TRIAGE 2

DAVID BEN-ZVI

Consider the real version of the horocycle correspondence. The upper half-space is $\mathbf{H} = G/K = SL_2\mathbf{R}/SO_2$.

The space of horocycles is G/MN. We have a correspondence

$$\begin{array}{ccc} & G/M & \\ \swarrow & & \searrow \\ G/MN & & G/K \end{array}$$

In the theory of modular forms, the induced integral transform

$$\mathrm{Fun}(G/K) \to \mathrm{Fun}(G/MN)$$

is known as the constant term and its adjoint

$$\mathrm{Fun}(G/MN) \to G/K$$

is the Eisenstein series map.

There is the wonderful compactification $G \subset \overline{G}$, whose boundary is $G/B \times G/B$. Near infinity we have $G/N \times G/N$. If one works G-equivariantly, the boundary we get is $G \backslash (G/N \times G/N)$.

1. Convolution algebras

Consider $X \xrightarrow{p} Y$. Then we have the convolution algebras $QC(X \times_Y X)$ and $\mathcal{D}(X \times_Y X)$.

For example, if $X = \cdot/K$ and $Y = \cdot/G$, then $K\backslash G/K = X \times_Y X$.

Question: is $(QC(X \times_Y X), *)$ Morita equivalent to $(QC(Y), \otimes)$?

Theorem. *Morita equivalence holds whenever X, Y are perfect stacks and $p : X \to Y$ faithfully flat. It is also true if X, Y are smooth and p proper.*

For example, $\mathrm{Vect}\, G$ is Morita equivalent to $\mathrm{Rep}\, G$.

$Z(QC(X \times_Y X)) \cong QC(\mathcal{L}Y)$.

Consider a G-space X. This is the same as $Z \to (\cdot/G)$. Indeed, $X = Z \times_{BG} \cdot$.

$$\mathbf{C}G\text{-mod} \cong \mathrm{Vect}(BG).$$

In the algebrac case one has

$$QC(G)\text{-mod} \cong \text{sheaves of categories over } BG.$$

We also have

$$\mathcal{D}(G)\text{-mod} \cong \text{same with a flat connection.}$$

Finally,

$$C_\bullet(G)\text{-mod} \cong \text{local systems on } BG.$$

One can take global sections (invariants).

Gaitsgory: $QC(G)\text{-mod} \to QC(BG)\text{-mod}$ is an equivalence.

This doesn't work for $\mathcal{D}$-modules.

However, for chains one has (GKM):

$$C_\bullet(G)\text{-mod} \xrightarrow{\sim} C^\bullet(BG)\text{-mod}\,.$$

This is a Koszul duality.

If $\mathcal{H} = \mathcal{D}(B\backslash G/B)$, one has a module structure on $\mathcal{D}(K\backslash G/B)$, which is equivalent to the category of $(\mathfrak{g}, K)$-modules. This is related to the representations of a real form of G.

Soergel conjecture: $\oplus_\sigma \mathcal{D}(K_\sigma\backslash G/B)$ is equivalent to $\oplus_\theta \mathcal{D}(K_\theta\backslash G^\vee/B^\vee)$.

Note, that both sides have actions of the Hecke categories $\mathcal{D}(B\backslash G/B)$ and $\mathcal{D}(B^\vee\backslash G^\vee/B^\vee)$. One can also consider weak quotient by B introduced before.

Theorem (Beilinson-Ginzburg-Soergel, Bezrukavnikov-Yun)**.** *$\mathcal{D}(B\backslash G/B) \xrightarrow{\sim} \mathcal{D}(B^\vee\backslash G^\vee/B^\vee)$.*

That means we have an equivalence of TFTs

$$Z_{\mathcal{H}_G} \cong Z_{\mathcal{H}_{G^\vee}}\,.$$