TRIAGE

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Once we know that QC is compactly generated, it becomes easy to prove that

$$QC(X \times Y) = QC(X) \otimes QC(Y) = \mathrm{Fun}(QC(X), QC(Y)).$$

Theorem (Thomason-Trobaugh)**.** *For X a quasi-compact separated schemes $QC(X)$ is compactly generated by perfect objects.*

Let $U = \sqcup U_i \to Z$ an open cover. A functor on Z is a function on $\sqcup U_i$ which agrees on $\sqcup U_{ij}$. We have

$$\sqcup U_{ij} \rightrightarrows \sqcup U_i \to Z.$$

For vector bundles we need to go one step up, for gerbes need to go to steps up.

Consider a map $X \to Y$. Then we have a Čech simplicial object

$$X \times_Y X \times_Y X \rightrightarrows X \times_Y X \rightrightarrows X \to Y.$$

Let $X \to Y$ be a faithfully flat cover. Then we have a descent theorem

$$QC(Y) = \lim QC(X \times_Y \ldots \times_Y X).$$

If a scheme gives a functor from rings to sets, a simplicial scheme will be a functor from rings to simplicial sets (spaces, infinity-grouopoids). Stacks map to the subcategory of ordinary grouopoids.

For example, we can look at the loop space $\mathcal{L}X = \mathrm{Map}(S^1, X)$. Here S^1 is thought of as a simplicial set. For instance, $\mathcal{L}(\cdot/G) = G/G$. $\cdot/G$ has the following Čech nerve:

$$G \times G \times G \rightrightarrows G \times G \rightrightarrows G \rightrightarrows \cdot \to \cdot/G.$$

Going from schemes to stacks corrects colimits. For example, the quotient $\cdot/G$ in schemes is just $\cdot$.

We also want to correct limits. Let $X = \mathrm{Spec}\, R$, then $\mathcal{L}X = \Delta \cap \Delta$. Ring-theoretically we get $R \otimes_{R \otimes R} R$. We derive it to get the Hochschild homology $HH_\bullet(R)$. So,

$$\mathcal{L}X = \mathrm{Spec}\, HH_\bullet(R).$$

To make sense of this definition, we want some kind of derived rings, e.g. dg-rings, simplicial rings etc.

So, a derived stack is a functor from dg-rings to spaces. Now we can rephrase the HKR theorem: $\mathcal{L}X = T_X[-1]$.

One can also define the derived mapping space $\mathrm{Map}(\Sigma, X) = X^\Sigma$ for a simplicial set Σ. For example, $\mathrm{Map}(\Sigma, BG)$ is the space of local systems $\mathrm{Loc}_G\Sigma$.