TRIAGE

DAVID BEN-ZVI

Recall the Grothendieck-Springer resolution:

$$\widetilde{\mathfrak{g}} = \{x \in \mathfrak{g}, \mathfrak{b} \in \mathcal{B} | \mathfrak{b} \ni x\}.$$

$$\begin{array}{ccccc}
T^*G/B & \longrightarrow & \widetilde{\mathfrak{g}} & \longleftarrow & \widetilde{\mathfrak{g}}^{reg} \\
\downarrow & & \downarrow & & \downarrow W \\
\mathcal{N} & \longrightarrow & \mathfrak{g} & \longleftarrow & \mathfrak{g}^{reg} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathfrak{h}/W & \longleftarrow & \mathfrak{h}/W^{reg}
\end{array}$$

The Beilinson-Bernstein theorem gives an equivalence

$$\mathcal{D}(G/B) \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} U\mathfrak{g}_0\text{-mod}.$$

Both categories carry smooth G-actions, i.e. they are module categories over $(\mathcal{D}(G), *)$ (a monoidal dg-category).

$\mathcal{D}(G/B)$ carries a left module structure over $\mathcal{D}(G)$ and a right module structure over $\mathcal{D}(B\backslash G/B) =: \mathcal{H}$. In fact, $\operatorname{End} \mathcal{D}(G/B) = \mathcal{D}(B\backslash G/B)$.

Recall from the first talk that $\mathcal{H}(G, K)$-mod is equivalent to $\mathbf{C}G$-mod generated by their K-invariants, i.e. $\mathbf{C}G$-mod generated by $\mathbf{C}[G/K]$.

Similarly, $\mathcal{H}$-mod is equivalent to the subcategory of $\mathcal{D}(G)$-mod generated by $\mathcal{D}(G/B)$.

Let X be a variety with a G-action. Then $\mathcal{D}(X)^K = \mathcal{D}(K\backslash X)$, which carries an action of $\mathcal{H}(G, K)$.

We get that $\mathcal{D}(K\backslash G/B) = \mathcal{D}(G/B)^K$. By Beilinson-Bernstein we get $\mathcal{D}(G/B)^K \cong (U\mathfrak{g}_0\text{-mod})^K$, which can be identified with $(\mathfrak{g}, K)$-mod.

We can identify the category $\mathcal{O}$ with

$$\mathcal{O}_0 = \mathcal{D}(B\backslash G/B),$$

where the right B-action gives a strongly N-equivariant $\mathcal{D}$-module, which is weakly H-equivariant.

$$\begin{array}{ccc} & G\backslash G\times G/B & \\ & \downarrow \quad \searrow & \\ & G/G \qquad B\backslash G/B & \end{array}$$

The space on the top can be identified with G/B, where B acts by conjugation. It has a subspace $B/B \cong \widetilde{G}/G$. We get

$$\pi_*(\mathbf{C}_{\widetilde{G}/G}) \subset D(G/G),$$

the Grothendieck-Springer sheaf.

To construct a character sheaf, take $\mathcal{D}_{G/G}$. To make the characteristic variety sit inside the nilpotent cone, mod out $\mathcal{D}_{G/G}$ by the set of all G-biinvariant differential operators on G. In this way we get the Harish-Chandra sheaf.

Theorem (Hotta-Kashiwara)**.** *The Grothendieck-Springer sheaf coincides with the Harish-Chandra system sheaf.*