TRIAGE

DAVID BEN-ZVI

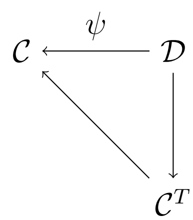
1. Reminder on Barr-Beck

Consider an adjunction

$$\phi : \mathcal{C} \rightleftarrows \mathcal{D} : \psi.$$

ϕ preserves all colimits (coproducts, coequalizers, ...), ψ preserves all limits (products, equalizers).

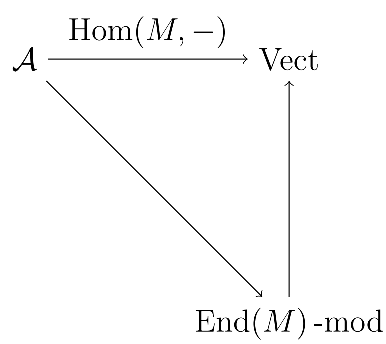
Adjoint functor theorem: in a good setting, preserving all colimits is equivalent to being a left adjoint.



The monad is $T = \psi\phi$. For $\mathcal{D} \to \mathcal{C}^T$ to be an equivalence, we need

- ψ is conservative, i.e. if $\psi(M \xrightarrow{f} N)$ is an iso, f is an iso. In the abelian or stable setting, can simply require $\psi M = 0 \Rightarrow M = 0$.
- ψ preserves some colimits (for example all)

Example: let $\mathcal{A}$ be an abelian category and $M \in \mathcal{A}$. Consider the functor $\mathcal{A} \to \mathrm{Vect}$ given by $\mathrm{Hom}(M, -)$.



For Barr-Beck to hold we need:

- $\mathcal{M}$ is a compact generator

- $\mathcal{M}$ is projective.

2. ∞-categories

It is a homotopical version of a category. There are notions of limits, colimits, adjunctions, adjoint functor theorem, Barr-Beck. Linear version: stable ∞-categories. It has a notion of algebras, modules, tensor and Hom of categories.

Dwyer-Kan construction: consider a category with a notion of weak equivalences. The construction produces a simplicial category $\mathcal{C}[W^{-1}]$. Morphisms are given by zig-zags, where half of the maps are weak equivalences.

The same thing happens with dg-category. In the usual derived category,

$$\mathrm{Hom}_{D(\mathcal{A})}(C^\bullet, D^\bullet) = H^\bullet(\mathrm{Hom}_{dg}(C^\bullet, D^\bullet)).$$

Dold-Kan: chain complexes are simplicial abelian groups. Fancy version: (pretriangulated) dg-categories are the same as $\mathbf{Z}$-linear stable ∞-categories.

dg-categories are linear (homological). But they don't form a dg-category. Need to talk about equivalence of categories, $\otimes$ and Hom, monoidal dg-categories...