TRIAGE

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Main theme of yesterday: TFT is an organizing principle for algebra/category theory. It starts as an invariant of n-manifolds.

Cobordism hypothesis allows one to reconstruct a TFT from what it assigns to a point. It says that if you start with a "finite enough" (fully dualizable) object, it uniquely determines an extended n-dimensional TFT.

Since we will be dealing with low-dimensional manifolds M, we think of $Z_A(M)$ as an invariant of A.

Examples of TFTs:

- 1d: V finite-dimensional vector space. $Z_V(S^1)$ is given by
$$1 \stackrel{coev}{\to} V \otimes V^* \stackrel{ev}{\to} 1.$$
This gives $Z_V(S^1) = \dim V$.
- 2d: Alg $\subset$ Cat. In this case we get the diagram
$$1 \stackrel{A}{\to} A \otimes A^{op} \stackrel{A}{\to} 1.$$
Then
$$Z_A(S^1) = A \otimes_{A \otimes A^{op}} A = A/[A, A].$$
For a category we define the Hochschild homology to be what the TFT assigns to S^1. The diagram in this case is
$$\mathrm{Vect} \to \mathrm{End}(\mathcal{C}) \cong \mathcal{C} \otimes \mathcal{C}^{op} \stackrel{\mathrm{tr}}{\to} \mathrm{Vect}\,.$$
If $\mathcal{C} = D^b(X)$ for X smooth projective, then we get the B-model. $Z_{\mathcal{C}}(S^1) = HH_\bullet(X)$.

Hochschild homology is the universal recipient for the trace. Moreover, for any module M
$$\mathrm{End}_A(M) \stackrel{\mathrm{tr}}{\to} HH_\bullet(A)$$
and tr(id) is the character of M.

For example, if $\mathcal{C} = \mathrm{Rep}\, G$ we get $HH_\bullet(\mathcal{C}) = \mathbf{C}[G/G]$

From the cobordism hypothesis we get an extra S^1-action on $HH_\bullet(A)$. Explicitly it is given by Connes B operator.

Theorem (HKR, Quillen)**.** $HH_\bullet(A) \cong \mathrm{Sym}\, T^*_A[1]$.

Consider the σ-model TFT. It studies maps into target X. Then $Z_X(M)$ is the linearization of $\mathrm{Map}(M, X)$.

If A is a commutative algebra, let $X = \operatorname{Spec} A$. Then

$$Z_A(S^1) = \mathrm{Map}(S^1, X) = \mathcal{L}X.$$

In 3d case we study monoidal categories. Alternatively, we look at 2-categories. We think of monoidal categories as algebra objects in $(\infty, 1)$-category of categories.

We also have the center

$$HH^\bullet(A) = Z(A) = \mathrm{Hom}_{A \otimes A^{op}}(A, A) = \mathrm{End}(\mathrm{id}_{A\text{-mod}}).$$

For a category $\mathcal{C}$ we can similarly define

$$Z(\mathcal{C}) = \mathrm{End}(\mathrm{id}_{\mathcal{C}}).$$

For a monoidal category we have the Drinfeld center

$$Z(\mathcal{C}, \otimes) = \mathrm{Hom}_{\mathcal{C}-\mathcal{C}^\vee}(\mathcal{C}, \mathcal{C}) = \mathrm{End}(\mathrm{id}_{\mathcal{C}\text{-mod}}).$$

In TFTs these come from $Z_A(S^1_0)$.

If we consider $SO(2)$ fixed points, we get $HH_\bullet \cong HH^\bullet$.