BEILINSON-BERNSTEIN LOCALIZATION OVER THE HARISH-CHANDRA CENTER

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Abstract. We present a simple proof of a strengthening of the derived Beilinson-Bernstein localization theorem using the formalism of descent in derived algebraic geometry. The arguments and results apply to arbitrary modules without the need to fix infinitesimal character. Roughly speaking, we demonstrate that all $\mathfrak{U}\mathfrak{g}$-modules are the invariants, or equivalently coinvariants, of the action of intertwining functors (a refined form of Weyl group symmetry). This is a quantum version of descent for the Grothendieck-Springer simultaneous resolution.

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1. Introduction

The Beilinson-Bernstein localization theorem is perhaps the quintessential result in geometric representation theory.

Let us recall the statement. Let $\mathcal{P}_\lambda = \mathcal{D}_\lambda(G/B)$ denote the ∞-category of λ-twisted $\mathcal{D}$-modules on the flag variety. More precisely, we consider $\mathcal{D}$-modules on G/N which are weakly H equivariant and have generalized monodromy λ along the torus fibers. (The notation $\mathcal{P}_\lambda$ is motivated by the analogy with principal series representations of real or p-adic groups.) Similarly we let $\mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda]}$ denote the ∞-category of $\mathfrak{U}\mathfrak{g}$-modules with generalized central character the W-orbit $[\lambda] \in \mathfrak{h}^*/W = \operatorname{Spec} \mathcal{Z}(\mathfrak{U}\mathfrak{g})$.

Theorem 1.1. *For $\lambda \in \mathfrak{h}^*$ regular, the global sections and localization provide inverse equivalences of derived categories*

$$\Delta : \mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda]} \leftrightarrow \mathcal{P}_\lambda$$

Remark 1.2 (Abelian vs. derived equivalence). The theorem further bounds the cohomological amplitude of Γ by the length of a Weyl group element w for which $w \cdot \lambda$ dominant, so that in particular for λ dominant we have an equivalence of the corresponding abelian categories. In this paper we will only consider the derived form of Beilinson-Bernstein localization, not keeping track of its effect on t-structures.

Despite its importance, which is hard to overestimate, one can identify three drawbacks of Theorem 1.1. First, the theorem applies to category of representations with a *fixed* infinitesimal character. This makes it difficult to apply to questions of harmonic analysis, which typically involve the geometry or topology of families of representations (for example the Plancherel formula and Baum-Connes conjecture). Second, it is sometimes inconvenient that to localize representations, which depend on a parameter $[\lambda] \in \mathfrak{h}^*/W$, we must choose a lift $\lambda \in \mathfrak{h}^*$. (Note that the geometry of the corresponding $\mathcal{D}$-modules – for example, the dimension of their support – depends on this choice.) Finally, the theorem does not apply as stated to singular infinitesimal characters, where the category of $\mathcal{D}$-modules is larger than the corresponding category of representations, though one can correct this problem [K, BMR1, BK] using localization on *partial* flag varieties.

In this paper we describe a natural refinement of the derived Beilinson-Bernstein localization theorem which simultaneously corrects these three drawbacks by invoking descent. A naive paraphrase of the result asserts "$\mathfrak{U}\mathfrak{g}$-mod consists of the trivial Weyl group isotypic component in monodromic $\mathcal{D}$-modules on G/N". We first present the result and then explain the paraphrase and a "classical limit" for quasicoherent sheaves on the Grothendieck-Springer resolution.

1.0.1. *Beilinson-Bernstein via Barr-Beck.* The natural setting in which to consider the localization problem for all infinitesimal characters at once is provided (see e.g. [B]) by the category $\mathcal{P} = \mathcal{D}_H(G/N)$ of weakly H-equivariant $\mathcal{D}$-modules on the "basic affine space" G/N, an H-bundle over the flag manifold (in other words we consider $\mathcal{D}$-modules which are locally constant along the torus fibers but with unspecified monodromy). The notation derives from the interpretation as a categorical variant of the universal principal series representation of a real group (or the universal unramified principal series representation of a p-adic group).

The action of $\mathfrak{U}\mathfrak{g}$ by differential operators on G/N gives rise to an adjunction

$$\Delta : \mathfrak{U}\mathfrak{g}\text{-mod} \leftrightarrow \mathcal{P} : \Gamma.$$

In fact this is naturally an *ambidextrous* adjunction: Δ is canonically identified with the *right* adjoint of localization as well. This is a reflection of the Calabi-Yau structure of the category $\mathcal{P}$, which is the key ingredient in the approach of [BMR1, BMR2] to establishing Beilinson-Bernstein equivalences (following an argument of [BKR] in the setting of the McKay correspondence, and extended to localization for quantum symplectic resolutions in [MN]). The composition $\mathcal{W} = \Delta \circ \Gamma$ thus acquires the structures of a monad and a comonad (in fact a Frobenius monad) - an algebra (respectively coalgebra and Frobenius algebra) object in endofunctors of $\mathcal{P}$. A crucial feature of Δ is its conservativity - no objects localize to zero (an easy consequence of the localization formalism). The Barr-Beck-Lurie theorem now guarantees that we can describe $\mathfrak{U}\mathfrak{g}$-mod as (co)modules for $\mathcal{W}$ in $\mathcal{P}$:

Theorem 1.3 (monadic Beilinson-Bernstein localization)**.**
(1) *The Weyl functor $\mathcal{W} = \Delta \circ \Gamma$ carries a canonical structure of Frobenius monad on $\mathcal{P}$.*
(2) *There are canonical equivalences*

$$\mathcal{P}_{\mathcal{W}} \simeq \mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathcal{P}^{\mathcal{W}}$$

between the categories of $\mathcal{W}$-modules and $\mathcal{W}$-comodules in $\mathcal{P}$ and the category of $\mathfrak{U}\mathfrak{g}$-modules.

1.0.2. *Beilinson-Bernstein via Hecke symmetry.* In order to interpret the formal statement 1.3 we need to identify the Weyl functor concretely via the symmetries of $\mathcal{P}$. The category $\mathcal{P}$ carries two fundamental commuting actions. First there's the action of the group G on the left. This action is canonically trivialized infinitesimally (i.e. the action of the formal group of G is trivialized), making $\mathcal{P}$ a *smooth G-category*, or equivalently, defines on $\mathcal{P}$ the structure of module category for the "smooth group algebra", the monoidal ∞-category $\mathcal{D}(G)$.

On the other hand there's the action of the Hecke category $\mathcal{H} = \mathcal{D}_{H\times H}(N\backslash G/N)$ on the right by intertwining functors, i.e., commuting with the action of G. $\mathcal{H}$ is the varying-monodromy version of the finite Hecke category $\mathcal{D}(B\backslash G/B)$, a categorical form of the finite Hecke algebra or Artin braid group of fundamental importance to Kazhdan-Lusztig theory (abelian variants of this category include the BGG Category $\mathcal{O}$ and the categories of Harish-Chandra bimodules and projective functors). In fact we prove that the Hecke category consists precisely of G-symmetries of $\mathcal{P}$:

Theorem 1.4. *There is a monoidal equivalence*

$$\mathcal{H} \simeq End_{\mathcal{D}(G)}(\mathcal{P}).$$

The group G also acts in a "smooth" (i.e. infinitesimally trivialized) fashion on the category $\mathfrak{U}\mathfrak{g}$-mod (via the adjoint action on $\mathfrak{g}$), and as was emphasized in [BD] (see also [FG]) the localization and global sections functors commute with these structures. Thus we expect the Weyl (co)monad $\mathcal{W}$ to come from a (co)algebra object in $\mathcal{H}$ (which can be easily seen independently of Theorem 1.4):

Definition 1.5. The *universal Weyl sheaf* $\mathcal{W} \in \mathcal{H}$ is the sheaf of differential operators on $N\backslash G/N$ with its canonical H-biequivariant structure.

Theorem 1.6 (Beilinson-Bernstein localization, Hecke version)**.**
(1) *The universal Weyl sheaf $\mathcal{W}$ carries a canonical structure of Frobenius algebra in $\mathcal{H}$, the image of the Weyl Frobenius monad $\mathcal{W}$ under the identification* $\mathrm{End}_{\mathcal{D}(G)}(\mathcal{P}) \simeq \mathcal{H}$.
(2) *There are canonical equivalences*

$$\mathcal{P}_{\mathcal{W}} \simeq \mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathcal{P}^{\mathcal{W}}$$

between the categories of $\mathcal{W}$-modules and $\mathcal{W}$-comodules in $\mathcal{P}$ and the category of $\mathfrak{U}\mathfrak{g}$-modules.

In Section 3.3 we spell out what this construction means for each infinitesimal character $[\lambda]$.

For generic λ, the dependence of Beilinson-Bernstein localization on choice of lift λ of regular infinitesimal characters $[\lambda]$ is well known to be governed by intertwining functors. Namely the Weyl group acts on the direct sum of the categories

$$\mathcal{D}_{[\lambda]}(G/B) := \bigoplus_{W} \mathcal{D}_{w\cdot\lambda}(G/B)$$

by functors, which are categorical analogs of the classical intertwining operators for principal series representations. Indeed for generic λ the Hecke category is very small, and there is only a single sheaf (standard=costandard) attached to any Schubert cell, and the corresponding intertwining functors satisfy the Weyl group relation. , Thus we may identify $\mathfrak{U}\mathfrak{g}\text{-mod}_{[\lambda]}$ with W-equivariant collections of $\mathcal{D}$-modules with W-related twists. In other words, for generic λ the Weyl sheaf $\mathcal{W}$ specializes to the group algebra of the Weyl group, acting on $\mathcal{P}_{[\lambda]}$ lifting the action on the λ-plane $\mathfrak{h}^*$.

As λ specializes, the geometry of the Weyl sheaf becomes more interesting. For λ regular integral, $\mathcal{W}$ still realized a Weyl group action, now through specified convolutions of standard and costandard sheaves on Schubert cells.

In the most singular case $\lambda = 0$, the sheaf $\mathcal{W}$ specializes to the maximal tilting sheaf, or projective cover of the skyscraper, an object that plays a central role in the work of Soergel [S]. Similar statements hold along other walls. In these cases the Weyl sheaf plays the role of idempotent, realizing the smaller category of $U_\lambda\mathfrak{g}$-modules as a summand inside $\mathcal{D}_\lambda(G/B)$, recovering a variant of the singular Beilinson-Bernstein localization as in [BK].

1.1. **Interpretation.** We would like to interpret Theorem 1.6 as a refined version of taking Weyl group invariants on $\mathcal{P}$.

First let us recall the Weyl character formula and Borel-Weil-Bott theorem in a K-theoretic form which we learned from [BH] (where it is refined to a result on equivariant KK-theory closely connected to the Baum-Connes conjecture for Lie groups). We can relate the representation ring of the compact group G_c and the equivariant K-theory of the flag manifold via maps

$$\Delta : K^*_{G_c}(pt) \leftrightarrow K^*_{G_c}(G/B) : \Gamma$$

where Γ is the equivariant index (Borel-Weil-Bott construction) and Δ is given by pullback followed by multiplication by the virtual bundle $\Omega^{\cdot}$. Note that for a representation $V \in K^*_{G_c}(pt)$, the virtual bundle $\Delta(V)$ can be identified with the complex that fiberwise computes $\mathfrak{n}$-homology (where $\mathfrak{n}$ is the radical of the Lie algebra stabilizer of a point in G/B), in other words with the K-theory image of the Beilinson-Bernstein localization of V.

The Weyl group acts from the right (nonholomorphically) on $G/B = G_c/T$, and the theorem asserts that $\Gamma \circ \Delta = |W| Id$ and, when restricted to the W-invariants[1], we also have $\Delta \circ \Gamma = |W| \operatorname{Id}$. Identifying the virtual character of $\Omega^{\cdot}$ with the Weyl denominator we recover the Weyl character formula. In fact Block and Higson prove (in the more refined setting of equivariant KK-theory) that the composition $\Delta \circ \Gamma$ is given by the standard idempotent in the group algebra of W (the projector to the trivial representation), realized as a sum of standard intertwining operators. Note that the group algebra $\mathbf{C}W$ is a Frobenius algebra, and so W invariants and coinvariants are both identified with the summand, image of the standard idempotent.

In our present categorical setting, the action of the Weyl group on K-theory is replaced by the action of the Hecke category $\mathcal{H} = \mathcal{D}_{H \times H}(N\backslash G/N)$ on $\mathcal{P}$. For integral λ the corresponding Hecke category $\mathcal{H}_\lambda = \mathcal{D}_\lambda(B\backslash G/B)$ of Kazhdan-Lusztig theory has Grothendieck group $\mathbf{C}W$ and in fact gives an action not of W but of the corresponding Artin braid group on categories of representations. Moreover a result of [BN] asserts that for any λ, the Hecke category $\mathcal{H}_\lambda = \mathcal{D}_\lambda(B\backslash G/B)$ is the categorified analog of a finite dimensional semisimple Frobenius algebra, i.e., it is the value on a point of an extended oriented two-dimensional topological field theory. (More precisely it is a two-dualizable Calabi-Yau algebra object in stable ∞-categories.)

The Frobenius monad $\mathcal{W}$ is the categorified analog of the standard idempotent, with $\mathcal{W}$-modules playing the role of W-invariants and $\mathcal{W}$-comodules, that of W-coinvariants. For fixed regular λ, the argument of [BMR1] utilizes this Frobenius (or Calabi-Yau) property and the indecomposability of $\mathcal{D}_\lambda(G/B)$ to deduce the Beilinson-Bernstein equivalence. For singular λ or in families, the indecomposability fails, but we recover $\mathfrak{U}\mathfrak{g} = \mathcal{P}^{\mathcal{W}}$ as a summand (isotypic component) of $\mathcal{P}$.

1.2. **Classical version.** The families version of Beilinson-Bernstein has a natural classical analog for quasicoherent sheaves on the Springer resolution, where its interpretation as a form of proper descent becomes evident.

First let $\pi : X \to Y$ denote any morphism of perfect stacks (for example of schemes), and $X \times_Y X$ the fiber product. Then $\mathcal{H} = \mathcal{Q}(X \times_Y X)$ is a monoidal category with respect to convolution, and we have a monoidal equivalence

$$\mathcal{Q}(X \times_Y X) \simeq \operatorname{End}_{\mathcal{Q}(Y)}(\mathcal{Q}(X)).$$

The adjunction (π^*, π_*) defines a comonad $T^\vee = \pi^*\pi_*$ on $\mathcal{Q}(X)$, and it is easy to see that this comonad is represented by the coalgebra object $\mathcal{A} = \mathcal{O}_{X \times_Y X} \in \mathcal{H}$. Namely, $\mathcal{A}$ is simply the groupoid coalgebra (functions on the groupoid with convolution coproduct) for the descent groupoid $X \times_Y X$ acting on X, enforcing invariance along the fibers of π. If π is faithfully flat, then by [L4, 7] descent holds so that

$$\mathcal{Q}(Y) \simeq \mathcal{Q}(X)^{T^\vee} = \mathcal{Q}(X)^{\mathcal{A}}.$$

On the other hand let's suppose that π is proper and surjective on field points. In this case the adjunction

$$\pi_* : \mathcal{Q}^!(X) \leftrightarrow \mathcal{Q}^!(Y) : \pi^!$$

on ind-coherent sheaves gives rise to a monad $T = \pi^!\pi_*$ on $\mathcal{Q}^!(X)$. We have a canonical action map

$$\mathcal{Q}^!(X \times_Y X) \longrightarrow \operatorname{End}_{\mathcal{Q}^!(Y)}(\mathcal{Q}^!(X)),$$

which is not typically an equivalence. However the monad T is realized by an algebra object

$$\mathcal{A}^! = \omega_{X \times_Y X/X} \in \mathcal{Q}^!(X \times_Y X),$$

[1]get sign right

the groupoid algebra (relative volume forms on the groupoid with convolution product) for the descent groupoid. Moreover the proper descent theorem of [P, Proposition A.2.8] and [G1, 7.2.2] implies that

$$\mathcal{Q}^!(Y) \simeq \mathcal{Q}^!(X)_T = \mathcal{Q}^!(X)_{\mathcal{A}^!}.$$

Let us now assume that X and Y are smooth, so that we have canonical identifications $\mathcal{Q}(X) \simeq \mathcal{Q}^!(X)$ and $\mathcal{Q}(Y) \simeq \mathcal{Q}^!(Y)$. Moreover assume that π is *crepant* or Calabi-Yau of dimension zero, i.e., that we are given a trivialization of the relative dualizing sheaf and thus an identification $\pi^* \simeq \pi^!$. We then find that T is a Frobenius monad, or that the groupoid algebra $\mathcal{O}_{X\times_Y X}$ is a Frobenius algebra object, and we have equivalences

$$\mathcal{Q}(X)^T \simeq \mathcal{Q}(Y) \simeq \mathcal{Q}(X)_T.$$

The prime example for these restrictive hypotheses is the Grothendieck-Springer simultaneous resolution $\pi : X = \tilde{\mathfrak{g}} \to Y = \mathfrak{g}$, with $X \times_Y X$ the Steinberg variety. In this case the above descent picture is precisely the classical limit of Beilinson-Bernstein localization. We describe the "classical Weyl sheaf" $\mathcal{O}_{X\times_Y X}$ and corresponding monadic picture explicitly in Section 2, in particular detailing the various specializations of the descent picture over different regions in the adjoint quotient.

We include an appendix describing (under the same general hypotheses) a Morita equivalence between the monoidal ∞-categories $\mathcal{Q}(Y)$ and $\mathcal{Q}(X \times_Y X)$, which is a categorified form of descent.

1.3. **Categorical Context.** We will work throughout in the language of derived algebraic geometry, following [L1, L2]; we refer the reader to [BFN, BN] for some gentle discussion of the context and tools.

We will work throughout over $k = \mathbb{C}$, or more generally a commutative dg $\mathbb{C}$-algebra. The word "category" stands in for either a k-linear pre-triangulated dg category or a k-linear stable ∞-category. Such categories fit into two related contexts, the symmetric monoidal ∞-categories St_k and st_k of stable presentable k-linear ∞-categories with morphisms continuous functors, and of small stable idempotent-complete k-linear ∞-categories with morphisms exact functors. Here we say a functor $f : \mathcal{C} \to \mathcal{C}'$ between stable ∞-categories is *continuous* if it preserves coproducts, *proper*[2] if it preserves compact objects, and *exact* if it preserves zero objects and finite colimits.

Taking ind-objects defines a faithful symmetric monoidal functor $Ind : \mathrm{st}_k \to \mathrm{St}_k$, with left inverse (on the subcategory of proper functors) given by passing to compact objects. Any $\mathcal{C} \in \mathrm{st}_k$ is dualizable, with dual the opposite category $\mathcal{C}^{op}$, so that the dual of $Ind\mathcal{C} \in \mathrm{St}_k$ is the restricted opposite category $(\mathrm{Ind}\,\mathcal{C})^\vee = \mathrm{Ind}(\mathcal{C}^{op})$.

We will make repeated use the theory of monads and of both monadic and comonadic forms of the Barr-Beck-Lurie theorem [L2]. For a monad T (comonad $T^\vee$) on a category $\mathcal{C}$ we denote by $\mathcal{C}_T$ ($\mathcal{C}^{T^\vee}$) the corresponding categories of modules (comodules).

Remark 1.7 (Frobenius monads). The adjunctions appearing in this paper are *ambidextrous*, i.e. we are given an identification of left and right adjoints, so that we have a single endofunctor with compatible algebra (monad) and coalgebra (comonad) structures. We interpret the result informally as giving a *Frobenius monad*, a Frobenius algebra object in endofunctors, though we do not formalize this notion in the ∞-categorical setting in this paper – see [St, La] for the discrete setting. A natural context in which to identify the appropriate ∞-categorical structures is that of the cobordism hypothesis with singularities [L3], where the notion of ambidextrous adjunction or Frobenius monad captures an *oriented* domain wall between topological field theories (as explained pictorially in [La] in the discrete setting).[3]

[2]or more properly, *quasi-proper*

[3]too vague?

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2. Grothendieck-Springer resolution

2.1. **Recollections.** We recall here the construction of the Grothendieck-Springer resolution of a reductive Lie algebra and the Steinberg variety.

Let G be a complex reductive group. For a Borel subgroup $B \subset G$, let $N \subset B$ denote its unipotent radical, and $H = B/N$ the universal Cartan torus. Denote by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$, and $\mathfrak{h}$ the respective Lie algebras. Let W denote the Weyl group of $\mathfrak{g}$, $\mathfrak{c} = \mathfrak{h}/\!/W$ the affine quotient, and $\mathcal{N} = \mathfrak{g} \times_{\mathfrak{c}} \{0\} \subset \mathfrak{g}$ the nilpotent cone. Fix a G-invariant inner product on $\mathfrak{g}$ to obtain an identification $\mathfrak{g}^* \simeq \mathfrak{g}$.

Let $\mathcal{B} = G/B$ be the flag variety, and $\widetilde{\mathcal{B}} = G/N$ the base affine space. The natural projection $\widetilde{\mathcal{B}} \to \mathcal{B}$ is a G-equivariant torsor for the natural H-action on $\widetilde{\mathcal{B}}$. Such torsors correspond to homomorphisms $B \to H$, and the base affine space $\widetilde{\mathcal{B}} \to \mathcal{B}$ corresponds to the tautological homomorphism $B \to B/N \simeq H$.

The cotangent bundle $T^*\mathcal{B} \to \mathcal{B}$ classifies pairs of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ together with an element $v \in (\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{n}$. The moment map for the natural G-action is given by the projection

$$\mu_{\mathcal{B}} : T^*\mathcal{B} \longrightarrow \mathfrak{g}^* \simeq \mathfrak{g} \qquad \mu_{\mathcal{B}}(\mathfrak{b}, v) = v$$

The cotangent bundle $T^*\widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ classifies pairs of an element $x_{\mathfrak{b}} \in \widetilde{\mathcal{B}}$ over a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ together with an element $v \in (\mathfrak{g}/\mathfrak{n})^* \simeq \mathfrak{b}$. The moment map for the natural $G \times H$-action is given by the projection

$$\mu_{\widetilde{\mathcal{B}}} : T^*\widetilde{\mathcal{B}} \longrightarrow \mathfrak{g}^* \times \mathfrak{h}^* \simeq \mathfrak{g} \times \mathfrak{h} \qquad \mu_{\widetilde{\mathcal{B}}}(x_{\mathfrak{b}}, v) = (v, [v])$$

where $[v] \in \mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ denotes the image of $v \in \mathfrak{b}$.

The cotangent bundles are related by Hamiltonian reduction along the H-action

$$T^*\mathcal{B} = T^*(\widetilde{\mathcal{B}}/H) \simeq (p_{\mathfrak{h}} \circ \mu_{\widetilde{\mathcal{B}}})^{-1}(0)/H$$

where $p_{\mathfrak{h}} : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ denotes projection.

We will be interested in the quotient $\widetilde{\mathfrak{g}} = (T^*\widetilde{\mathcal{B}})/H$ classifying a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ together with an element $v \in (\mathfrak{g}/\mathfrak{n})^* \simeq \mathfrak{b}$. The moment map for the G-action on $T^*\widetilde{\mathcal{B}}$ descends to the Grothendieck-Springer resolution

$$\mu_{\widetilde{\mathfrak{g}}} : \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \qquad \mu_{\widetilde{\mathfrak{g}}}(\mathfrak{b}, v) = v$$

The Grothendieck-Springer resolution $\mu_{\widetilde{\mathfrak{g}}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is projective, generically finite and G-equivariant. Moreover its relative dualizing sheaf is canonically trivial (and hence the same is true of any base change of $\mu_{\widetilde{\mathfrak{g}}}$). To see this last claim, recall we have fixed a G-invariant inner product on $\mathfrak{g}$ to obtain a G-equivariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$. This in turn induces an isomorphism of lines $\wedge^{\dim \mathfrak{g}}\mathfrak{g} \simeq \wedge^{\dim \mathfrak{h}}\mathfrak{h}$. Thus a trivialization of $\wedge^{\dim \mathfrak{h}}\mathfrak{h}$ trivializes the canonical bundle of $\mathfrak{g}$. Furthermore, the partial moment map $\widetilde{\mathfrak{g}} \to \mathfrak{h}$ is smooth with symplectic fibers, hence a trivialization of $\wedge^{\dim \mathfrak{h}}\mathfrak{h}$ also trivializes the canonical bundle of $\widetilde{\mathfrak{g}}$.

2.1.1. *The Steinberg variety.* The Grothendieck-Steinberg variety is the fiber product $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ classifying triples of a pair of Borel subalgebras $\mathfrak{b}_1, \mathfrak{b}_2 \subset \mathfrak{g}$ together with an element $v \in \mathfrak{b}_1 \cap \mathfrak{b}_2$. (Note here the derived fiber product coincides with the naive fiber product. A derived enhancement of the usual Steinberg variety is recovered as the fiber over $0 \in \mathfrak{c}$, i.e., replacing $\widetilde{\mathfrak{g}}$ by $\widetilde{\mathcal{N}}$.[4]) It has a microlocal interpretation involving the double coset spaces

$$Z = B\backslash G/B \simeq G\backslash \mathcal{B} \times \mathcal{B} \qquad \widetilde{Z} = N\backslash G/N \simeq G\backslash \widetilde{\mathcal{B}} \times \widetilde{\mathcal{B}}.$$

Namely, returning to the identification

$$\widetilde{\mathfrak{g}} = (T^*\widetilde{\mathcal{B}})/H \qquad \widetilde{\mathcal{B}} = G/N$$

we have a similar identification

$$G\backslash(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}) \simeq (T^*\widetilde{Z})/H \times H \ ,$$

or (after de-equivariantization)

$$\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \simeq (pt \times_{BG} T^*\widetilde{Z})/H \times H.$$

From this viewpoint, the fiber product in the construction of the Grothendieck-Steinberg variety $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ arises as the moment map equation for Hamiltonian reduction along the diagonal G-action for $T^*\widetilde{\mathcal{B}} \times T^*\widetilde{\mathcal{B}}$.

2.2. **Descent pattern.** The Hamiltonian action of G on $\widetilde{\mathfrak{g}}$ endows the ∞-category $\mathcal{Q}(\widetilde{\mathfrak{g}})$ with two important compatible structures: an algebraic action of G, and a module structure over $\mathcal{Q}(\mathfrak{g})$ (given by pullback under μ). Let us identify the symmetries of $\mathcal{Q}(\widetilde{\mathfrak{g}})$ preserving these structures.

First the symmetries relative to $\mathcal{Q}(\mathfrak{g})$ are given (thanks to [BFN]) as integral transforms with kernels on the fiber product: we have a monoidal equivalence

$$\Phi : \mathcal{Q}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{Q}(\mathfrak{g})}(\mathcal{Q}(\widetilde{\mathfrak{g}})) \qquad \Phi_{\mathcal{K}}(-) = p_{2*}(\mathcal{K} \otimes p_1^*(-))$$

where $p_1, p_2 : \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ denote the projections. In particular, the identity functor corresponds to the integral kernel $\Delta_{\widetilde{\mathfrak{g}}*}\mathcal{O}_{\widetilde{\mathfrak{g}}} \in \mathcal{Q}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}})$ obtained by pushforward along the diagonal map

$$\Delta_{\widetilde{\mathfrak{g}}} : \widetilde{\mathfrak{g}} \longrightarrow \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$$

On the other hand the endomorphisms of $\mathcal{Q}(\widetilde{\mathfrak{g}})$ commuting with G are simply G-equivariant kernels on the square,

$$\mathrm{End}_G(\mathcal{Q}(\widetilde{\mathfrak{g}})) \simeq \mathcal{Q}(G\backslash(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}))$$

and finally the endomorphisms as a Hamiltonian G-category (combining the two structures) are given by equivariant kernels on the fiber product,

$$\mathrm{End}_{G-Ham}(\mathcal{Q}(\widetilde{\mathfrak{g}})) \simeq \mathcal{Q}(G\backslash(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}})),$$

i.e., the monoidal ∞-category of equivariant quasicoherent sheaves on the Steinberg variety.

2.2.1. *Descent monad.* Consider the standard adjunction and Grothendieck duality adjunction on stable ∞-categories of quasi-coherent sheaves

$$\mu_{\widetilde{\mathfrak{g}}}^* : \mathcal{Q}(\mathfrak{g}) \rightleftarrows \mathcal{Q}(\widetilde{\mathfrak{g}}) : \mu_{\widetilde{\mathfrak{g}}*} \qquad \mu_{\widetilde{\mathfrak{g}}*} : \mathcal{Q}(\widetilde{\mathfrak{g}}) \rightleftarrows \mathcal{Q}(\mathfrak{g}) : \mu_{\widetilde{\mathfrak{g}}}^!$$

Since the relative dualizing sheaf of $\mu_{\widetilde{\mathfrak{g}}}$ is canonically trivial, we have a canonical equivalence $\mu_{\widetilde{\mathfrak{g}}}^! \simeq \mu_{\widetilde{\mathfrak{g}}}^*$, but we distinguish them to avoid confusion. By the projection formula, we can view these as adjunctions of $\mathcal{Q}(\mathfrak{g})$-module categories. The adjunctions are also evidently G-equivariant, i.e., preserve the Hamiltonian G-structure.

[4] right? forget what's actually derived

Let $T = \mu^!_{\widetilde{\mathfrak{g}}}\mu_{\widetilde{\mathfrak{g}}*}$ denote the resulting monad, or in other words, algebra object in the monoidal category of linear endomorphisms $\mathrm{End}_{\mathcal{Q}(\mathfrak{g})}(\mathcal{Q}(\widetilde{\mathfrak{g}}))$. Likewise let $T^\vee = \mu^*_{\widetilde{\mathfrak{g}}}\mu_{\widetilde{\mathfrak{g}}*}$ denote the corresponding comonad. Since $\mu^*_{\widetilde{\mathfrak{g}}}$ is conservative and continuous, $\mu^!_{\widetilde{\mathfrak{g}}}$ is as well. Thus the Barr-Beck Theorem provides canonical identifications of $\mathcal{Q}(\mathfrak{g})$-module categories

$$\mathcal{Q}(\mathfrak{g}) \simeq \mathcal{Q}(\widetilde{\mathfrak{g}})_T \simeq \mathcal{Q}(\widetilde{\mathfrak{g}})^{T^\vee}$$

We have canonical equivalences

$$T(-) = \mu^!_{\widetilde{\mathfrak{g}}}\mu_{\widetilde{\mathfrak{g}}*}(-) \simeq p_{2*}p_1^!(-) \simeq p_{2*}(p_1^*(-) \otimes p_1^!\mathcal{O}_{\widetilde{\mathfrak{g}}}) \simeq p_{2*}p_1^*(-)$$

by base change, existence of a dualizing sheaf for p_1, and triviality of the dualizing sheaf for p_1. For the latter two assertions, we have used that the canonical equivalence $\mu^!_{\widetilde{\mathfrak{g}}} \simeq \mu^*_{\widetilde{\mathfrak{g}}}$ provides such an equivalence for any base change of $\mu_{\widetilde{\mathfrak{g}}}$. In summary, the integral kernel giving rise to the underlying functor $T \simeq T^\vee$ of our (co)monad is simply the structure sheaf $\mathcal{O}_{\widetilde{\mathfrak{g}}\times_{\mathfrak{g}}\widetilde{\mathfrak{g}}}$ (which has a canonical G-equivariant structure).

The comonadic structure on $T^\vee$ corresponds to the tautological coalgebra structure (with respect to convolution) on the structure sheaf $\mathcal{O}_{\widetilde{\mathfrak{g}}\times_{\mathfrak{g}}\widetilde{\mathfrak{g}}}$. Likewise the monadic structure on T corresponds to the tautological algebra structure on the relative dualizing sheaf $\omega_{\widetilde{\mathfrak{g}}\times_{\mathfrak{g}}\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{g}}} = p_1^!\mathcal{O}_{\widetilde{\mathfrak{g}}}$, which we have trivialized. Namely, these structures devolve from standard identities applied to the equivalence $\mu^!_{\widetilde{\mathfrak{g}}} \simeq \mu^*_{\widetilde{\mathfrak{g}}}$, its base changes, the counit morphism

$$\mu_{\widetilde{\mathfrak{g}}*}\mu^!_{\widetilde{\mathfrak{g}}}\mathcal{O}_{\mathfrak{g}} \longrightarrow \mathcal{O}_{\mathfrak{g}}$$

and the unit morphism

$$\Delta_{\widetilde{\mathfrak{g}}*}\mathcal{O}_{\widetilde{\mathfrak{g}}} \longrightarrow \mathcal{O}_{\widetilde{\mathfrak{g}}\times_{\mathfrak{g}}\widetilde{\mathfrak{g}}}$$

adjoint to the identity morphism

$$\mathcal{O}_{\widetilde{\mathfrak{g}}} \xrightarrow{\ \sim\ } \mathcal{O}_{\widetilde{\mathfrak{g}}} \simeq \Delta^!_{\widetilde{\mathfrak{g}}}p_1^!\mathcal{O}_{\widetilde{\mathfrak{g}}} \simeq \Delta^!_{\widetilde{\mathfrak{g}}}\mathcal{O}_{\widetilde{\mathfrak{g}}\times_{\mathfrak{g}}\widetilde{\mathfrak{g}}}$$

2.3. **Specified eigenvalues.** We describe here the above descent picture to distinguished loci within $\mathfrak{g}$.

2.3.1. *Regular locus.* Over the open regular locus $\mathfrak{g}^r \subset \mathfrak{g}$, we have a fiber square

$$\begin{array}{ccc}
\widetilde{\mathfrak{g}}^r = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \mathfrak{g}^r & \xrightarrow{\mu_{\widetilde{\mathfrak{g}}^r}} & \mathfrak{g}^r \\
\downarrow & & \downarrow \\
\mathfrak{h} & \xrightarrow[\pi]{} & \mathfrak{c} = \mathfrak{h}/\!/W_{\mathfrak{g}}
\end{array}$$

Thus descent over $\mathfrak{g}^r \subset \mathfrak{g}$ is simply the base change of descent over the geometric invariant theory quotient.

2.3.2. *Regular semisimple locus.* Over the open regular semisimple locus $\mathfrak{g}^{rs} \subset \mathfrak{g}$, we have a fiber square

$$\begin{array}{ccc}
\widetilde{\mathfrak{g}}^{rs} = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \mathfrak{g}^{rs} & \xrightarrow{\mu_{\widetilde{\mathfrak{g}}^{rs}}} & \mathfrak{g}^{rs} \\
\downarrow & & \downarrow \\
\mathfrak{h}^r & \xrightarrow[\pi^r]{} & \mathfrak{c}^r = \mathfrak{h}^r/W_{\mathfrak{g}}
\end{array}$$

In other words, we have a free W-action and quotient identification

$$W_{\mathfrak{g}} \times \widetilde{\mathfrak{g}}^{rs} \longrightarrow \widetilde{\mathfrak{g}}^{rs} \qquad \mathfrak{g}^{rs} \simeq \widetilde{\mathfrak{g}}^{rs}/W$$

Thus descent over $\mathfrak{g}^{rs} \subset \mathfrak{g}$ is simply equivariance for the Weyl group W.

2.3.3. *Nilpotent cone.* Over the nilpotent cone $\mathcal{N} = \mathfrak{g} \times_{\mathfrak{c}} \{0\} \subset \mathfrak{g}$, we have the base change

$$\mu_{\widetilde{\mathfrak{g}}_0} : \widetilde{\mathfrak{g}}_0 = \widetilde{\mathfrak{g}} \times_{\mathfrak{c}} \{0\} \longrightarrow \mathcal{N}$$

where $\widetilde{\mathfrak{g}}_0$ is a non-reduced scheme with underlying reduced scheme the usual Springer resolution $\widetilde{\mathcal{N}} \simeq T^*\mathcal{B}$ classifying a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ together with an element $v \in (\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{n}$.

By construction, descent along $\mu_{\widetilde{\mathfrak{g}}_0}$ is governed by the restricted algebra object

$$\mathcal{O}_{\widetilde{\mathfrak{g}}_0 \times_{\mathcal{N}} \widetilde{\mathfrak{g}}_0} \simeq \mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}}|_{\mathcal{N}}$$

Remark 2.1. To work instead with the traditional Springer resolution

$$\mu_{\widetilde{\mathcal{N}}} : \widetilde{\mathcal{N}} \longrightarrow \mathcal{N}$$

we must pass to ind-coherent sheaves. In applying the Barr-Beck Theorem, we use that the adjunction

$$\mu_{\widetilde{\mathfrak{g}}_0 *} : \mathcal{Q}(\widetilde{\mathfrak{g}}_0) \rightleftarrows \mathcal{Q}(\mathcal{N}) : \mu^!_{\widetilde{\mathfrak{g}}_0}$$

comprises a proper left adjoint and hence continuous right adjoint. But in contrast, this does not hold for the adjunction

$$\mu_{\widetilde{\mathcal{N}} *} : \mathcal{Q}(\widetilde{\mathcal{N}}) \rightleftarrows \mathcal{Q}(\mathcal{N}) : \mu^!_{\widetilde{\mathcal{N}}}$$

For example, $\widetilde{\mathcal{N}}$ is smooth, hence all skyscraper sheaves on it are compact, but $\mathcal{N}$ is singular, hence many skyscraper sheaves on it are not compact. Rather we must pass to ind-coherent sheaves and work with the analogous adjunction

$$\mu_{\widetilde{\mathcal{N}} *} : \operatorname{Ind}\operatorname{Coh}(\widetilde{\mathcal{N}}) \rightleftarrows \operatorname{Ind}\operatorname{Coh}(\mathcal{N}) : \mu^!_{\widetilde{\mathcal{N}}}$$

Here by construction, the adjunction comprises a proper left adjoint and hence continuous right adjoint.

3. Beilinson-Bernstein localization

Now we will repeat the constructions of the previous section after quantization of the natural Poisson structures, i.e., turning on the noncommutative deformation from cotangent bundles to $\mathcal{D}$-modules.

3.1. **Quantization.** Let $\mathfrak{U}\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$, and $\mathfrak{Z}\mathfrak{g} \subset \mathfrak{U}\mathfrak{g}$ the Harish Chandra center.

Let $\mathfrak{U}\mathfrak{g}$-mod denote the stable ∞-category of $\mathfrak{U}\mathfrak{g}$-modules. Informally speaking, $\mathfrak{U}\mathfrak{g}$-mod consists of noncommutative modules on the Poisson manifold $\mathfrak{g} \simeq \mathfrak{g}^*$.

Let $\mathfrak{U}\mathfrak{g}$-perf $\subset$ $\mathfrak{U}\mathfrak{g}$-mod denote the small stable full ∞-subcategory of perfect modules so that $\mathfrak{U}\mathfrak{g}$-mod $\simeq$ Ind($\mathfrak{U}\mathfrak{g}$-perf).

Lemma 3.1. *There are canonical equivalences*

$$\mathfrak{U}\mathfrak{g}\text{-perf} \simeq \mathfrak{U}\mathfrak{g}\text{-perf}^{op} \qquad \mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathfrak{U}\mathfrak{g}\text{-mod}^{\vee}$$

Proof. First, viewing $\mathfrak{U}\mathfrak{g}$ as a $\mathfrak{U}\mathfrak{g}$-bimodule, we define the duality identification

$$\mathfrak{U}\mathfrak{g}\text{-perf}^{op} \xrightarrow{\sim} \mathfrak{U}\mathfrak{g}^{op}\text{-perf} \qquad M \longmapsto \mathcal{H}om_{\mathfrak{U}\mathfrak{g}}(M, \mathfrak{U}\mathfrak{g}[\dim \mathfrak{g}])$$

Now let $\mathfrak{g}^{op}$ denote the vector space $\mathfrak{g}$ with the opposite Lie bracket

$$[\cdot,\cdot]_{\mathfrak{g}^{op}} = -[\cdot,\cdot]_{\mathfrak{g}}$$

The negation map $\mathfrak{g} \to \mathfrak{g}$, $v \mapsto -v$ provides a canonical isomorphism $\mathfrak{g} \simeq \mathfrak{g}^{op}$ and hence a canonical isomorphism $\mathfrak{U}\mathfrak{g} \simeq \mathfrak{U}\mathfrak{g}^{op}$. This establishes the first assertion, and the second then follows from the standard identity

$$\mathfrak{U}\mathfrak{g}\text{-mod}^{\vee} \simeq \operatorname{Ind}(\mathfrak{U}\mathfrak{g}\text{-perf}^{op})$$

□

Remark 3.2. The equivalence $\mathfrak{U}\mathfrak{g}$-perf $\simeq \mathfrak{U}\mathfrak{g}$-perf^{op} is a twisted form of the Serre duality equivalence $\mathrm{Perf}(\mathfrak{g}^*) \simeq \mathrm{Perf}(\mathfrak{g}^*)^{op}$. Namely, the former invokes the negation on the vector space $\mathfrak{g}^*$ while the latter does not.

Let $\mathcal{D}_{\widetilde{\mathcal{B}}} \in \mathcal{Q}(\widetilde{\mathcal{B}})$ denote the sheaf of differential operators on $\widetilde{\mathcal{B}}$. Let $\mathcal{D}_H(\widetilde{\mathcal{B}})$ denote the stable ∞-category of H-weakly equivariant $\mathcal{D}$-modules on $\widetilde{\mathcal{B}}$. Its objects are H-equivariant quasi-coherent complexes on $\widetilde{\mathcal{B}}$ equipped with a compatible H-equivariant action of $\mathcal{D}_{\widetilde{\mathcal{B}}}$.

Let $\widetilde{\mathcal{D}}_{\mathcal{B}} \in \mathcal{Q}(\mathcal{B})$ denote the sheaf of H-invariant differential operators on $\widetilde{\mathcal{B}}$. It admits the natural presentation $\widetilde{D}_{\mathcal{B}} \simeq \mathcal{D}_{\mathcal{B}} \otimes_{\mathfrak{Z}\mathfrak{g}} \mathfrak{U}\mathfrak{h}$, where $\mathfrak{U}\mathfrak{h}$ acts by vector fields on the right, and $\mathfrak{Z}\mathfrak{g} \subset \mathfrak{U}\mathfrak{g}$ by differential operators on the left. Then $\mathcal{D}_H(\widetilde{\mathcal{B}})$ is equivalently the stable ∞-category of quasi-coherent complexes on $\mathcal{B}$ equipped with a compatible action of $\widetilde{\mathcal{D}}_{\mathcal{B}}$.

Informally speaking, $\mathcal{D}_H(\widetilde{\mathcal{B}})$ consists of non-commutative modules on the Poisson manifold $\widetilde{\mathfrak{g}} = (T^*\widetilde{\mathcal{B}})/H$.

Let $\mathcal{D}^c_H(\widetilde{\mathcal{B}}) \subset \mathcal{D}_H(\widetilde{\mathcal{B}})$ denote the small stable full ∞-subcategory of coherent modules so that $\mathcal{D}_H(\widetilde{\mathcal{B}}) \simeq \mathrm{Ind}(\mathcal{D}^c_H(\widetilde{\mathcal{B}}))$.

Lemma 3.3. *Verdier duality provides canonical equivalences*

$$\mathcal{D}^c_H(\widetilde{\mathcal{B}}) \simeq \mathcal{D}^c_H(\widetilde{\mathcal{B}})^{op} \qquad \mathcal{D}_H(\widetilde{\mathcal{B}}) \simeq \mathcal{D}_H(\widetilde{\mathcal{B}})^\vee$$

Proof. The first assertion is Verdier duality, and the second follows from the standard identity $\mathcal{D}_H(\widetilde{\mathcal{B}})^\vee \simeq \mathrm{Ind}(\mathcal{D}^c_H(\widetilde{\mathcal{B}})^{op})$. □

Remark 3.4. When keeping track of additional structures, it is useful to keep in mind the opposite base affine space $\widetilde{\mathcal{B}}^{op} = N\backslash G$. The inverse map $G \to G$, $g \mapsto g^{-1}$ provides a canonical isomorphism $\widetilde{\mathcal{B}} \simeq \widetilde{\mathcal{B}}^{op}$.

Consider the localization adjunction

$$\gamma^* : \mathfrak{U}\mathfrak{g}\text{-mod} \rightleftarrows \mathcal{D}_H(\widetilde{\mathcal{B}}) : \gamma_*$$

$$\gamma^*(M) = \mathcal{D}_{\widetilde{\mathcal{B}}} \otimes_{\mathfrak{U}\mathfrak{g}} M \qquad \gamma_*(\mathcal{M}) = \mathrm{Hom}(\mathcal{D}_{\widetilde{\mathcal{B}}}, \mathcal{M})$$

Informally speaking, this is a quantization of the standard adjunction for the Grothendieck-Springer resolution $\mu_{\widetilde{\mathfrak{g}}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$.

Proposition 3.5. *The right adjoint γ_* is continuous and proper, and hence itself admits a continuous right adjoint $\gamma^!$. Furthermore, there is a canonical identification $\gamma^! \simeq \gamma^*$.*

Proof. It suffices to check that the right adjoint γ_* fits into a commutative square on compact objects

$$\begin{array}{ccc} \mathcal{D}^c_H(\widetilde{\mathcal{B}}) & \simeq & \mathcal{D}^c_H(\widetilde{\mathcal{B}})^{op} \\ \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*^{op}} \\ \mathfrak{U}\mathfrak{g}\text{-perf} & \simeq & \mathfrak{U}\mathfrak{g}\text{-perf}^{op} \end{array}$$

But this is evident from its construction and that of the horizontal duality equivalences. □

3.1.1. *Linearity.* Let $\mathfrak{Z}\mathfrak{g} \subset \mathfrak{U}\mathfrak{g}$ be the Harish Chandra center, and $\mathfrak{U}\mathfrak{h}$ the universal enveloping algebra of $\mathfrak{h}$. We also have the canonical embedding $\mathfrak{Z}\mathfrak{g} \subset \mathfrak{U}\mathfrak{h}$ as the ρ-shifted Weyl invariants.

Observe that $\mathfrak{U}\mathfrak{g}$-mod is naturally $\mathfrak{Z}\mathfrak{g}$-linear, and $\mathcal{D}_H(\widetilde{\mathcal{B}})$ is naturally $\mathfrak{U}\mathfrak{h}$-linear and hence $\mathfrak{Z}\mathfrak{g}$-linear. The following is evident from the constructions.

Lemma 3.6. *The adjunctions*

$$\gamma^* : \mathfrak{U}\mathfrak{g}\text{-mod} \rightleftarrows \mathcal{D}_H(\widetilde{\mathcal{B}}) : \gamma_* \qquad \gamma_* : \mathcal{D}_H(\widetilde{\mathcal{B}}) \rightleftarrows \mathfrak{U}\mathfrak{g}\text{-mod} : \gamma^!$$

are naturally $\mathfrak{Z}\mathfrak{g}$-linear.

3.1.2. *Symmetries.* We now introduce quantum analogs of the Hamiltonian G-actions on $\widetilde{\mathfrak{g}}$ and $\mathfrak{g}^*$. Recall from [BD, FG] the notion of a smooth or *infinitesimally trivialized* G-action on a category, of which the prime examples are the categories $\mathcal{D}(X)$ for a G-space X and $\mathfrak{Ug}$-mod. We adopt the terminology $\mathcal{D}(G)$*-module* for any smooth G-category.

Remark 3.7. Let $\mathcal{D}(G)$ denote the stable ∞-category of $\mathcal{D}$-modules on G. It carries a comonoidal convolution structure, induced by the operation of !-pullback under the group multiplication. Using the self-duality of $\mathcal{D}(G)$ as a stable ∞-category we can transport this into a *monoidal* convolution structure on $\mathcal{D}(G)$ (given on holonomic $\mathcal{D}$-modules by !-pushforward). One can show that smooth G-categories are indeed identified with $\mathcal{D}(G)$-modules for this algebra structure, though we will not need this fact in this paper.

Consider the action of G on itself by right multiplication. Observe that $\mathfrak{Ug}$-mod is equivalent to the resulting stable ∞-category $\mathcal{D}_G(G)$ of G-weakly equivariant $\mathcal{D}$-modules on G. This is the quantum analog of the identification $T^*G/G \simeq \mathfrak{g}^*$. Likewise the quantum analog of $\widetilde{\mathfrak{g}} = (T^*\widetilde{\mathcal{B}})/H$ is $\mathcal{D}_H(\widetilde{\mathcal{B}})$. The left actions of G on G and $\widetilde{\mathcal{B}}$ (or convolution on the left with $\mathcal{D}(G)$) endow $\mathfrak{Ug}$-mod and $\mathcal{D}_H(\widetilde{\mathcal{B}})$ with $\mathcal{D}(G)$-module structures.

The following is evident from the constructions.

Lemma 3.8. *The adjunctions*

$$\gamma^* : \mathfrak{Ug}\text{-mod} \rightleftarrows \mathcal{D}_H(\widetilde{\mathcal{B}}) : \gamma_* \qquad \gamma_* : \mathcal{D}_H(\widetilde{\mathcal{B}}) \rightleftarrows \mathfrak{Ug}\text{-mod} : \gamma^!$$

are naturally $\mathcal{D}(G)$*-linear.*

Consider the stack $\widetilde{Z} = N\backslash G/N$.

Let $\mathcal{D}_{H\times H}(\widetilde{Z})$ denote the stable ∞-category of $H \times H$-weakly equivariant $\mathcal{D}$-modules on $\widetilde{Z}$. Convolution equips $\mathcal{D}_{H\times H}(\widetilde{Z})$ with a natural monoidal structure, and $\mathcal{D}(\widetilde{\mathcal{B}})$ with a natural right $\mathcal{D}_{H\times H}(\widetilde{Z})$-module structure commuting with its natural left $\mathcal{D}(G)$-module structure. These are quantum analogs of sheaves on the equivariant Grothendieck-Steinberg variety

$$\mathcal{Q}(G\backslash(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}})) \simeq \mathcal{Q}_{H\times H}(T^*Z)$$

and their monoidal action on $\mathcal{Q}(\widetilde{\mathfrak{g}})$ by symmetries of the Hamiltonian G-action. In fact we have the following:

Theorem 3.9. *Convolution provides an anti-monoidal equivalence*

$$\Phi : \mathcal{D}_{H\times H}(\widetilde{Z}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}(G)}(\mathcal{D}_H(\widetilde{\mathcal{B}}))$$

3.2. **Universal Weyl sheaf.** We will work with the adjunction

$$\gamma_* : \mathcal{D}_H(\widetilde{\mathcal{B}}) \rightleftarrows \mathfrak{Ug}\text{-mod} : \gamma^!$$

Let $T = \gamma^!\gamma_*$ denote the resulting monad, or in other words, algebra object in the monoidal category of endomorphisms $\mathrm{End}_{\mathcal{D}(G)}(\mathcal{D}_H(\widetilde{\mathcal{B}}))$.

Lemma 3.10. $\gamma^!$ *is conservative.*

Proof. From the standard calculation

$$\gamma_*(\mathcal{D}_{\widetilde{\mathcal{B}}}) = \mathrm{Hom}(\mathcal{D}_{\widetilde{\mathcal{B}}}, \mathcal{D}_{\widetilde{\mathcal{B}}})^H \simeq \mathfrak{Ug} \otimes_{\mathfrak{Z}\mathfrak{g}} \mathfrak{Uh}$$

we find

$$\mathrm{Hom}(\mathcal{D}_{\widetilde{\mathcal{B}}}, \gamma^!(M)) \simeq \mathrm{Hom}(\mathfrak{Ug} \otimes_{\mathfrak{Z}\mathfrak{g}} \mathfrak{Uh}, M)$$

Factoring with the canonical integration morphism $\mathfrak{Ug} \otimes_{\mathfrak{Z}\mathfrak{g}} \mathfrak{Uh} \to \mathfrak{Ug}$, we conclude that $M \not\simeq 0$ implies $\gamma^!(M) \not\simeq 0$. □

Since $\gamma^!$ is continuous and conservative, the Barr-Beck Theorem provides a canonical identification

$$\mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathrm{Mod}_T(\mathcal{D}_H(\widetilde{\mathcal{B}}))$$

We would like to explicitly describe the integral kernel giving rise to T under the equivalence

$$\Phi : \mathcal{D}_{H\times H}(\widetilde{Z}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}(G)}(\mathcal{D}_H(\widetilde{\mathcal{B}}))$$

Definition 3.11. The *universal Weyl sheaf* $\mathcal{W} \in \mathcal{D}_{H\times H}(\widetilde{Z})$ is the sheaf of differential operators on $\widetilde{Z}$ with its canonical $H \times H$-weakly equivariant structure.

By quantum Hamiltonian reduction (and under the identification $\widetilde{Z} \simeq G\backslash(G/N \times G/N)$), the pullback of $\mathcal{W}$ along the natural quotient map

$$r : G/N \times G/N \longrightarrow N\backslash G/N$$

is the G-strongly equivariant $\mathcal{D}$-module

$$r^*\mathcal{W} = \mathcal{D}_{G/N\times G/N}/(\mathfrak{g})$$

where $(\mathfrak{g}) \subset \mathcal{D}_{G/N\times G/N}$ is the left ideal generated by vector fields arising from the diagonal G-action on $G/N \times G/N$. Thus $\mathcal{W}$ is the quantum analog of the structure sheaf of $G\backslash\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \simeq T^*Z/(H \times H)$, or equivalently of the structure sheaf $\mathcal{O}_{\widetilde{\mathfrak{g}}\times_{\mathfrak{g}}\widetilde{\mathfrak{g}}}$ with its G-equivariant structure.

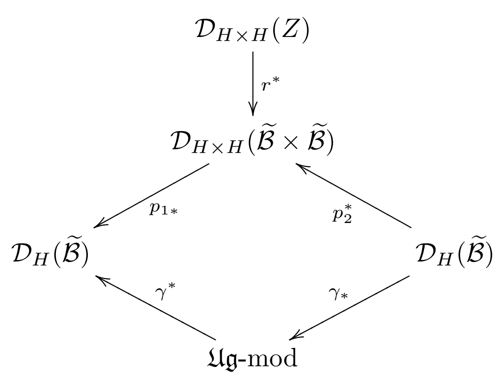
Theorem 3.12. *The monoidal equivalence*

$$\Phi : \mathcal{D}_{H\times H}(\widetilde{Z}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}(G)}(\mathcal{D}_H(\widetilde{\mathcal{B}}))$$

takes the universal Weyl sheaf $\mathcal{W}$ to endofunctor $T = T^\vee$. In particular $\mathcal{W}$ carries the structure of Frobenius algebra in $\mathcal{D}_{H\times H}(\widetilde{Z})$, and we have equivalences

$$\mathfrak{U}\mathfrak{g}\text{-mod} \simeq \mathcal{D}_H(\widetilde{B})_{\mathcal{W}} \simeq \mathcal{D}_H(\widetilde{B})^{\mathcal{W}}.$$

Proof. This is simply a microlocal version of the base change argument in the commutative case, as summarized in the following (informal but suggestive) diagram:



On the one hand, given an object $\mathcal{M} \in \mathcal{D}_H(\widetilde{\mathcal{B}})$, we find

$$T(\mathcal{M}) \simeq \gamma^*\gamma_*\mathcal{M} \simeq \widetilde{\mathcal{D}}_{\mathcal{B}} \otimes_{\mathfrak{U}\mathfrak{g}} \mathrm{Hom}_{\widetilde{\mathcal{D}}_{\mathcal{B}}}(\widetilde{\mathcal{D}}_{\mathcal{B}}, \mathcal{M})$$

Thus it is the $\mathfrak{U}\mathfrak{g}$-coinvariants of the intermediate functor

$$\widetilde{\mathcal{D}}_{\mathcal{B}} \otimes \mathrm{Hom}_{\widetilde{\mathcal{D}}_{\mathcal{B}}}(\widetilde{\mathcal{D}}_{\mathcal{B}}, \mathcal{M})$$

This can be viewed as the integral transform with integral kernel $\widetilde{\mathcal{D}}_{\mathcal{B}} \boxtimes \widetilde{\mathcal{D}}_{\mathcal{B}}$.

On the other hand, the universal Weyl sheaf $\mathcal{W}$ is simply the $\mathfrak{U}\mathfrak{g}$-coinvariants of the integral kernel $\widetilde{\mathcal{D}}_{\mathcal{B}} \boxtimes \widetilde{\mathcal{D}}_{\mathcal{B}}$. Thus since all functors are continuous, taking $\mathfrak{U}\mathfrak{g}$-coinvariants can be equivalently performed on the integral kernel or on the result of the integral transform. □

3.3. **Specified central character.** The commutative algebra $\mathfrak{U}\mathfrak{h} \otimes \mathfrak{U}\mathfrak{h} = \mathcal{O}(\mathfrak{h}^* \times \mathfrak{h}^*)$ acts by central endomorphisms on the stable ∞-category $\mathcal{D}_{H\times H}(\widetilde{Z})$. The action factors through the closed subscheme $\Gamma \subset \mathfrak{h}^* \times \mathfrak{h}^*$ given by the union of the graphs of Weyl elements

$$\Gamma = \coprod_{w\in W} \Gamma_w \qquad \Gamma_w = \{(\lambda, w\lambda) \in \mathfrak{h}^* \times \mathfrak{h}^*\}$$

To better understand the universal Weyl sheaf $\mathcal{W} \in \mathcal{D}_{H\times H}(\widetilde{Z})$, let us restrict it to distinguished loci inside of $\mathfrak{h}^* \times \mathfrak{h}^*$.

3.3.1. *Regular semisimple locus.* Over the regular semisimple locus, the restriction of $\mathcal{W} \in \mathcal{D}_{H\times H}(\widetilde{Z})$ provides an action of the group algebra of the Weyl group W on the restriction of $\mathcal{D}_H(\widetilde{\mathcal{B}})$.

3.3.2. *Fixed central character.* The composite projection to either factor

$$\Gamma \hookrightarrow \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$$

is a finite flat map. All of what follows is symmetric in the two projections, so for concreteness, let us focus on the projection to the first factor.

Let us consider the fibers of the stable ∞-category $\mathcal{D}_{H\times H}(\widetilde{Z})$ along the projection to the first factor. Let us also forget the H-weak equivariance along the second factor. For $\lambda \in \mathfrak{h}^*$, we can identify the result with N-strongly equivariant $\mathcal{D}_\lambda$-modules on the flag variety

$$\mathcal{D}_\lambda(N\backslash\mathcal{B}) \simeq \mathcal{D}_{H\times H}(\widetilde{Z}) \otimes_{\mathcal{O}(\mathfrak{h})} \mathbf{C}_\lambda$$

This in turn is the full subcategory of those $\mathcal{D}_\lambda$-modules that are locally constant along Schubert cells.

Now let us identify the fiber $\mathcal{W}_\lambda = \mathcal{W} \otimes_{\mathcal{O}(\mathfrak{h})} \mathfrak{h}^*$ of the universal Weyl sheaf. This is a regular holonomic $\mathcal{D}_\lambda$-module on the flag variety $\mathcal{B}$ locally constant along Schubert cells.

For concreteness, we will consider several specific case: (1) λ generic (regular and not at all integral), (2) λ regular and integral, and (3) λ trivial.

(1) When λ is generic, we have a direct sum decomposition

$$\mathcal{W}_\lambda \simeq \oplus_{w\in W} \mathcal{W}_{\lambda,w\lambda}$$

Each summand admits the description as a standard or costandard extension off of a Schubert cell

$$\mathcal{W}_{\lambda,w\lambda} \simeq j_{w!}\mathcal{O}_{\lambda,w} \simeq j_{w*}\mathcal{O}_{\lambda,w}$$

where $j_w : X_w \to \mathcal{B}$ is the inclusion of the w-Schubert cell, and $\mathcal{O}_{\lambda,w}$ is the λ-twisted constant sheaf on X_w.

(2) Suppose λ is regular and integral, and let $\eta \in W$ be the Weyl element such that $\lambda = \eta\lambda_0$ for some dominant λ_0.

We can tensor by the line bundle $\mathcal{O}(\lambda)$ to obtain an identification

$$\mathcal{D}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}_\lambda(\mathcal{B}) \qquad M \longmapsto M \otimes \mathcal{O}(\lambda)$$

This is convenient since we will describe $\mathcal{W}_\lambda$ in terms of the natural monoidal structure on $\mathcal{D}(B\backslash\mathcal{B})$. Namely, under the above identification, we have a direct sum decomposition

$$\mathcal{W}_\lambda \simeq \oplus_{w\in W} \mathcal{W}_{\lambda,w\lambda}$$

Each summand admits the description as the convolution of standard extensions $T_{w*} = j_{w*}\mathcal{O}_w$ and costandard extensions $T_{w!} = j_{w!}\mathcal{O}_w$ off of Schubert cells

$$\mathcal{W}_{\lambda,w\lambda} \simeq T_{\eta!} * T_{w\eta^{-1}*}$$

(3) When $\lambda = 0$ is trivial, the fiber $\mathcal{W}_0$ is the maximal tilting sheaf on $\mathcal{B}$. Namely, within $\mathcal{D}_\lambda(N\backslash\mathcal{B})$, it is the projective cover of the skyscraper sheaf at the closed Schubert cell.

References

[AG] D. Arinkin and D. Gaitsgory, Singular support of coherent sheaves, and the geometric Langlands conjecture. arXiv:1201.6343.

[BK] E. Backelin and K. Kremnizer, On Singular Localization of $\mathfrak{g}$-modules, arXiv:1011.0896.

[B] A. Beilinson, Localization of representations of reductive Lie algebras. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 699710, PWN, Warsaw, 1984.

[BB] A. Beilinson and J. Bernstein, A proof of Jantzen conjectures. I. M. Gelfand Seminar, 1–50, Adv. Soviet Math., 16, Part 1, Amer. Math. Soc., Providence, RI, 1993.

[BD] A. Beilinson and V. Drinfeld, Quantization of Hitchin Hamiltonians and Hecke Eigensheaves. Preprint, available at math.uchicago.edu/~mitya.

[BG] A. Beilinson and V. Ginzburg, Wall-crossing functors and $\mathcal{D}$-modules. arXiv:alg-geom/9709022. Represent. Theory 3 (1999), 131.

[BGS] A. Beilinson, V. Ginzburg and W. Soergel, Koszul duality patterns in representation theory. *Jour. Amer. Math. Soc.* **9** (1996), no. 2, 473–527.

[BFN] D. Ben-Zvi, J. Francis and D. Nadler, Integral transforms and Drinfeld centers in derived algebraic geometry. Preprint arXiv:0805.0157. *Jour. Amer. Math. Soc.* **23** (2010), 909–966.

[BN] D. Ben-Zvi and D. Nadler, The character theory of a complex group. e-print arXiv:math/0904.1247

[BMR1] R. Bezrukavnikov, I. Mirkovic and D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic. arXiv:math/0205144.

[BMR2] R. Bezrukavnikov, I. Mirkovic and D. Rumynin, Singular localization and intertwining functors for reductive Lie algebras in prime characteristic. arXiv:math/0602075.

[Be] R. Bezrukavnikov, Noncommutative counterparts of the Springer resolution. arXiv:math.RT/0604445. International Congress of Mathematicians. Vol. II, 1119–1144, Eur. Math. Soc., Zürich, 2006.

[BH] J. Block and N. Higson, Weyl character formula in KK-theory. Available at http://web.me.com/ndh2/math/Papers.html.

[BKR] T. Bridgeland, A. King and M. Reid, The McKay correspondence as an equivalence of derived categories. J. Amer. Math. Soc. 14 (2001), no. 3, 535554

[CG] N. Chriss and V. Ginzburg, Representation theory and complex geometry. Birkhäuser Boston, Inc., Boston, MA, 1997. x+495 pp.

[FG] E. Frenkel and D. Gaitsgory, Local geometric Langlands correspondence and affine Kac-Moody algebras. Algebraic geometry and number theory, 69–260, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006.

[G1] D. Gaitsgory, Notes on Geometric Langlands: ind-coherent sheaves. arXiv:1105.4857

[G2] D. Gaitsgory, Sheaves of Categories over Prestacks. Preprint, 2012.

[K] M. Kashiwara, Representation theory and D-modules on flag varieties. Orbites unipotentes et reprsentations, III. Astrisque No. 173-174 (1989), 9, 55109.

[La] A. Lauda, Frobenius algebras and ambidextrous adjunctions. Theory Appl. Categ. 16 (2006), No. 4, 84122.

[L1] J. Lurie, Higher topos theory. arXiv:math.CT/0608040. Annals of Mathematics Studies, 170. Princeton University Press, Princeton, NJ, 2009.

[L2] J. Lurie, Higher Algebra. Available at http://www.math.harvard.edu/~lurie/

[L3] J. Lurie, On the classification of topological field theories. Available at http://www.math.harvard.edu/~lurie/ Current developments in mathematics, 2008, 129V280, Int. Press, Somerville, MA, 2009.

[L4] J. Lurie, Derived Algebraic Geometry XI: Descent theorems. Available at http://www.math.harvard.edu/~lurie/

[MN] K. McGerty and T. Nevins, Derived equivalence for quantum symplectic resolutions, arXiv:1108.6267.

[P] A. Preygel, Thom-Sebastiani and Duality for Matrix Factorizations. arXiv:1101.5834

[S] W. Soergel, Kategorie O, perverse Garben und Moduln ber den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. 3 (1990), no. 2, 421445.

[St] R. Street, Frobenius monads and pseudomonoids. J. Math. Phys. 45 (2004), no. 10, 39303948.

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