPherson [Mac74] proved a conjecture of Deligne and Grothendieck stating that there exists a natural transformation $c_* : \mathcal{F} \rightarrow H_*$ such that if Y is non-singular, then $c_*(\mathbb{1}_Y) = c(T_Y) \cap [Y]$. The naturality of c_* means that if $f : Y \rightarrow X$ is a proper morphism, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(Y) & \xrightarrow{c_*} & H_*(Y) \\ \downarrow f_* & & \downarrow f_* \\ \mathcal{F}(X) & \xrightarrow{c_*} & H_*(X) \end{array}$$

That is,

$$(7) \quad f_*(c_*(\varphi)) = c_*(f_*(\varphi))$$

in H_*X , for all constructible functions φ . Resolution of singularities and the normalization requirement easily imply that c_* is unique.

If Y is a compact complex variety, the class $c_*(\mathbb{1}_Y)$ coincides with a class defined earlier by M. H. Schwartz [Sch65a, Sch65b]; this class is the *Chern-Schwartz-MacPherson* (CSM) class of Y . Taking f to be a constant map, the commutativity of the above diagram implies that $\int c_*(\mathbb{1}_Y) = \chi(Y)$, so this class provides a natural generalization of the Poincaré-Hopf theorem to possibly singular varieties. Abusing language a little, we denote by $c_{\text{SM}}(W) := c_*(1_W) \in H_*Y$ the CSM class of any constructible set W in a variety Y ; by additivity of Euler characteristics, $\int c_{\text{SM}}(W) = \chi(W)$.

Our main tool will be the observation that if Z is a nonsingular variety and $W \subseteq Z$ is an open subvariety such that $Z \setminus W$ is a SNC divisor with components D_i , then

$$(8) \quad c_{\text{SM}}(W) = \frac{c(T_Z)}{(1 + [D_1]) \cdots (1 + [D_n])} \in H_*Z$$

(cf. [GP02, Proposition 15.3], [Alu99, Théorème 1]).

In fact this observation may be used to extend the scope of the natural transformation c_* to arbitrary algebraically closed fields of characteristic 0, with values in the Chow group A_* . In this generality, c_* may be constructed as follows. Every constructible function on Y can be written as a linear combination of characteristic functions $\mathbb{1}_W$ for W locally closed and non-singular in Y , so it suffices to describe $c_{\text{SM}}(W) = c_*(\mathbb{1}_W)$ for such W . By resolution of singularities, there exists a desingularization $\pi : Z \rightarrow \overline{W}$ of the closure \overline{W} of W in Y such that $D := \pi^{-1}(\overline{W} \setminus W)$ is a SNC divisor in Z . Then one may take the push-forward of (8) to Y as the *definition* of $c_*(\mathbb{1}_W)$: one can show that over algebraically closed fields of characteristic 0 the resulting c_* is independent of the choices and satisfies the Deligne-Grothendieck axioms mentioned above ([Alu06a, Alu06b]).

3.2. A recursive formula for CSM classes of Schubert cells. We will now apply identity (8) to calculate the Chern-Schwartz-MacPherson class of a Schubert cell $X(w)^\circ \subseteq X := G/B$. This class may be viewed as an element of $H_*(G/B)$, and in fact of $H_*(X(w))$, and hence it can be written as an integer linear combination of classes $[X(v)]$ for $v \leq w$ in the Bruhat order, as we observed in §2.1. We will give an algorithm which yields this linear combination. All the necessary ingredients were developed in §2.3 and we keep the notation of that section. In particular we recall the fiber diagram (5):

$$\begin{array}{ccccc} Z & \xrightarrow{\theta_1} & G/B \times_{G/P_{i_k}} G/B & \xrightarrow{pr_1} & G/B \\ \pi \downarrow & & \downarrow pr_2 & & \downarrow p_{i_k} \\ Z' & \xrightarrow{\theta'} & G/B & \xrightarrow{p_{i_k}} & G/P_{i_k} \end{array}$$

Let $s_{i_1} \cdots s_{i_k}$ be any word (reduced or otherwise), and let $Z := Z_{i_1, \dots, i_k}$ be the corresponding Bott-Samelson variety. Recall from §2.3 that $G/B \times_{G/P_{i_k}} G/B = \mathbb{P}(\mathcal{E})$ for a canonically defined vector rank-2 vector bundle \mathcal{E} on G/B ; thus $Z = \mathbb{P}(\mathcal{E}')$ is the projectivization of the pull-back $\mathcal{E}' = \theta'^*(\mathcal{E})$. In $H^2(Z_{i_1, \dots, i_k})$ we have the tautological class $h_k = c_1(\mathcal{O}_{\mathcal{E}'(1)})$, as well as the pull-backs of h_j from Z_{i_1, \dots, i_j} for $j < k$; we will omit the pull-back notation.

Let D_Z be the SNC divisor defined in §2.3, and denote by $Z^\circ = Z_{i_1, \dots, i_k}^\circ$ the complement $Z \setminus D_Z$. By (8),

$$(9) \quad c_{\text{SM}}(Z^\circ) = \frac{c(T_Z)}{(1 + [D_1]) \cdots (1 + [D_k])}$$

where D_1, \dots, D_k are the components of D_Z .

Lemma 3.1. *With notation as above, the following holds in $H_*(Z)$:*

$$c_{SM}(Z^\circ) = (1 + h_k) \cdot \pi^*(c_{SM}(Z'^\circ)) = \prod_{j=1}^k (1 + h_j) \cdot [Z].$$

Proof. The first formula follows from (9) and Proposition 2.4 (c), and the second formula is an immediate consequence. \square

Now let $X(w)$ be the Schubert variety determined by $w \in W$, that is, the closure of $X(w)^\circ$ in X , and fix a reduced decomposition $s_{i_1} \cdots s_{i_k}$ of w and the corresponding Bott-Samelson variety Z . As recalled in Proposition 2.3, the composition $pr_1 \circ \theta_1$ gives a proper birational morphism (hence a desingularization) $\theta : Z \rightarrow X(w)$, restricting to an isomorphism on $\theta^{-1}(X(w)^\circ) = Z^\circ$.

Lemma 3.2. *The CSM class of the Schubert cell $X(w)^\circ$ is given by*

$$c_{SM}(X(w)^\circ) = \theta_*((1 + h_k) \cdot \pi^*(c_{SM}(Z'^\circ))).$$

Proof. By construction, $\theta_*(\mathbb{1}_{Z^\circ}) = \mathbb{1}_{X(w)^\circ}$; therefore, the functoriality of CSM classes (7) implies $c_{SM}(X(w)^\circ) = \theta_*(c_{SM}(Z^\circ))$, and the stated formula then follows from Lemma 3.1. \square

Lemma 3.2 motivates the study of the quantity $\theta_*((1 + h_k) \cdot \pi^*(\gamma))$ for $\gamma \in H_*(Z')$. The next theorem gives the key formulas needed for explicit calculations, in terms of the operators introduced in §2.2.

Theorem 3.3. *Let $\gamma \in H_*(Z')$. Then the following holds in $H_*(X)$:*

- (a) $\theta_*(\pi^*(\gamma)) = \partial_{i_k}(\theta'_*(\gamma))$;
- (b) $\theta_*(h_k \cdot \pi^*(\gamma)) = -s_{i_k}(\theta'_*(\gamma))$.

Therefore,

$$\theta_*((1 + h_k) \cdot \pi^*(\gamma)) = \mathcal{T}_{i_k}(\theta'_*(\gamma))$$

where $\mathcal{T}_i : H^*(X) \rightarrow H^*(X)$ is the operator given by $\mathcal{T}_i = \partial_i - s_i$.

Before proving the theorem, we note that if $\gamma = c_{SM}(Z'^\circ)$, then $\theta'_*(\gamma) = c_{SM}(X(w')^\circ)$ where $w' = s_{i_1} \cdots s_{i_{k-1}} = ws_{i_k}$. Therefore Theorem 3.3 gives a recursive formula to calculate the CSM classes:

Corollary 3.4. *Let $w \in W$ be a non-identity element and let s_k be a simple reflection such that $\ell(ws_k) < \ell(w)$. Then the following recursive identity holds:*

$$c_{SM}(X(w)^\circ) = \mathcal{T}_k(c_{SM}(X(ws_k)^\circ)),$$

with the initial condition that $c_{SM}(X(id)^\circ) = c_{SM}(pt) = [pt]$.

The explicit action of the operator \mathcal{T}_k on Schubert classes $[X(u)]$ is obtained by combining identities (2) and (4) above. The resulting formula together with other properties of the operator \mathcal{T}_k will be presented in §4 below.

Proof of Theorem 3.3. Both the left and right squares in (5) are fiber squares, and p_{i_k} is flat and $\theta = pr_1\theta_1$ is proper, so

$$\theta_*\pi^*(\gamma) = p_{i_k}^*(p_{i_k})_*\theta'_*(\gamma) = \partial_{i_k}(\theta'_*(\gamma))$$

by [Ful98, Proposition 1.7] and the definition of ∂_{i_k} given in §2.2. This proves (a).

For (b), let $\underline{\gamma} := \theta'_*(\gamma) \in H^*(G/B)$ and

$$\tilde{h}_k = c_1(\mathcal{O}_{\mathcal{E}}(1)) \in H^2(G/B \times_{G/P_{i_k}} G/B),$$

so that $h_k = \theta_1^*(\tilde{h}_k)$. Then

$$\begin{aligned} \theta_*(h_k \cdot \pi^*(\gamma)) &= (pr_1)_*(\theta_1)_*(\theta_1^*(\tilde{h}_k) \cdot \pi^*(\gamma)) \\ &= (pr_1)_*(\tilde{h}_k \cdot (\theta_1)_*\pi^*(\gamma)) \\ &= (pr_1)_*(\tilde{h}_k \cdot pr_2^*(\theta')_*(\gamma)) \\ &= (pr_1)_*(\tilde{h}_k \cdot pr_2^*(\gamma)). \end{aligned}$$

In the second equality we used the projection formula, and in the third we used the fact that the left square in (5) is a fibre square and that pr_2 is flat and θ' is proper. Now recall that G/B is the projectivization $\mathbb{P}(E)$ of a vector bundle E over G/P_{i_k} , and \mathcal{E} is $p_{i_k}^*(E) \otimes \mathcal{O}_E(1)$ as a bundle over G/B . We can compute the tautological subbundle $\mathcal{O}_{\mathcal{E}}(-1)$ of $pr_2^*(\mathcal{E})$, a bundle over $G/B \times_{G/P_{i_k}} G/B$, by using [Ful98, Appendix B.5.5]:

$$\mathcal{O}_{\mathcal{E}}(-1) = \mathcal{O}_{p_{i_k}^*(E) \otimes \mathcal{O}_E(1)}(-1) = pr_2^* \mathcal{O}_{p_{i_k}^*(E)}(-1) \otimes pr_2^* \mathcal{O}_E(1) = pr_1^* \mathcal{O}_E(-1) \otimes pr_2^* \mathcal{O}_E(1).$$

Letting $\eta = c_1(\mathcal{O}_E(1))$, this implies

$$\tilde{h}_k = c_1(pr_1^* \mathcal{O}_E(1)) + c_1(pr_2^* \mathcal{O}_E(-1)) = pr_1^*(\eta) - pr_2^*(\eta),$$

and the projection formula gives

$$\begin{aligned} (pr_1)_*(\tilde{h}_k \cdot pr_2^*(\gamma)) &= (pr_1)_*((pr_1^*(\eta) - pr_2^*(\eta)) \cdot pr_2^*(\gamma)) \\ &= \eta \cdot (pr_1)_*pr_2^*(\gamma) - (pr_1)_*pr_2^*(\eta \cdot \gamma) \\ &= \eta \cdot p_{i_k}^*(p_{i_k})_*(\gamma) - p_{i_k}^*(p_{i_k})_*(\eta \cdot \gamma) \end{aligned}$$

where the last equality follows since the second square in (5) is also a fiber square and p_{i_k} is both flat and proper. By definition, $\partial_{i_k} = p_{i_k}^*(p_{i_k})_*$. Putting all together, we have shown that

$$\theta_*(h_k \cdot \pi^*\gamma) = \eta \cdot \partial_{i_k}(\gamma) - \partial_{i_k}(\eta \cdot \gamma).$$

Since $\mathcal{O}_E(1)$ has degree 1 on the fibres of p_{i_k} , and p_{i_k} has relative dimension 1, we have

$$\partial_{i_k}(\eta) = p_{i_k}^*(p_{i_k})_*(\eta) = [G/B].$$

We use this and part (b) of Proposition 2.1 to get

$$\begin{aligned} \partial_{i_k}(\eta \cdot \gamma) &= \partial_{i_k}(\eta) \cdot \gamma + \eta \cdot \partial_{i_k}(\gamma) - c_1(\mathcal{L}_{\alpha_{i_k}}) \cdot \partial_{i_k}(\eta) \cdot \partial_{i_k}(\gamma) \\ &= \gamma + \eta \cdot \partial_{i_k}(\gamma) - c_1(\mathcal{L}_{\alpha_{i_k}}) \cdot \partial_{i_k}(\gamma) \end{aligned}$$

and finally

$$\theta_*(h_k \cdot \pi^*\gamma) = -\gamma + c_1(\mathcal{L}_{\alpha_{i_k}}) \cdot \partial_{i_k}(\gamma) = (-\text{id} + c_1(\mathcal{L}_{\alpha_{i_k}}))(\gamma) = -s_{i_k}(\gamma)$$

by (3), concluding the proof of (b). \square

3.3. Chern classes of Schubert cells in G/P . Fix a parabolic subgroup $P \subset G$ containing the Borel subgroup B . Let $W_P \subseteq W$ be the subgroup generated by the simple reflections in P . It is known (see e.g., [Hum90, §1.10]) that each coset in W/W_P has a unique minimal length representative; we denote by W^P the set of these representatives. If $w \in W$, then one can define a length function $\ell : W/W_P \rightarrow \mathbb{N}$ by $\ell(wW_P) := \ell(w')$ where $w' \in W^P$ is in the coset of w .

The space G/P is a projective manifold of dimension $\ell(w_0W_P)$, where w_0 is the longest element in W . For each $w \in W^P$ there is a Schubert cell $X(wW_P)^\circ := BwP/P$ of dimension $\ell(wW_P)$, and the corresponding Schubert variety $X(wW_P) := \overline{BwP/P}$; see e.g., [BL00, §2.6]. The fundamental classes $[X(wW_P)] \in H_{2\ell(wW_P)}(G/P)$ ($w \in W^P$) form a \mathbb{Z} -basis

for the homology $H_*(G/P)$. The natural projection $p : G/B \rightarrow G/P$ satisfies $p(X(w)) = X(wW_P)$ and the induced map in homology is given by

$$(10) \quad p_*[X(w)] = \begin{cases} [X(wW_P)] & \text{if } \ell(w) = \ell(wW_P); \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.5. *With notation as above,*

$$(11) \quad c_{SM}(X(wW_P)^\circ) = p_*(c_{SM}(X(w)^\circ)) \in H_*(G/P)$$

for all $w \in W$. Further, if $u \leq w$ and $\ell(u) = \ell(uW_P)$, then the coefficient of $[X(u)]$ in $c_{SM}(X(w)^\circ)$ equals the coefficient of $[X(uW_P)]$ in $c_{SM}(X(wW_P)^\circ)$.

Proof. The topological Euler characteristic χ of the fibers of the restriction of p to $X(w)^\circ$ is constant, hence the push-forward $p_*(\mathbb{1}_{X(w)^\circ})$ equals $\chi \cdot \mathbb{1}_{X(wW_P)^\circ}$. By functoriality of CSM classes (7) this implies that $p_*(c_{SM}(X(w)^\circ)) = \chi \cdot c_{SM}(X(wW_P)^\circ)$. Since the coefficient of $[pt]$ in both CSM classes equals 1, it follows that $\chi = 1$. The last claim follows from (10). \square

Thus the CSM classes of Schubert cells in G/P are determined by the corresponding classes in G/B . For example, the CSM classes of Schubert cells in the ordinary Grassmannian, determined explicitly in [AM09], can also be computed in principle using the recursive formula obtained in Corollary 3.4; see Example 4.4 for a concrete example. Further, the push-forward formula (10) implies that if the positivity conjecture discussed in §5 is true for the CSM classes of Schubert cells in G/B , then the analogous conjecture must be true for CSM classes of Schubert cells in G/P for any parabolic P containing B .

4. THE OPERATORS \mathcal{T}_k AND A WEYL GROUP REPRESENTATION ON $H^*(G/B)$

In this section we analyze the operator $\mathcal{T}_k = \partial_k - s_k : H_*(X) \mapsto H_*(X)$ which gives the recursion for CSM classes of Schubert cells as proven in Corollary 3.4. We start by recording the main algebraic properties of the operators \mathcal{T}_k .

Proposition 4.1. *The following identities hold:*

- (a) $\mathcal{T}_k^2 = 1$.
- (b) *The operators \mathcal{T}_k satisfy the braid relations, i.e., $(\mathcal{T}_i \mathcal{T}_j)^{m_{i,j}} = 1$ where $m_{i,j}$ is the order of the element $s_i s_j \in W$. Also, if $w = s_{i_1} \cdots s_{i_k}$ is a representation of an element $w \in W$ as a word in simple reflections, then the operator $\mathcal{T}_w := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_k}$ is independent of the choice of the word representing w .*
- (c) *For any $u, v \in W$, $\mathcal{T}_u \cdot \mathcal{T}_v = \mathcal{T}_{uv}$.*

Proof. First we note that

$$\mathcal{T}_k^2 = (\partial_k - s_k)^2 = s_k^2 = 1$$

since $\partial_k^2 = 0$ and $\partial_k s_k + s_k \partial_k = 0$ by Lemma 2.2. This proves part (a). To prove the first part of (b), it suffices to show that the relations hold after applying the operators to the classes $c_{SM}(X(w)^\circ)$, since these form a basis for $H_*(X)$. These relations follow then immediately from the fact that for all $w \in W$ and all simple reflections s_k ,

$$(12) \quad \mathcal{T}_k(c_{SM}(X(w)^\circ)) = c_{SM}(X(ws_k)^\circ)$$

as a consequence of Corollary 3.4 and part (a). The independence of \mathcal{T}_w on the specific word for w is also an immediate consequence of (12). Finally, (c) follows from the independence of \mathcal{T}_w on the word representing w . \square

The proposition implies that the operators \mathcal{T}_w define a representation of the Weyl group W on $H_*(G/B)$. We record an immediate consequence of the identity (12) from the proof of Proposition 4.1.

Corollary 4.2. *Let u, w be two Weyl group elements. Then the identity*

$$\mathcal{T}_u(c_{SM}(X(w)^\circ)) = c_{SM}(X(wu^{-1})^\circ)$$

holds in $H_*(G/B)$. In particular, $c_{SM}(X(w)^\circ) = \mathcal{T}_{w^{-1}}([pt])$.

Combining the actions of ∂_k and s_k on Schubert classes found in the identities (2) and (3) from §2.2 we obtain the following explicit formula for \mathcal{T}_k :

Proposition 4.3.

$$\mathcal{T}_k([X(w)]) = \begin{cases} -[X(w)] & \text{if } \ell(ws_k) < \ell(w) \\ [X(ws_k)] + [X(w)] + \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)] & \text{if } \ell(ws_k) > \ell(w) \end{cases}$$

where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(w) = \ell(ws_k s_\beta)$.

Example 4.4. Using Corollary 4.2 and Proposition 4.3 it is straightforward to implement computations of CSM classes of Schubert cells in symbolic manipulation packages such as Maple. For instance, we obtain that the CSM class for the open cell in the flag manifold $\text{Fl}(4)$ (in type A) is:

$$\begin{aligned} c_{SM}(X(4321)^\circ) &= [X(4321)] + [X(4312)] + [X(4231)] + [X(3421)] + 2[X(4213)] + 2[X(4132)] \\ &\quad + [X(3412)] + 2[X(3241)] + 2[X(2431)] + [X(4123)] + 5[X(3214)] + 5[X(3142)] \\ &\quad + 3[X(2413)] + [X(2341)] + 5[X(1432)] + 3[X(3124)] + 4[X(2314)] + 6[X(2143)] \\ &\quad + 4[X(1423)] + 3[X(1342)] + 3[X(2134)] + 4[X(1324)] + 3[X(1243)] + [X(1234)] \end{aligned}$$

where we use the standard identification of the elements of W with permutations in indexing the $4!$ Schubert classes.

Note that the terms corresponding to the ‘Grassmannian permutations’ $(a_1 a_2 b_1 b_2)$ with $a_1 < a_2$ and $b_1 < b_2$ are

$$[X(3412)] + 3[X(2413)] + 4[X(1423)] + 4[X(2314)] + 4[X(1324)] + [X(1234)]$$

and push-forward as prescribed by identity (11) in §3.3 to the CSM class for the open cell in $G(2, 4)$ (cf. the row corresponding to $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ in [AM09, Example 1.2]).

Remark 4.5. Even if $\ell(ws_k) > \ell(w)$, $\mathcal{T}_k([X(w)])$ is in general not a positive combination of Schubert classes. For example, let $G = \text{SL}_4(\mathbb{C})$, and let $w = w_0 s_3$, where w_0 is the longest element in $W = S_4$, the symmetric group with 4 letters. Using again the standard identification of the elements of W with permutations, so that $w_0 = (4321)$ and $s_3 = (1243)$, then $w = (4312)$ and

$$\mathcal{T}_3([X(4312)]) = [X(4312)] + [X(4321)] - [X(4231)].$$

Nevertheless, substantial evidence suggests that the classes $\mathcal{T}_k(\gamma)$, and hence all classes $\mathcal{T}_w(\gamma)$, are positive linear combinations of Schubert classes if γ is a positive combination of CSM classes $c_{SM}(X(u)^\circ)$; see §5.

5. POSITIVITY OF CSM CLASSES

Fix $w \in W$ and consider the CSM class $c_{\text{SM}}(X(w)^\circ)$. As we have shown, if $s_{i_1} \cdots s_{i_k}$ is a reduced decomposition for w , then

$$(13) \quad c_{\text{SM}}(X(w)^\circ) = \theta_*(c_{\text{SM}}(Z^\circ)) = \theta_* \left(\prod_{j=1}^k (1 + h_j) \cdot [Z] \right)$$

where $\theta : Z := Z_{i_1, \dots, i_k} \rightarrow X(W)$ is the Bott-Samelson resolution (Lemma 3.1). We have also shown that

$$c_{\text{SM}}(X(w)^\circ) = \mathcal{T}_{i_k} \cdots \mathcal{T}_{i_1}([pt])$$

(Corollary 3.4). Since $c_{\text{SM}}(X(w)^\circ) \in H_*(X(w))$, we have

$$(14) \quad c_{\text{SM}}(X(w)^\circ) = \sum_{u \leq w} c(w; u)[X(u)]$$

where $c(u; w)$ are well-defined integers. In fact, $c(w; w) = 1$ since the map θ is birational, and $c(w; \text{id}) = 1$ since $X(w)^\circ \cong \mathbb{A}^{\ell(w)}$ and $\chi(\mathbb{A}^{\ell(w)}) = 1$.

The operator \mathcal{T}_k does not preserve positivity: $\mathcal{T}_k([X(s_k)^\circ]) = -[X(s_k)^\circ]$ by Proposition 4.3, and in fact $\mathcal{T}_k([X(w)^\circ])$ may have negative contributions from Schubert classes even if $\ell(ws_k) > \ell(w)$ (Remark 4.5). Examples also show that $c_{\text{SM}}(Z^\circ)$ is not necessarily a positive combination of strata of the normal crossing divisor $D_Z := Z \setminus Z^\circ$ (Z_{12321}° is the smallest such example). So one should not expect any positivity properties of the CSM class *a priori*. Nevertheless, we conjecture that these classes *are* positive:

Conjecture 1. *For all $u \leq w$, the coefficient $c(w; u)$ from the expansion (14) is strictly positive.*

Note that with notation as above, the class of the Schubert variety $X(w)$ is given by

$$(15) \quad c_{\text{SM}}(X(w)) = \sum_{u \leq w} \left(\sum_{u \leq v \leq w} c(v; u) \right) [X(u)];$$

indeed, $\mathbb{1}_{X(w)} = \sum_{v \leq w} \mathbb{1}_{X(v)^\circ}$. So Conjecture 1 would imply that these classes are also necessarily effective.

A positivity result analogous to Conjecture 1 was conjectured by the authors in [AM09] for Schubert cells in the Grassmannian $\text{Gr}(p, n)$ of subspaces of dimension p in \mathbb{C}^n . This conjecture was proved in [AM09] in the case $p = 2$, in [Mih15] for $p = 3$, and several classes of coefficients were proved to be positive by B. Jones [Jon10] and J. Stryker [Str11]. The full conjecture has recently been proven by June Huh [Huh]. By Proposition 3.5, the CSM classes of Schubert cells in any homogeneous space G/P are in fact push-forwards of CSM classes of Schubert cells in G/B ; therefore Conjecture 1 would simultaneously imply the positivity of all CSM classes of Schubert cells in all G/P , and in particular it would yield an alternative proof of Huh's theorem.

By the same token, Huh's theorem provides some evidence for Conjecture 1, since it implies that $c(w; u) > 0$ in type A when u and w are both Grassmannian permutations (cf. Example 4.4). In fact, Conjecture 1 in type A is also supported by explicit computations of several thousand cases. At the time of this writing, we have verified that the CSM classes of all Schubert cells $\text{Fl}(n)$ are positive for $n \leq 6$; for all words of length ≤ 17 in $\text{Fl}(7)$; and for all words of length ≤ 14 in $\text{Fl}(8)$.

In the rest of this section we discuss more evidence for Conjecture 1 in all types. We prove positivity in the following cases:

- $c(w; u) > 0$ if $u < w$ and $\ell(w) - \ell(u) = 1$ (Corollary 5.2);
- $c(w; u) > 0$ for all $u \leq w$ if w admits a decomposition into distinct simple reflections (Corollary 5.4).

These two results will follow from more general considerations which seem independently interesting: the first one is an explicit computation of the codimension 1 term in the CSM class of a Schubert cell (Proposition 5.1), and the second one highlights one case in which the operator \mathcal{T}_k does preserve positivity (Proposition 5.3).

Proposition 5.1. *Let $\rho = \omega_1 + \dots + \omega_r$ be the sum of the fundamental weights, and let $w \in W$. Then*

$$c_{SM}(X(w)^\circ) = [X(w)] + c_1(\mathcal{L}_\rho) \cdot [X(w)] + \text{lower dimensional terms.}$$

Proof. Let $w = s_{i_1} \dots s_{i_k}$ be a reduced decomposition, and let $Z := Z_{i_1, \dots, i_k}$ with SNC divisor D_Z , as in §2.3. By [BK05, Prop. 2.2.2],

$$K_Z = \mathcal{O}_Z(-D_Z) \otimes \theta^*(c_1(\mathcal{L}_{-\rho}))$$

and hence $c_1(T_Z) = [D_Z] + \theta^*(c_1(\mathcal{L}_\rho))$. On the other hand, $c_1(T_Z) = [D_Z] + \sum_{i=1}^k h_i$ by Proposition 2.4 (c). Therefore $h_1 + \dots + h_k = \theta^*(c_1(\mathcal{L}_\rho))$, and the stated identity follows from (13) and the projection formula. \square

Corollary 5.2. *The coefficient $c(w; u) > 0$ if $u < w$ with $\ell(u) = \ell(w) - 1$.*

Proof. Recall that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ (the Kronecker symbol) and in particular $\langle \rho, \alpha_j^\vee \rangle > 0$ for all simple roots α_j . Consider $u < w$ such that $\ell(u) = \ell(w) - 1$. Then $u = ws_\beta$ for some positive root $\beta \in R^+$ (see e.g. [Hum90, §5.11]). By the Chevalley formula (1), the coefficient of $[X(ws_\beta)]$ in $c_1(\mathcal{L}_\rho) \cap [X(w)]$ equals $\langle \rho, \beta^\vee \rangle > 0$, concluding the proof. \square

Proposition 5.3. *Let $w \in W$ be a Weyl group element, and assume w admits a decomposition into simple reflections other than s_k .*

- (a) *The homology class $T_k([X(v)])$ is a non-negative linear combination of Schubert classes $[X(u)]$ with $u \leq vs_k$. In fact,*

$$\mathcal{T}_k([X(v)]) = [X(vs_k)] + [X(v)] + \sum_{u < vs_k, u \neq v} d_k(v; u)[X(u)]$$

with $d_k(v; u) \geq 0$ for all $u < vs_k$.

- (b) *Assume in addition that s_k commutes with all simple reflections in a decomposition of v . Then $d_k(v; u) = 0$ for $u < vs_k$, $u \neq v$, that is:*

$$\mathcal{T}_k([X(v)]) = [X(vs_k)] + [X(v)].$$

Proof. Let $S_v := \{s_{i_1}, \dots, s_{i_t}\}$ be the set of reflections appearing in a reduced decomposition of v ; this set is independent of the choice of reduced decomposition, since it is preserved by the braid relations in W (see e.g., [Hum90, §5.1]). Since every decomposition of v into simple reflections can be reduced to a reduced decomposition, the hypothesis of the proposition implies that $s_k \notin S_v$ in part (a), and that further s_k commutes with all $s_{i_j} \in S$ in part (b).

Since $vs_k > v$, by Proposition 4.3 we have

$$(16) \quad \mathcal{T}_k([X(v)]) = [X(vs_k)] + [X(v)] + \sum \langle \alpha_k, \beta^\vee \rangle [X(vs_k s_\beta)]$$

where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(v) = \ell(vs_k s_\beta)$. We have to prove that, under the hypothesis of the proposition, $\langle \alpha_k, \beta^\vee \rangle \geq 0$ for all β in the range of summation. In fact, all the β in this range satisfy $vs_k s_\beta < vs_k$, and we will verify that

$\langle \alpha_k, \beta^\vee \rangle \geq 0$ for all such reflections β . By [Hum90, §5.7] the condition $vs_k > v$ implies that $v(\alpha_k) > 0$, and $vs_k s_\beta < vs_k$ implies that $vs_k(\beta) < 0$, i.e.,

$$(17) \quad v(\beta - \langle \beta, \alpha_k^\vee \rangle \alpha_k) = v(\beta) - \langle \beta, \alpha_k^\vee \rangle v(\alpha_k) < 0.$$

If $v(\beta) > 0$, then we are done, because $v(\alpha_k) > 0$. So we assume $v(\beta) < 0$, which is equivalent to $vs_\beta < v$, and it follows that vs_β admits a reduced expression only using reflections in S_v . We deduce that s_k does not appear in a reduced expression for s_β , and hence that the simple root α_k does not appear in the support of the positive root β . Since $s_{\alpha_k} \notin S_v$, it follows that α_k does not appear in the support of $v(\beta)$; and α_k appears with coefficient $+1$ in $v(\alpha_k)$. Then (17) forces $\langle \beta, \alpha_k^\vee \rangle \geq 0$ as claimed. This proves part (a).

To prove part (b) we use a similar argument. By (16), it suffices to show that $\langle \alpha_k, \beta^\vee \rangle = 0$ for all reflections s_β such that $vs_k s_\beta < vs_k$. This relation implies that s_β has a reduced decomposition containing only simple reflections in an expression for vs_k . But $\beta \neq \alpha_k$, and no simple reflections s_{α_j} with α_j adjacent to α_k in the Dynkin diagram for G can appear in the decomposition of s_β : otherwise such reflections would appear in S_v , contradicting the commutativity hypothesis. This implies that the support of β does not contain any simple root adjacent to α_k , thus $\langle \alpha_k, \beta^\vee \rangle = 0$, concluding the proof. \square

Corollary 5.4. *Let $w \in W$ be a Weyl group element, and assume w admits a decomposition into simple reflections other than s_k . If $c(w; v) > 0$ for all $v \leq w$, then $c(ws_k; u) > 0$ for all $u \leq ws_k$.*

In particular, if $w \in W$ admits a decomposition into distinct simple reflections, then $c(w; u) > 0$ for all $u \leq w$.

Proof. The second statement follows from the first by an immediate induction. To prove the first statement, note that if s_k does not appear in a decomposition for w , then it does not appear in a reduced decomposition for w , and hence it does not appear in a decomposition for v . Thus the hypothesis of Proposition 5.3 applies to all $v \leq w$. By Corollary 3.4 we have

$$\begin{aligned} c_{\text{SM}}(X(ws_k)^\circ) &= \mathcal{T}_k(c_{\text{SM}}(w)^\circ) = \mathcal{T}_k \sum_{v \leq w} c(w; v) [X(v)] \\ &= \sum_{v \leq w} c(w; v) \left([X(vs_k)] + [X(v)] + \sum_{u' < vs_k, u' \neq v} d_k(v; u') [X(u')] \right) \end{aligned}$$

with $c(w; v) > 0$ by hypothesis and $d_k(v; u') \geq 0$ by Proposition 5.3. The statement is immediate from this expression, since $u \leq ws_k$ implies that u has a reduced expression which is a subexpression of one for ws_k , thus either $u = v \leq w$ or $u = vs_k$ with $v \leq w$. \square

A particular case of Corollary 5.4 is particularly vivid: if $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition and the simple reflections s_{i_1}, \dots, s_{i_k} commute with one another, then an induction argument based on Proposition 5.3(b) implies that

$$(18) \quad c_{\text{SM}}(X(w)^\circ) = \sum_{u \leq w} [X(u)].$$

Notice that if w satisfies this condition, then so does every v preceding it in the Bruhat order. Then (15) and (18) give the CSM class of the Schubert variety

$$c_{\text{SM}}(X(w)) = \sum_{u \leq w} 2^{\ell(w) - \ell(u)} [X(u)],$$

for every $w \in W$ decomposing into commuting simple reflections.

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