CHARACTERS, TRACES AND FIXED POINTS

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1. Ordinary characters

Let A be a dualizable object in a symmetric monoidal ∞-category.

$$\mathbf{1} \stackrel{coev}{\to} A \otimes A^\vee \stackrel{ev}{\to} \mathbf{1}.$$

The composition $\mathrm{ev} \circ \mathrm{coev} =: \dim A$. For vector spaces dim is the usual dimension. For algebras $\dim A = HH_\bullet A$.

Here dim is a map $\mathcal{C}^{fd} \to \Omega\mathcal{C}$.

For any $\phi : A \to A$ one gets $\operatorname{tr} \phi \in \operatorname{End}(\mathbf{1})$:

$$\mathbf{1} \to A \otimes A^\vee \stackrel{\phi \otimes 1}{\to} A \otimes A^\vee \to \mathbf{1}.$$

$\operatorname{tr}(\phi\psi) \cong \operatorname{tr}(\psi\phi)$ whenever we can compose.

Consider an adjunction

$$\phi : A \rightleftarrows B : \psi.$$

We have $\mathbf{1}_A \to \psi\phi$ and $\phi\psi \to \mathbf{1}_B$. Then we have

$$\dim A = \operatorname{tr}(\mathbf{1}_A) \stackrel{\mathrm{tr}}{\to} \operatorname{tr}(\psi\phi) \stackrel{\sim}{\to} \operatorname{tr}(\phi\psi) \to \operatorname{tr}(\mathbf{1}_B) = \dim B.$$

The composition is what we call $\dim \phi$. So, dim is a functor from the category of dualizable objects with continuous morphisms to endomorphisms of $\mathbf{1}$.

Replace A by a category $\mathcal{C}$. Then an object of $\mathcal{C}$ is the same as a morphism $\mathbf{1} \to \mathcal{C}$.

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{V} & \mathcal{C} \\ & \searrow^{\pi_* V} & \downarrow \pi_* \\ & & \mathcal{D} \end{array}$$

We get

Theorem (Formal Grothendieck-Riemann-Roch)**.** $\dim(\pi_* V) = (\dim \pi_*)(\dim V)$.

Categorical setting: $\mathcal{C}$ dualizable (e.g. compactly generated) dg-category. $\dim \mathcal{C} = HH_\bullet \mathcal{C}$. For an object $V \in \mathcal{C}$ we have $[V] \in HH_\bullet(\mathcal{C})$.

Let Corr be the following category:

- objects are varieties/stacks
- morphisms are correspondences

$$X \leftarrow Z \rightarrow Y.$$

We also have higher morphisms between correspondences, which we require to be proper.

Any X is dualizable:

$$\cdot \leftarrow X \rightarrow X \times X \leftarrow X \rightarrow \cdot.$$

The composition is $\dim X = X \times_{X \times X} X = \mathcal{L}X$.

For $X \stackrel{\pi}{\to} Y$ a map, we get a map

$$\dim \pi : \mathcal{L}X \stackrel{\mathcal{L}\pi}{\to} \mathcal{L}Y.$$

Sheaf theory is a functor Shv : Corr $\to$ Cat. For X a variety, we assign its category of sheaves $\mathrm{Sh}(X)$.

Note, that from the correspondences

$$X \stackrel{\mathrm{id}}{\leftarrow} X \stackrel{f}{\to} Y$$

and

$$X \stackrel{f}{\leftarrow} Y \stackrel{\mathrm{id}}{\to} Y$$

we get pushforward and pullback functors.

Examples of sheaf theories:

(1) $Q^!$, which coincides with $QC(X)$ for X smooth. Ref: Gaitsgory, ind-coherent sheaves. The category of sheaves is $\mathrm{Ind}(\mathrm{Coh}(X))$.
(2) $\mathcal{D}$. Ref: Gaitsgory-Rozenblyum.

$$\dim(\mathrm{Sh}(X)) = \mathrm{Sh}(\cdot \leftarrow \mathcal{L}X \rightarrow \cdot) = \Gamma(\mathcal{O}_{\mathcal{L}X}).$$

For $\pi : X \to Y$ one gets

$$\dim \pi_* : \mathcal{O}(\mathcal{L}X) \stackrel{\mathcal{L}\pi_*}{\to} \mathcal{O}(\mathcal{L}Y).$$

By the HKR theorem $\mathcal{O}(\mathcal{L}X) = \Omega^\bullet(X)$. The pushforward is given by integration of differential forms.

For $V \in \mathrm{Shv}(X)$ w have $[V] \in \dim(\mathrm{Shv}(X)) = HH_\bullet(X)$.

Now let X be a G-space and $Z = X/G$. Consider $X/G \stackrel{\pi}{\to} \cdot/G$. we get a map

$$\mathcal{L}(X/G) \to \mathcal{L}(\cdot/G) = G/G.$$

One has

$$\mathcal{L}(X/G) = \{g \in G, x \in X^g\}/G.$$

Let V be an equivariant vector bundle on X. This is the same as $\mathcal{V}$ a vector bundle on $Z = X/G$.

Then $\pi_* \mathcal{V}$ is a (virtual) representation of G.

Theorem (Atiyah-Bott-Leftschetz)**.** $[\pi_* V]$ *is given by integration over fixed points.*

Now let $X = G/B$, then $Z = G\backslash X = \cdot B \to \cdot/G$.

Then we have

$$\mathcal{L}(\cdot/B) = B/B \to G/G = \mathcal{L}(\cdot/G).$$

We get the Frobenius character formula this way.

If B is the Borel, we have

$$B/B = \tilde{G}/G.$$

In this way we get the Weyl character formula.

2. Categorical characters

Let $\mathrm{Shv} : \mathrm{Corr} \to 2\,\mathrm{Cat}$. For example, $X \mapsto QCoh(X)$-mod or $\mathcal{D}(X)$-mod.

Let X be a G-space. We have an action $\mathcal{D}(G)$ on $\mathcal{D}(X)$. What is its character?

We get a map

$$\mathcal{D}(\mathcal{L}X/G) \to \mathcal{D}(G/G).$$

This sends

$$\mathcal{O} \overset{\mathcal{L}\pi_*}{\mapsto} \mathcal{L}\pi_* \mathcal{O}.$$

Let $X = G/B$. Then $\mathcal{D}(G/B)$ is a $\mathcal{D}(G)$-module. To calculate its character, consider

$$\mathcal{L}(G\backslash X) = \mathcal{L}(B\backslash \cdot) = B/B = \tilde{G}/G.$$

The character formula tells us that the character is the pushforward of the constant sheaf on $\tilde{G}/G$ pushed forward to G/G. This is precisely the Springer sheaf.

Theorem. $\dim_{\mathcal{D}(G)\text{-mod}}(\mathcal{D}(G/B)) = \mathcal{S}$.

Consider $M \in (\mathfrak{g}, K)$-mod. Via Beilinson-Bernstein this is an element of $\mathcal{D}(K\backslash G/B)$. One can interpret this as a $\mathcal{D}(G)$-map

$$\mathcal{D}(G/K) \to \mathcal{D}(G/B).$$

We can write

$$\mathcal{D}(K\backslash G/B) = \mathrm{Hom}_{\mathcal{D}(G)}(\mathcal{D}(G/B), \mathcal{D}(G/K)).$$

Use the functoriality of the dimension, then

$$\mathcal{S} = \dim(\mathcal{D}(G/B)) \stackrel{\dim(M)}{\to} \dim(\mathcal{D}(G/K)) = \mathcal{S}_K.$$

This is a map of $\mathcal{D}(G/G)$-modules.

Hotta-Kashiwara tell us that $\mathcal{S}$ is the Harish-Chandra system

$$HC_\lambda : \mathcal{D}_{G/G}/\langle z - \lambda(z)\rangle.$$

Here $z \in Z(U\mathfrak{g}) = {}^G\mathcal{D}(G)^G$.

So, $\dim(M) = \chi_M$ gives a solution of the Harish-Chandra system valued in $\mathcal{S}_K$.

Harish-Chandra defined characters for admissible $G-\mathbf{R}$-representations.

Characters are well-defined as distributions on G, which are invariant by conjugation. If V has an infinitesimal character (i.e. $Z \stackrel{\lambda}{\to} \mathbf{C}$), then the corresponding distribution is annihilated by the Harish-Chandra system, i.e.

$$z\chi = \lambda(z)\chi.$$

Analysis tells us that generically it is an analytic function.

3. Geometric Langlands

Let Σ be a Riemann surface over $\mathbf{C}$. We are studying $\mathrm{Bun}_G \Sigma$ the moduli space of G-bundles on Σ.

Take a bundle $\mathcal{P} \in \mathrm{Bun}_G \Sigma$. We can trivialize $\mathcal{P}$ outside of some number of points z_i. So, we can describe the bundles by gluing data in $LG = G(\mathbf{C}((z)))$.

Theorem (Weil)**.** $\mathrm{Bun}_G \Sigma = G(\mathbf{C}(\Sigma))\backslash \prod_i' G(\mathbf{C}((z_i)))/G(\mathbf{C}[[z_i]])$.

For G semisimple we can use just one point:

Theorem (Drinfeld-Simpson)**.**

$$\mathrm{Bun}_G \Sigma = G(\Sigma\backslash x)\backslash LG/LG_+.$$

Here $LG/LG_+ = \mathrm{Gr}_G$ is the affine Grassmannian.

For $x \in \Sigma$ we have a Hecke correspondence $LG_+\backslash LG/LG_+$. It acts by changing the trivialization around one puncture.

For example, $\mathcal{D}(\mathrm{Bun}_G \Sigma)$ carries an action of the spherical Hecke category $\mathcal{H}_{sph} = \mathcal{D}(LG_+\backslash LG/LG_+)$.

There is a slight variant with a parabolic reduction: we can look at G-bundles on Σ with a reduction to B at $x \in \Sigma$.

Let $I \subset LG_+$ be the Iwahori subgroup of loops whose value at the closed point is inside of B.

Then the category of $\mathcal{D}$-modules on such bundles carries an action of the affine Hecke category $\mathcal{D}(I\backslash LG/I)$.

Problem: decompose $\mathcal{D}(\operatorname{Bun}_G X)$ as a module for the Hecke symmetries.

Note, that instead of studying the module structure of $\mathcal{D}(K\backslash G/K)$ on $\mathcal{D}(X/K)$, one can study the $\mathcal{D}(G)$-module structure on $\mathcal{D}(X)$.

In classical local Langlands one studies the function space $L^2(\Gamma\backslash G/K)$, which carries an action of the Hecke algebra on $K\backslash G/K$, which is actually commutative.

On the abelian level one has the geometric Satake correspondence, which identifies $\mathcal{H}_{sph} = \mathcal{D}(LG_+\backslash LG/LG_+)$ with $\operatorname{Rep} G^\vee$. This is a commutative algebra.

We can take the "spectrum" of this commutative algebra

$$\otimes_{x\in X} \operatorname{Rep} G^\vee = QC(?).$$

It turns out that $? = \operatorname{Loc}_{G^\vee} \Sigma$.

One can define $\operatorname{Spec} \mathcal{C}(R) = \operatorname{Hom}_\otimes(\mathcal{C}, R\text{-mod})$, this is naturally a stack.

Problem: decompose $\mathcal{D}(\operatorname{Bun}_G X)$ as a module for the Hecke symmetries.

$$\mathcal{D}(\operatorname{Bun}_G X) \cong QC(\operatorname{Loc}_{G^\vee} \Sigma).$$

There is a 4-dimensional TFT (due to Kapustin-Witten) Z_G.

$$Z_G(\Sigma) = \mathcal{D}(\operatorname{Bun}_G \Sigma).$$

$$Z_G(S^2) = \mathcal{D}(LG_+\backslash LG/LG_+) = \mathcal{H}_{sph}.$$

Consider the cobordism given by taking the cylinder $\Sigma \times I$ and cutting out a ball. This is a cobordism

$$\Sigma \sqcup S^2 \to \Sigma,$$

which gives the action map

$$\mathcal{D}(\operatorname{Bun}_G \Sigma) \otimes \mathcal{H}_{sph} \to \mathcal{D}(\operatorname{Bun}_G \Sigma).$$

We can move spheres around, this gives an E_3-structure (symmetric monoidal) on the spherical Hecke category.

Arthur-Selberg trace formula, which is a version of the Frobenius character formula. Roughly, it calculates the character of $L^2(\Gamma\backslash G)$ as a G-module.

Let $M = \mathcal{D}(\operatorname{Bun}_G \Sigma)$ as a module over the spherical Hecke algebra. Its character lives in $HH_\bullet(\mathcal{H}_{sph})$.

One can show that $Z(X \times S^1) = \dim Z(X)$. $\mathcal{H}_{aff}$ is conjecturally 2-dualizable and gives a 2d TFT called "affine character theory".

Question: what is $HH_\bullet(\mathcal{D}(\mathrm{Bun}_G\,\Sigma))$ as a plain category? This is the same as $H^\bullet(\mathcal{L}\,\mathrm{Bun}_G\,\Sigma)$. In turn, the loop space space can be identified with the space $\mathcal{H}iggs_G\Sigma$ of G-bundles with an automorphism.

We have the Hitchin map $\mathcal{H}iggs_G\Sigma \xrightarrow{N} \mathrm{Map}(\Sigma, H/W)$. The dimension as a Hecke module of $\mathcal{D}(\mathrm{Bun}_G\,\Sigma)$ is $N_*\mathbf{C}$.

We want to compute $HH_\bullet(\mathcal{H}_{aff})$, i.e. affine character sheaves. In some sense this should be some version of $\mathcal{D}(LG/LG)$.

Instead of working with the usual conjugation, we will look at the q-twisted conjugation, i.e. the fiber over q of conjugacy classes in $LG \rtimes \mathbf{C}^\times$.

A q-twisted quotient LG/LG is the same as a G-bundle on $E_q = \mathbf{C}^\times/q^{\mathbf{Z}}$. So, $HH_\bullet(\mathcal{H}_{aff})$ should have something to do with $Z_G(T^2)$.

Theorem. $HH_\bullet(\mathcal{H}_{aff}) \cong \mathcal{D}_{nil}(\mathrm{Bun}_G\,E)$.

Note, that the category on the right is locally-constant on the moduli space of elliptic curves.

Corollary. *(Nilpotent) Geometric Langlands for an elliptic curve.*

This uses a result of Bezrukavnikov, which identifies $\mathcal{H}_{aff}$ with some gadget on the Langlands dual side.