A SEMI-SMALL DECOMPOSITION OF THE CHOW RING OF A MATROID

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ABSTRACT. We give a semi-small orthogonal decomposition of the Chow ring of a matroid M. The decomposition is used to give simple proofs of Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations for the Chow ring, recovering the main result of [AHK18]. We also introduce the augmented Chow ring of M and show that a similar semi-small orthogonal decomposition holds for the augmented Chow ring.

1. INTRODUCTION

A *matroid* M on a finite set E is a nonempty collection of subsets of E, called *flats* of M, that satisfies the following properties:

(1) The intersection of any two flats is a flat.

(2) For any flat F, any element in $E \setminus F$ is contained in exactly one flat that is minimal among the flats strictly containing F.

Throughout, we suppose in addition that M is a *loopless* matroid:

(3) The empty subset of E is a flat.

We write $\mathcal{L}(\mathrm{M})$ for the lattice of all flats of M. Every maximal flag of proper flats of M has the same cardinality d, called the *rank* of M. A matroid can be equivalently defined in terms of its *independent sets* or the *rank function*. For background in matroid theory, we refer to [Oxl11] and [Wel76].

The first aim of the present paper is to decompose the Chow ring of M as a module over the Chow ring of the deletion $\mathrm{M} \setminus i$ (Theorem 1.2). The decomposition resembles the decomposition of the cohomology ring of a projective variety induced by a semi-small map. In Section 4, we use the decomposition to give a simple proof of the Kähler package for the Chow ring: Poincaré duality,

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the hard Lefschetz theorem, and the Hodge–Riemann relations. This recovers the main result of [AHK18].

The second aim of the present paper is to introduce the augmented Chow ring of M, which contains the graded Möbius algebra of M as a subalgebra. We give an analogous semi-small decomposition of the augmented Chow ring of M as a module over the augmented Chow ring of the deletion $\mathrm{M}\backslash i$ (Theorem 1.5), and use this to prove the Kähler package for the augmented Chow ring. These results play a major role in the follow-up paper [BHM$^+$], where we prove the Top-Heavy conjecture along with the nonnegativity of the coefficients of the Kazhdan–Lusztig polynomial of a matroid.

Remark 1.1. The main objects of study in [BHM$^+$] are combinatorial abstractions of intersection cohomology groups of singular algebraic varieties. In contrast, the objects of study in this paper are combinatorial abstractions of cohomology groups (or Chow rings) of smooth projective varieties.

1.1. Let $\underline{S}_\mathrm{M}$ be the ring of polynomials with variables labeled by the nonempty proper flats of M:

$$\underline{S}_\mathrm{M} \coloneqq \mathbb{Q}[x_F \,|\, F \text{ is a nonempty proper flat of } \mathrm{M}].$$

The *Chow ring* of M, introduced by Feichtner and Yuzvinsky in [FY04], is the quotient algebra[1]

$$\underline{\mathrm{CH}}(\mathrm{M}) \coloneqq \underline{S}_\mathrm{M}/(\underline{I}_\mathrm{M} + \underline{J}_\mathrm{M}),$$

where $\underline{I}_\mathrm{M}$ is the ideal generated by the linear forms

$$\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F, \quad \text{for every pair of distinct elements } i_1 \text{ and } i_2 \text{ of } E,$$

and $\underline{J}_\mathrm{M}$ is the ideal generated by the quadratic monomials

$$x_{F_1} x_{F_2}, \quad \text{for every pair of incomparable nonempty proper flats } F_1 \text{ and } F_2 \text{ of } \mathrm{M}.$$

When E is nonempty, the Chow ring of M admits a *degree map*

$$\underline{\deg}_\mathrm{M} : \underline{\mathrm{CH}}^{d-1}(\mathrm{M}) \longrightarrow \mathbb{Q}, \qquad x_\mathcal{F} \coloneqq \prod_{F \in \mathcal{F}} x_F \longmapsto 1,$$

where $\mathcal{F}$ is any complete flag of nonempty proper flats of M (Definition 2.15). For any integer k, the degree map defines the *Poincaré pairing*

$$\underline{\mathrm{CH}}^k(\mathrm{M}) \times \underline{\mathrm{CH}}^{d-k-1}(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1, \eta_2) \longmapsto \underline{\deg}_\mathrm{M}(\eta_1 \eta_2).$$

[1] A slightly different presentation for the Chow ring of M was used in [FY04] in a more general context. The present description was used in [AHK18], where the Chow ring of M was denoted $A(\mathrm{M})$. For a comparison of the two presentations, see [BES].

If M is realizable over a field,[2] then the Chow ring of M is isomorphic to the Chow ring of a smooth projective variety over the field (Remark 2.16).

Let i be an element of E, and let $\mathrm{M}\backslash i$ be the *deletion* of i from M. By definition, $\mathrm{M}\backslash i$ is the matroid on $E\backslash i$ whose flats are the sets of the form $F\backslash i$ for a flat F of M. The Chow rings of M and $\mathrm{M}\backslash i$ are related by the graded algebra homomorphism

$$\underline{\theta}_i = \underline{\theta}_i^{\mathrm{M}} : \underline{\mathrm{CH}}(\mathrm{M}\backslash i) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M}), \qquad x_F \longmapsto x_F + x_{F\cup i},$$

where a variable in the target is set to zero if its label is not a flat of M. As we will see in Section 3, this homomorphism is induced by a projection from the Bergman fan of M to the Bergman fan of $\mathrm{M}\backslash i$. Let $\underline{\mathrm{CH}}_{(i)}$ be the image of the homomorphism $\underline{\theta}_i$, and let $\underline{\mathcal{S}}_i$ be the collection

$$\underline{\mathcal{S}}_i = \underline{\mathcal{S}}_i(\mathrm{M}) = \big\{F \,|\, F \text{ is a nonempty proper subset of } E\backslash i \text{ such that } F \in \mathcal{L}(\mathrm{M}) \text{ and } F \cup i \in \mathcal{L}(\mathrm{M})\big\}.$$

The element i is said to be a *coloop* of M if the ranks of M and $\mathrm{M}\backslash i$ are not equal. Thus, $\underline{\mathcal{S}}_i$ is the collection of all nonempty proper subsets F of $E\backslash i$ such that $F \cup i$ is a flat of M and i is a coloop in $\mathrm{M}^{F\cup i}$.

Theorem 1.2. If i is not a coloop of M, there is a direct sum decomposition of $\underline{\mathrm{CH}}(\mathrm{M})$ into indecomposable graded $\underline{\mathrm{CH}}(\mathrm{M}\backslash i)$-modules

$$\underline{\mathrm{CH}}(\mathrm{M}) = \underline{\mathrm{CH}}_{(i)} \oplus \bigoplus_{F\in\underline{\mathcal{S}}_i} x_{F\cup i}\underline{\mathrm{CH}}_{(i)}. \tag{$\underline{\mathrm{D}}_1$}$$

All pairs of distinct summands are orthogonal for the Poincaré pairing of $\underline{\mathrm{CH}}(\mathrm{M})$. If i is a coloop of M, there is a direct sum decomposition of $\underline{\mathrm{CH}}(\mathrm{M})$ into indecomposable graded $\underline{\mathrm{CH}}(\mathrm{M}\backslash i)$-modules[3]

$$\underline{\mathrm{CH}}(\mathrm{M}) = \underline{\mathrm{CH}}_{(i)} \oplus x_{E\backslash i}\underline{\mathrm{CH}}_{(i)} \oplus \bigoplus_{F\in\underline{\mathcal{S}}_i} x_{F\cup i}\underline{\mathrm{CH}}_{(i)}. \tag{$\underline{\mathrm{D}}_2$}$$

All pairs of distinct summands except for the first two are orthogonal for the Poincaré pairing of $\underline{\mathrm{CH}}(\mathrm{M})$.

We write $\mathrm{rk}_{\mathrm{M}} : 2^E \to \mathbb{N}$ for the rank function of M. For any proper flat F of M, we set[4]

$\mathrm{M}^F \coloneqq$ the localization of M at F, a loopless matroid on F of rank equal to $\mathrm{rk}_{\mathrm{M}}(F)$,

$\mathrm{M}_F \coloneqq$ the contraction of M by F, a loopless matroid on $E\backslash F$ of rank equal to $d - \mathrm{rk}_{\mathrm{M}}(F)$.

[2]We say that M is *realizable* over a field $\mathbb{F}$ if there exists a linear subspace $V \subseteq \mathbb{F}^E$ such that $S \subseteq E$ is independent if and only if the projection from V to $\mathbb{F}^S$ is surjective. Almost all matroids are not realizable over any field [Nel18].

[3]When $E = \{i\}$, we treat the symbol $x_\varnothing$ as zero in the right-hand side of ($\underline{\mathrm{D}}_2$).

[4]The symbols M^F and M_F appear inconsistently in the literature, sometimes this way and sometimes interchanged. The localization is frequently called the restriction. On the other hand, the contraction is also sometimes called the restriction, especially in the context of hyperplane arrangements, so we avoid the word restriction to minimize ambiguity.

The lattice of flats of M^F can be identified with the lattice of flats of M that are contained in F, and the lattice of flats of M_F can be identified with the lattice of flats of M that contain F. The $\underline{\mathrm{CH}}(\mathrm{M}\backslash i)$-module summands in the decompositions ($\underline{\mathrm{D}}_1$) and ($\underline{\mathrm{D}}_2$) admit isomorphisms

$$\underline{\mathrm{CH}}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}\backslash i) \;\text{ and }\; x_{F\cup i}\underline{\mathrm{CH}}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)[-1],\text{[5]}$$

(Propositions 3.4 and 3.5). In addition, if i is a coloop of M, then

$$x_{E\backslash i}\underline{\mathrm{CH}}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}\backslash i)[-1].$$

Remark 1.3. When M is the Boolean matroid on E, the graded dimension of $\underline{\mathrm{CH}}(\mathrm{M})$ is given by the Eulerian numbers $\left\langle {d \atop k} \right\rangle$, and the decomposition ($\underline{\mathrm{D}}_2$) specializes to the known quadratic recurrence relation

$$s_d(t) = s_{d-1}(t) + t\sum_{k=0}^{d-2}\binom{d-1}{k} s_k(t)s_{d-k-1}(t), \qquad s_0(t)=1,$$

where $s_k(t)$ is the k-th Eulerian polynomial [Pet15, Theorem 1.5].

1.2. We also give similar decompositions for the augmented Chow ring of M, which we now introduce. Let S_M be the ring of polynomials in two sets of variables

$$S_\mathrm{M} \coloneqq \mathbb{Q}[y_i \,|\, i \text{ is an element of } E] \,\otimes\, \mathbb{Q}[x_F \,|\, F \text{ is a proper flat of } \mathrm{M}].$$

The *augmented Chow ring* of M is the quotient algebra

$$\mathrm{CH}(\mathrm{M}) \coloneqq S_\mathrm{M}/(I_\mathrm{M} + J_\mathrm{M}),$$

where I_M is the ideal generated by the linear forms

$$y_i - \sum_{i\notin F} x_F, \quad \text{for every element } i \text{ of } E,$$

and J_M is the ideal generated by the quadratic monomials

$$x_{F_1}x_{F_2}, \quad \text{for every pair of incomparable proper flats } F_1 \text{ and } F_2 \text{ of } \mathrm{M}, \text{ and}$$
$$y_i\, x_F, \quad \text{for every element } i \text{ of } E \text{ and every proper flat } F \text{ of } \mathrm{M} \text{ not containing } i.$$

The augmented Chow ring of M admits a *degree map*

$$\deg_\mathrm{M} : \mathrm{CH}^d(\mathrm{M}) \longrightarrow \mathbb{Q}, \qquad x_{\mathcal{F}} \coloneqq \prod_{F\in\mathcal{F}} x_F \longmapsto 1,$$

where $\mathcal{F}$ is any complete flag of proper flats of M (Definition 2.15). For any integer k, the degree map defines the *Poincaré pairing*

$$\mathrm{CH}^k(\mathrm{M}) \times \mathrm{CH}^{d-k}(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1,\eta_2) \longmapsto \deg_\mathrm{M}(\eta_1\eta_2).$$

[5]For a graded vector space V, we write $V[m]$ for the graded vector space whose degree k piece is equal to V^{k+m}.

If M is realizable over a field, then the augmented Chow ring of M is isomorphic to the Chow ring of a smooth projective variety over the field (Remark 2.16).

Remark 1.4. The subring of the augmented Chow ring generated by the elements y_i is isomorphic to the *graded Möbius algebra* $\mathrm{H}(\mathrm{M})$ (Proposition 2.18), and we have isomorphisms

$$\underline{\mathrm{CH}}(\mathrm{M}) \cong \mathrm{CH}(\mathrm{M})/\mathrm{ideal}(y_i)_{i\in E} \cong \mathrm{CH}(\mathrm{M}) \otimes_{\mathrm{H}(\mathrm{M})} \mathbb{Q}.$$

The $\mathrm{H}(\mathrm{M})$-module structure of $\mathrm{CH}(\mathrm{M})$ will be studied in detail in the forthcoming paper [BHM+]. In this paper, the graded Möbius algebra will not appear outside of Proposition 2.18.

As before, we write $\mathrm{M}\backslash i$ for the matroid obtained from M by deleting the element i. The augmented Chow rings of M and $\mathrm{M}\backslash i$ are related by the graded algebra homomorphism

$$\theta_i = \theta_i^{\mathrm{M}} : \mathrm{CH}(\mathrm{M}\backslash i) \longrightarrow \mathrm{CH}(\mathrm{M}), \qquad x_F \longmapsto x_F + x_{F\cup i},$$

where a variable in the target is set to zero if its label is not a flat of M. As we will see in Section 3, this homomorphism is induced by a projection from the augmented Bergman fan of M to the augmented Bergman fan of $\mathrm{M}\backslash i$. Let $\mathrm{CH}_{(i)}$ be the image of the homomorphism θ_i, and let $\mathcal{S}_i$ be the collection

$$\mathcal{S}_i = \mathcal{S}_i(\mathrm{M}) \coloneqq \big\{F \,|\, F \text{ is a proper subset of } E\backslash i \text{ such that } F \in \mathcal{L}(\mathrm{M}) \text{ and } F \cup i \in \mathcal{L}(\mathrm{M})\big\}.$$

Equivalently, $\mathcal{S}_i$ can be defined as the collection of all proper subsets F of $E\backslash i$ such that $F \cup i$ is a flat of M and i is a coloop in $\mathrm{M}^{F\cup i}$.

Theorem 1.5. If i is not a coloop of M, there is a direct sum decomposition of $\mathrm{CH}(\mathrm{M})$ into indecomposable graded $\mathrm{CH}(\mathrm{M}\backslash i)$-modules

$$\mathrm{CH}(\mathrm{M}) = \mathrm{CH}_{(i)} \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\mathrm{CH}_{(i)}. \tag{D$_1$}$$

All pairs of distinct summands are orthogonal for the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$. If i is a coloop of M, there is a direct sum decomposition of $\mathrm{CH}(\mathrm{M})$ into indecomposable graded $\mathrm{CH}(\mathrm{M}\backslash i)$-modules

$$\mathrm{CH}(\mathrm{M}) = \mathrm{CH}_{(i)} \oplus x_{E\backslash i}\mathrm{CH}_{(i)} \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i}\mathrm{CH}_{(i)}. \tag{D$_2$}$$

All pairs of distinct summands except for the first two are orthogonal for the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$.

The $\mathrm{CH}(\mathrm{M}\backslash i)$-module summands in the decompositions (D$_1$) and (D$_2$) admit isomorphisms

$$\mathrm{CH}_{(i)} \cong \mathrm{CH}(\mathrm{M}\backslash i) \ \text{ and } \ x_{F\cup i}\mathrm{CH}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \mathrm{CH}(\mathrm{M}^F)[-1],$$

(Propositions 3.4 and 3.5). In addition, if i is a coloop of M,

$$x_{E\backslash i}\mathrm{CH}_{(i)} \cong \mathrm{CH}(\mathrm{M}\backslash i)[-1].$$

1.3. Let B be the Boolean matroid on E. By definition, every subset of E is a flat of B. The Chow rings of B and M are related by the surjective graded algebra homomorphism

$$\underline{\mathrm{CH}}(\mathrm{B}) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M}), \qquad x_S \longmapsto x_S,$$

where a variable in the target is set to zero if its label is not a flat of M. Similarly, we have a surjective graded algebra homomorphism

$$\mathrm{CH}(\mathrm{B}) \longrightarrow \mathrm{CH}(\mathrm{M}), \qquad x_S \longmapsto x_S,$$

where a variable in the target is set to zero if its label is not a flat of M. As in [AHK18, Section 4], we may identify the Chow ring $\underline{\mathrm{CH}}(\mathrm{B})$ with the ring of piecewise polynomial functions modulo linear functions on the normal fan $\underline{\Pi}_{\mathrm{B}}$ of the standard permutohedron in $\mathbb{R}^E$. Similarly, the augmented Chow ring $\mathrm{CH}(\mathrm{B})$ can be identified with the ring of piecewise polynomial functions modulo linear functions of the normal fan Π_{B} of the stellahedron in $\mathbb{R}^E$ (Definition 2.4). A convex piecewise linear function on a complete fan is said to be *strictly convex* if there is a bijection between the cones in the fan and the faces of the graph of the function.

In Section 4, we use Theorems 1.2 and 1.5 to give simple proofs of Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations for $\underline{\mathrm{CH}}(\mathrm{M})$ and $\mathrm{CH}(\mathrm{M})$.

Theorem 1.6. Let $\underline{\ell}$ be a strictly convex piecewise linear function on $\underline{\Pi}_{\mathrm{B}}$, viewed as an element of $\underline{\mathrm{CH}}^1(\mathrm{M})$.

(1) (Poincaré duality theorem) For every nonnegative integer $k < \frac{d}{2}$, the bilinear pairing

$$\underline{\mathrm{CH}}^k(\mathrm{M}) \times \underline{\mathrm{CH}}^{d-k-1}(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1, \eta_2) \longmapsto \underline{\deg}_{\mathrm{M}}(\eta_1\eta_2)$$

is nondegenerate.

(2) (Hard Lefschetz theorem) For every nonnegative integer $k < \frac{d}{2}$, the multiplication map

$$\underline{\mathrm{CH}}^k(\mathrm{M}) \longrightarrow \underline{\mathrm{CH}}^{d-k-1}(\mathrm{M}), \quad \eta \longmapsto \underline{\ell}^{d-2k-1}\eta$$

is an isomorphism.

(3) (Hodge–Riemann relations) For every nonnegative integer $k < \frac{d}{2}$, the bilinear form

$$\underline{\mathrm{CH}}^k(\mathrm{M}) \times \underline{\mathrm{CH}}^k(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1, \eta_2) \longmapsto (-1)^k \underline{\deg}_{\mathrm{M}}(\underline{\ell}^{d-2k-1}\eta_1\eta_2)$$

is positive definite on the kernel of multiplication by $\underline{\ell}^{d-2k}$.

Let ℓ be a strictly convex piecewise linear function on Π_{B}, viewed as an element of $\mathrm{CH}^1(\mathrm{M})$.

(4) (Poincaré duality theorem) For every nonnegative integer $k \leqslant \frac{d}{2}$, the bilinear pairing

$$\mathrm{CH}^k(\mathrm{M}) \times \mathrm{CH}^{d-k}(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1, \eta_2) \longmapsto \deg_{\mathrm{M}}(\eta_1\eta_2)$$

is nondegenerate.

(5) (Hard Lefschetz theorem) For every nonnegative integer $k \leqslant \frac{d}{2}$, the multiplication map

$$\mathrm{CH}^k(\mathrm{M}) \longrightarrow \mathrm{CH}^{d-k}(\mathrm{M}), \quad \eta \longmapsto \ell^{d-2k}\eta$$

is an isomorphism.

(6) (Hodge–Riemann relations) For every nonnegative integer $k \leqslant \frac{d}{2}$, the bilinear form

$$\mathrm{CH}^k(\mathrm{M}) \times \mathrm{CH}^k(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad (\eta_1, \eta_2) \longmapsto (-1)^k \deg_{\mathrm{M}}(\ell^{d-2k}\eta_1\eta_2)$$

is positive definite on the kernel of multiplication by ℓ^{d-2k+1}.

Theorem 1.6 holds non-vacuously, as there are strictly convex piecewise linear functions on Π_{B} and $\underline{\Pi}_{\mathrm{B}}$ (Proposition 2.6). The first part of Theorem 1.6 on $\underline{\mathrm{CH}}(\mathrm{M})$ recovers the main result of [AHK18].[6] The second part of Theorem 1.6 on $\mathrm{CH}(\mathrm{M})$ is new.

Remark 1.7. By Remark 2.10 and Proposition 2.12, we can reformulate Theorem 1.6 as follows: Both the Bergman fan and the augmented Bergman fan satisfy Poincaré duality, and they satisfy the hard Lefschetz theorem and the Hodge–Riemann relations with respect to any strictly convex piecewise linear function.

1.4. In Section 5, we use Theorems 1.2 and 1.5 to obtain decompositions of $\underline{\mathrm{CH}}(\mathrm{M})$ and $\mathrm{CH}(\mathrm{M})$ related to those appearing in [AHK18, Theorem 6.18]. Let $\underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M})$ be the subalgebra of $\underline{\mathrm{CH}}(\mathrm{M})$ generated by the element

$$\underline{\alpha}_{\mathrm{M}} \coloneqq \sum_{i \in G} x_G \in \underline{\mathrm{CH}}^1(\mathrm{M}),$$

where the sum is over all nonempty proper flats G of M containing a given element i in E, and let $\mathrm{H}_{\alpha}(\mathrm{M})$ be the subalgebra of $\mathrm{CH}(\mathrm{M})$ generated by the element

$$\alpha_{\mathrm{M}} \coloneqq \sum_{G} x_G \in \mathrm{CH}^1(\mathrm{M}),$$

where the sum is over all proper flats G of M. We define graded subspaces $\underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M})$ and $\mathrm{J}_{\alpha}(\mathrm{M})$ by

$$\underline{\mathrm{J}}^k_{\underline{\alpha}}(\mathrm{M}) \coloneqq \begin{cases} \underline{\mathrm{H}}^k_{\underline{\alpha}}(\mathrm{M}) & \text{if } k \neq d-1, \\ 0 & \text{if } k = d-1, \end{cases} \qquad \mathrm{J}^k_{\alpha}(\mathrm{M}) \coloneqq \begin{cases} \mathrm{H}^k_{\alpha}(\mathrm{M}) & \text{if } k \neq d, \\ 0 & \text{if } k = d. \end{cases}$$

A degree computation shows that the elements $\underline{\alpha}^{d-1}_{\mathrm{M}}$ and α^d_{M} are nonzero (Proposition 2.32). We will construct an injective $\underline{\mathrm{CH}}(\mathrm{M})$-module homomorphism (Propositions 2.25 and 2.27)

$$\underline{\psi}^F_{\mathrm{M}} : \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M}), \quad \prod_{F'} x_{F' \setminus F} \otimes \prod_{F''} x_{F''} \longmapsto x_F \prod_{F'} x_{F'} \prod_{F''} x_{F''},$$

[6] Independent proofs of Poincaré duality for $\underline{\mathrm{CH}}(\mathrm{M})$ were given in [BES] and [BDF]. The authors of [BES] also prove the degree 1 Hodge–Riemann relations for $\underline{\mathrm{CH}}(\mathrm{M})$.

and an injective $\mathrm{CH}(\mathrm{M})$-module homomorphism (Propositions 2.21 and 2.23)

$$\psi^F_{\mathrm{M}} : \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \longrightarrow \mathrm{CH}(\mathrm{M}), \quad \prod_{F'} x_{F' \setminus F} \otimes \prod_{F''} x_{F''} \longmapsto x_F \prod_{F'} x_{F'} \prod_{F''} x_{F''}.$$

Theorem 1.8. Let $\mathcal{Q} = \mathcal{Q}(\mathrm{M})$ be the set of all nonempty proper flats of M, and let $\underline{\mathcal{Q}} = \underline{\mathcal{Q}}(\mathrm{M})$ be the set of all proper flats of M with rank at least two.

(1) We have a decomposition of $\underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M})$-modules

$$\underline{\mathrm{CH}}(\mathrm{M}) = \underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M}) \oplus \bigoplus_{F \in \underline{\mathcal{Q}}} \underline{\psi}^F_{\mathrm{M}}\, \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F). \tag{$\underline{\mathrm{D}}_3$}$$

All pairs of distinct summands are orthogonal for the Poincaré pairing of $\underline{\mathrm{CH}}(\mathrm{M})$.

(2) We have a decomposition of $\mathrm{H}_\alpha(\mathrm{M})$-modules

$$\mathrm{CH}(\mathrm{M}) = \mathrm{H}_\alpha(\mathrm{M}) \oplus \bigoplus_{F \in \mathcal{Q}} \psi^F_{\mathrm{M}}\, \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{J}_\alpha(\mathrm{M}^F). \tag{D_3}$$

All pairs of distinct summands are orthogonal for the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$.

Remark 1.9. The decomposition ($\underline{\mathrm{D}}_3$) and the decomposition induced by [AHK18, Theorem 6.18] are isomorphic as decompositions of graded vector spaces, but the latter one is not an orthogonal decomposition. In each case, applying the decomposition to M_F for all F, we get a basis of $\underline{\mathrm{CH}}(\mathrm{M})$ that is permuted by any automorphism of M. For example, when M is a matroid of rank 5, the decomposition ($\underline{\mathrm{D}}_3$) in degree 3 gives

$$\underline{\mathrm{CH}}^3(\mathrm{M}) = \mathbb{Q}\underline{\alpha}^3_{\mathrm{M}} \oplus \Big(\bigoplus_{\mathrm{rk}\, F=2} \mathbb{Q}\, \underline{\psi}^F_{\mathrm{M}}(\underline{\alpha}^2_{\mathrm{M}_F} \otimes 1) \Big) \oplus \Big(\bigoplus_{\mathrm{rk}\, F=3} \mathbb{Q}\, \underline{\psi}^F_{\mathrm{M}}(\underline{\alpha}_{\mathrm{M}_F} \otimes \underline{\alpha}_{\mathrm{M}^F}) \Big) \oplus \Big(\bigoplus_{\mathrm{rk}\, F=4} \mathbb{Q}\, \underline{\psi}^F_{\mathrm{M}}(1 \otimes \underline{\alpha}^2_{\mathrm{M}^F}) \Big),$$

while the decomposition [AHK18, Theorem 6.18] reads

$$\underline{\mathrm{CH}}^3(\mathrm{M}) = \mathbb{Q}\underline{\alpha}^3_{\mathrm{M}} \oplus \Big(\bigoplus_{\mathrm{rk}\, F=2} \mathbb{Q}\, \underline{\psi}^F_{\mathrm{M}}(\underline{\alpha}^2_{\mathrm{M}_F} \otimes 1) \Big) \oplus \Big(\bigoplus_{\mathrm{rk}\, F=3} \mathbb{Q}\, x_F\, \underline{\psi}^F_{\mathrm{M}}(1 \otimes \underline{\alpha}_{\mathrm{M}^F}) \Big) \oplus \Big(\bigoplus_{\mathrm{rk}\, F=4} \mathbb{Q}\, x_F^2\, \underline{\psi}^F_{\mathrm{M}}(1 \otimes 1) \Big).$$

Only the former is orthogonal to the common decomposition in the complimentary degree

$$\underline{\mathrm{CH}}^1(\mathrm{M}) = \mathbb{Q}\underline{\alpha}_{\mathrm{M}} \oplus \Big(\bigoplus_{\mathrm{rk}\, F=2} \mathbb{Q} x_F \Big) \oplus \Big(\bigoplus_{\mathrm{rk}\, F=3} \mathbb{Q} x_F \Big) \oplus \Big(\bigoplus_{\mathrm{rk}\, F=4} \mathbb{Q} x_F \Big).$$

In general, the basis of $\underline{\mathrm{CH}}(\mathrm{M})$ given by the decomposition ($\underline{\mathrm{D}}_3$) is different from the ones in [FY04, Corollary 1] and [BES, Corollary 3.3.3].

Remark 1.10. When M is the Boolean matroid on E, the decomposition ($\underline{\mathrm{D}}_3$) specializes to a linear recurrence relation for the Eulerian polynomials

$$0 = 1 + \sum_{k=0}^{d} \binom{d}{k} \frac{t - t^{d-k}}{1-t} s_k(t), \qquad s_0(t) = 1.$$

1.5. For realizable matroids, the Chow ring, the augmented Chow ring, and the Möbius algebra are indeed Chow rings (or cohomology rings when realized over $\mathbb{C}$) of certain algebraic varieties. We explain the geometric motivations of the paper in the remaining part of the introduction.

First, we recall the definition and some relevant properties of semi-small maps. Let $f : X \to Y$ be a map between smooth complex projective varieties. For every integer k, we say that the map f is *semi-small* if there is no irreducible subvariety $T \subseteq X$ such that $2 \dim T - \dim f(T) > \dim X$. For example, if f is the blowup along a smooth center Z, then f is semi-small if and only if $Z \subseteq Y$ is of codimension at most two. Semi-small maps play an essential role in the proof of the decomposition theorem by de Cataldo and Migliorini [dCM02, dCM09]. Using the language of the decomposition theorem, a map $f : X \to Y$ of smooth projective varieties is semi-small if and only if the derived pushforward $Rf_*(\mathbb{Q}_X[\dim X])$ is a perverse sheaf on Y. If f is semi-small, then for any ample class A in $H^2(Y;\mathbb{Q})$, its pullback $f^*(A)$ behaves like an ample class on X [dCM02, Propositions 2.2.7 and 2.3.1]. More precisely, the cohomology ring $H^\bullet(X;\mathbb{Q})$ satisfies the hard Lefschetz theorem and the Hodge–Riemann relations with respect to $f^*(A)$.

When M is realized by a hyperplane arrangement $\mathcal{A} = \{H_i\}_{i\in E}$ in the projectivization of a d-dimensional vector space V over a field $\mathbb{F}$, there are smooth projective varieties $X_{\mathcal{A}}$ and $\underline{X}_{\mathcal{A}}$ over $\mathbb{F}$ whose Chow rings are isomorphic to $\underline{\mathrm{CH}}(\mathrm{M})$ and $\mathrm{CH}(\mathrm{M})$ respectively, and we call them the wonderful model and the augmented wonderful model of $\mathcal{A}$. The wonderful model $\underline{X}_{\mathcal{A}}$ was used in [HK12] to prove the log-concavity of the coefficients of the characteristic polynomial of realizable matroids.

Let us recall the construction of $\underline{X}_{\mathcal{A}}$. For any nonempty proper subset $S \subseteq E$, we set

$$H_S = \bigcap_{i\in S} H_i \quad \text{and} \quad H_S^\circ = H_S \setminus \bigcup_{S\subsetneq T} H_T.$$

Then H_S° is nonempty if and only if S is a nonempty proper flat of M, and in this case, the dimension of H_S° is equal to $\mathrm{crk}(S) - 1$. The divisor $\bigcup_{i\in E} H_i$ admits a stratification $\bigcup_{i\in E} H_i = \bigsqcup_F H_F^\circ$, where the disjoint union is over all nonempty proper flats F of M. To construct the wonderful model $\underline{X}_{\mathcal{A}}$, we first blow up the points H_F in $\mathbb{P}(V)$ for all corank one flats F, then we blow up the strict transforms of H_F for corank two flats F, and so on. The resulting smooth projective variety is $\underline{X}_{\mathcal{A}}$. See Remark 2.16 for an alternative description of the wonderful model.

Denote by $\mathcal{A}\backslash i$ the hyperplane arrangement obtained from $\mathcal{A}$ by deleting the hyperplane H_i. Then there is regular map $\underline{X}_{\mathcal{A}} \to \underline{X}_{\mathcal{A}\backslash i}$, which models the decompositions ($\underline{\mathrm{D}}_1$) and ($\underline{\mathrm{D}}_2$). When i is not a coloop of M, the map is semi-small because it can be written as a sequence of blowups of smooth subvarieties of codimension two parametrized by $\underline{\mathcal{S}}_i$. In fact, for any F in $\underline{\mathcal{S}}_i$, when the strict transform of H_F is blown up in the construction of $\underline{X}_{\mathcal{A}\backslash i}$, the preimage of $H^\circ_{F\cup i}$ is of codimension two. The wonderful model $\underline{X}_{\mathcal{A}}$ can be obtained from $\underline{X}_{\mathcal{A}\backslash i}$ by blowing up the closure of the preimage of $H^\circ_{F\cup i}$ for all F in $\underline{\mathcal{S}}_i$. When i is a coloop, the map $\underline{X}_{\mathcal{A}} \to \underline{X}_{\mathcal{A}\backslash i}$ is generically a $\mathbb{P}^1$

bundle, which corresponds to the first two summands $\underline{\mathrm{CH}}_{(i)}$ and $x_{E\setminus i}\underline{\mathrm{CH}}_{(i)}$ in the decomposition ($\underline{\mathrm{D}}_2$).

The decomposition ($\underline{\mathrm{D}}_3$) is modeled by the composition of all the blowup maps $\underline{X}_{\mathcal{A}} \to \mathbb{P}(V)$ in the construction of $\underline{X}_{\mathcal{A}}$. Moreover, the class $\underline{\alpha}$ in $\underline{\mathrm{CH}}^1(\mathrm{M}_{\mathcal{A}})$ is equal to the pullback of the hyperplane class of $\mathbb{P}(V)$, via the identification of $\underline{\mathrm{CH}}(\mathrm{M}_{\mathcal{A}})$ and the Chow ring of $\underline{X}_{\mathcal{A}}$. For each i in E, choose a linear form f_i that defines the hyperplane H_i. Then the rational map

$$\left[\frac{1}{f_i}\right]_{i\in E} : \mathbb{P}(V) \dashrightarrow \mathbb{P}(\mathbb{F}^E)$$

extends to a regular map $\underline{X}_{\mathcal{A}} \to \mathbb{P}(\mathbb{F}^E)$. The pullback of the hyperplane class of $\mathbb{P}(\mathbb{F}^E)$ is equal to the class $\underline{\beta}$, which will be defined in Section 2.6. The image of $\underline{X}_{\mathcal{A}}$ in $\mathbb{P}(\mathbb{F}^E)$ is the projective reciprocal plane, which is singular in general, and its intersection cohomology groups are closely related to the Kazhdan–Lusztig polynomial of M (see [EPW16]). The decomposition of $\underline{\mathrm{CH}}(\mathrm{M})$ as a module over the subalgebra $\mathbb{Q}[\underline{\beta}]$ involves intersection cohomology groups of singular projective varieties and will be studied in [BHM$^+$].

To construct the augmented wonderful model $X_{\mathcal{A}}$, we start with the projective space

$$\mathbb{P}(V \oplus \mathbb{F}) = V \cup \mathbb{P}(V).$$

For any nonempty proper flat F, we consider the previously defined H_F as a linear subspace of $\mathbb{P}(V)$. Similar to the construction of $\underline{X}_{\mathcal{A}}$, we first blow up all H_F in $\mathbb{P}(V)$ for all corank one flats F, then we blow up the strict transforms of H_F for all corank two flats F. We continue this process until we blow up H_F for all rank one flats F. Since all the blowup centers are in the hyperplane at infinity, the affine space V remains an open subset of $X_{\mathcal{A}}$. Moreover, the strict transform of the hyperplane at infinity is exactly isomorphic to $\underline{X}_{\mathcal{A}}$, which explains the pullback map $\varphi^{\varnothing}_{\mathrm{M}} : \mathrm{CH}(\mathrm{M}) \to \underline{\mathrm{CH}}(\mathrm{M})$ and the pushforward map $\psi^{\varnothing}_{\mathrm{M}} : \underline{\mathrm{CH}}(\mathrm{M}) \to \mathrm{CH}(\mathrm{M})$.

For any i in E, there is a regular map $X_{\mathcal{A}} \to X_{\mathcal{A}\setminus i}$ which models the decompositions (D_1) and (D_2). Moreover, when i is not a coloop, the map $X_{\mathcal{A}} \to X_{\mathcal{A}\setminus i}$ is semi-small. The decomposition (D_3) is modeled by the composition of all blowup maps $X_{\mathcal{A}} \to \mathbb{P}(V \oplus \mathbb{F})$, and the class α is equal to the pullback of the hyperplane class, via identifying $\mathrm{CH}(\mathrm{M})$ with the Chow ring of $X_{\mathcal{A}}$.[7] There is a natural map $X_{\mathcal{A}} \to (\mathbb{P}^1)^E$, which induces the classes y_i for i in E. In fact, the linear map

$$(f_i)_{i\in E} : V \to \mathbb{F}^E$$

extends to a regular map $X_{\mathcal{A}} \to (\mathbb{P}^1)^E$. The image of the map $X_{\mathcal{A}} \to (\mathbb{P}^1)^E$ is the variety $Y_{\mathcal{A}}$ in [HW17], now called the *Schubert variety* of $\mathcal{A}$. The operational Chow ring of $Y_{\mathcal{A}}$ is isomorphic to $\mathrm{H}(\mathrm{M})$. The decomposition of $\mathrm{CH}(\mathrm{M})$ as an $\mathrm{H}(\mathrm{M})$-module involves the intersection cohomology groups of singular varieties, and will be studied in [BHM$^+$].

[7] There is no natural map from $X_{\mathcal{A}}$ to $\mathbb{P}(\mathbb{F}^E)$ or to $\mathbb{P}(\mathbb{F}^E \oplus \mathbb{F})$, and hence the symbol β_{M} will not be defined.

The decomposition theorem for proper toric maps was studied in [dCMM18], and the combinatorial generalization to fans was studied in [Kar19]. Since the Bergman fan and the augmented Bergman fan are not complete, our results are of a different nature.

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2. The Chow ring and the augmented Chow ring of a matroid

In this section, we collect the various properties of the algebras $\underline{\mathrm{CH}}(\mathrm{M})$ and $\mathrm{CH}(\mathrm{M})$ that we will need in order to prove Theorems 1.2–1.8. In Section 2.1, we review the definition and basic properties of the *Bergman fan* and introduce the closely related *augmented Bergman fan* of a matroid. Section 2.2 is devoted to understanding the stars of the various rays in these two fans, while Section 2.3 is where we compute the space of balanced top-dimensional weights on each fan. Feichtner and Yuzvinsky showed that the Chow ring of a matroid coincides with the Chow ring of the toric variety associated with its Bergman fan [FY04, Theorem 3], and we establish the analogous result for the augmented Chow ring in Section 2.4. Section 2.5 is where we show that the augmented Chow ring contains the graded Möbius algebra. In Section 2.6, we use the results of Section 2.2 to construct various homomorphisms that relate the Chow and augmented Chow rings of different matroids.

Remark 2.1. It is worth noting why we need to interpret $\underline{\mathrm{CH}}(\mathrm{M})$ and $\mathrm{CH}(\mathrm{M})$ as Chow rings of toric varieties. First, the study of balanced weights on the Bergman fan and augmented Bergman fan allow us to show that $\underline{\mathrm{CH}}^{d-1}(\mathrm{M})$ and $\mathrm{CH}^{d}(\mathrm{M})$ are nonzero, which is not easy to prove directly from the definitions. The definition of the pullback and pushforward maps in Section 2.6 is made cleaner by thinking about fans, though it would also be possible to define these maps by taking Propositions 2.20, 2.21, 2.24, 2.25, 2.28, and 2.29 as definitions. Finally, and most importantly, the fan perspective will be essential for understanding the ample classes that appear in Theorem 1.6.

2.1. Fans.

Let E be a finite set, and let M be a loopless matroid of rank d on the ground set E. We write rk_M for the rank function of M, and write cl_M for the closure operator of M, which for a set S returns the smallest flat containing S. The *independence complex* I_M of M is the simplicial complex of independent sets of M. A set $I \subseteq E$ is independent if and only if the rank of $\mathrm{cl}_\mathrm{M}(I)$ is $|I|$. The vertices of I_M are the elements of the ground set E, and a collection of vertices is a face of I_M when the corresponding set of elements is an independent set of M. The *Bergman complex* $\underline{\Delta}_\mathrm{M}$ of M is the order complex of the poset of nonempty proper flats of M. The vertices of $\underline{\Delta}_\mathrm{M}$ are the nonempty proper flats of M, and a collection of vertices is a face of $\underline{\Delta}_\mathrm{M}$ when the corresponding set of flats is a flag. The independence complex of M is pure of dimension $d-1$, and the Bergman complex of M is pure of dimension $d-2$. For a detailed study of the simplicial complexes I_M and $\underline{\Delta}_\mathrm{M}$, we refer to [Bjö92]. We introduce the *augmented Bergman complex* Δ_{M} of M as a simplicial complex that interpolates between the independence complex and the Bergman complex of M.

Definition 2.2. Let I be an independent set of M, and let $\mathcal{F}$ be a flag of proper flats of M. When I is contained in every flat in $\mathcal{F}$, we say that I is *compatible* with $\mathcal{F}$ and write $I \leqslant \mathcal{F}$. The *augmented Bergman complex* Δ_{M} of M is the simplicial complex of all compatible pairs $I \leqslant \mathcal{F}$, where I is an independent set of M and $\mathcal{F}$ is a flag of proper flats of M.

A vertex of the augmented Bergman complex Δ_{M} is either a singleton subset of E or a proper flat of M. More precisely, the vertices of Δ_{M} are the compatible pairs either of the form $\{i\} \leqslant \varnothing$ or of the form $\varnothing \leqslant \{F\}$, where i is an element of E and F is a proper flat of M. The augmented Bergman complex contains both the independence complex I_{M} and the Bergman complex $\underline{\Delta}_{\mathrm{M}}$ as subcomplexes. In fact, Δ_{M} contains the order complex of the poset of proper flats of M, which is the cone over the Bergman complex with the cone point corresponding to the empty flat. It is straightforward to check that Δ_{M} is pure of dimension $d-1$.

Proposition 2.3. The Bergman complex and the augmented Bergman complex of M are both connected in codimension 1.

Proof. The statement about the Bergman complex is a direct consequence of its shellability [Bjö92]. We prove the statement about the augmented Bergman complex using the statement about the Bergman complex.

The claim is that, given any two facets of Δ_{M}, one may travel from one facet to the other by passing through faces of codimension at most 1. Since the Bergman complex of M is connected in codimension 1, the subcomplex of Δ_{M} consisting of faces of the form $\varnothing \leqslant \mathcal{F}$ is connected in codimension 1. Thus it suffices to show that any facet of Δ_{M} can be connected to a facet of the form $\varnothing \leqslant \mathcal{F}$ through codimension 1 faces.

Let $I \leqslant \mathcal{F}$ be a facet of Δ_{M}. If I is nonempty, choose any element i of I, and consider the flag of flats $\mathcal{G}$ obtained by adjoining the closure of $I \backslash i$ to $\mathcal{F}$. The independent set $I \backslash i$ is compatible with the flag $\mathcal{G}$, and the facet $I \leqslant \mathcal{F}$ is adjacent to the facet $I \backslash i \leqslant \mathcal{G}$. Repeating the procedure, we can connect the given facet to a facet of the desired form through codimension 1 faces. □

Let $\mathbb{R}^E$ be the vector space spanned by the standard basis vectors $\mathbf{e}_i$ corresponding to the elements $i \in E$. For an arbitrary subset $S \subseteq E$, we set

$$\mathbf{e}_S \coloneqq \sum_{i \in S} \mathbf{e}_i.$$

For an element $i \in E$, we write ρ_i for the ray generated by the vector $\mathbf{e}_i$ in $\mathbb{R}^E$. For a subset $S \subseteq E$, we write ρ_S for the ray generated by the vector $-\mathbf{e}_{E \backslash S}$ in $\mathbb{R}^E$, and write $\underline{\rho}_S$ for the image of this ray in $\mathbb{R}^E/\langle \mathbf{e}_E\rangle$, which is generated by the image of $\mathbf{e}_S$.[8] Using these rays, we construct fan models of the Bergman complex and the augmented Bergman complex as follows.

Definition 2.4. The *Bergman fan* $\underline{\Pi}_{\mathrm{M}}$ of M is a simplicial fan in the quotient space $\mathbb{R}^E/\langle \mathbf{e}_E\rangle$ with rays $\underline{\rho}_F$ for nonempty proper flats F of M. The cones of $\underline{\Pi}_{\mathrm{M}}$ are of the form

$$\underline{\sigma}_{\mathscr{F}} \coloneqq \mathrm{cone}\{\mathbf{e}_F\}_{F\in\mathscr{F}} = \mathrm{cone}\{-\mathbf{e}_{E\setminus F}\}_{F\in\mathscr{F}},$$

where $\mathscr{F}$ is a flag of nonempty proper flats of M.

The *augmented Bergman fan* Π_{M} of M is a simplicial fan in $\mathbb{R}^E$ with rays ρ_i for elements i in E and ρ_F for proper flats F of M. The cones of the augmented Bergman fan are of the form

$$\sigma_{I\leqslant\mathscr{F}} \coloneqq \mathrm{cone}\{\mathbf{e}_i\}_{i\in I} + \mathrm{cone}\{-\mathbf{e}_{E\setminus F}\}_{F\in\mathscr{F}},$$

where $\mathscr{F}$ is a flag of proper flats of M and I is an independent set of M compatible with $\mathscr{F}$. We write σ_I for the cone $\sigma_{I\leqslant\mathscr{F}}$ when $\mathscr{F}$ is the empty flag of flats of M.

Remark 2.5. If E is nonempty, then the Bergman fan $\underline{\Pi}_{\mathrm{M}}$ is the star of the ray $\rho_\varnothing$ in the augmented Bergman fan Π_{M}. If E is empty, then $\underline{\Pi}_{\mathrm{M}}$ and Π_{M} both consist of a single 0-dimensional cone.

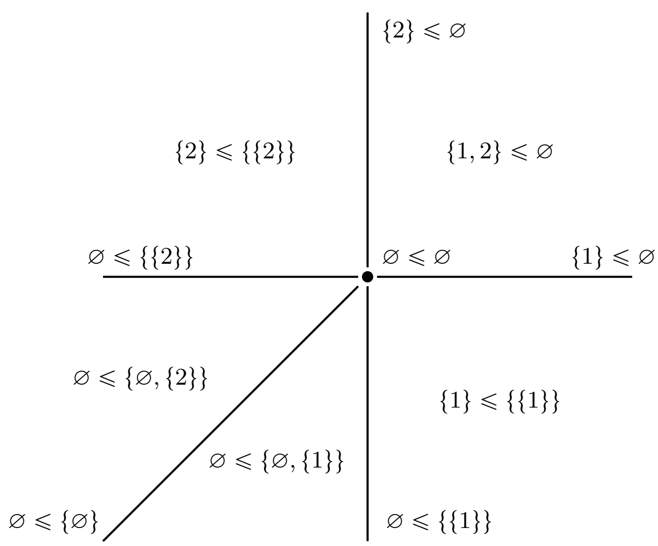

FIGURE 1. The augmented Bergman fan of the rank 2 Boolean matroid on $\{1, 2\}$.

Let N be another loopless matroid on E. The matroid M is said to be a *quotient* of N if every flat of M is a flat of N. The condition implies that every independent set of M is an independent set of N [Kun86, Proposition 8.1.6]. Therefore, when M is a quotient of N, the augmented Bergman fan

[8]The reason why ρ_S is the correct lift of $\underline{\rho}_S$ for our purposes will become clear in Section 2.2.

of M is a subfan of the augmented Bergman fan of N, and the Bergman fan of M is a subfan of the Bergman fan of N. In particular, we have inclusions of fans $\Pi_{\mathrm{M}} \subseteq \Pi_{\mathrm{B}}$ and $\underline{\Pi}_{\mathrm{M}} \subseteq \underline{\Pi}_{\mathrm{B}}$, where B is the *Boolean matroid* on E defined by the condition that E is an independent set of B.

Proposition 2.6. The Bergman fan and the augmented Bergman fan of B are each normal fans of convex polytopes. In particular, there are strictly convex piecewise linear functions on $\underline{\Pi}_{\mathrm{B}}$ and Π_{B}.

The above proposition can be used to show that the augmented Bergman fan and the Bergman fan of M are, in fact, fans.

Proof. The statement for the Bergman fan is well-known: The Bergman fan of B is the normal fan of the standard permutohedron in $\mathbf{e}_E^{\perp} \subseteq \mathbb{R}^E$. See, for example, [AHK18, Section 2]. The statement for the augmented Bergman fan Π_{B} follows from the fact that it is an iterated stellar subdivision of the normal fan of the simplex

$$\mathrm{conv}\{\mathbf{e}_i, \mathbf{e}_E\}_{i \in E} \subseteq \mathbb{R}^E.$$

More precisely, Π_{B} is isomorphic to the fan $\Sigma_{\mathcal{P}}$ in [AHK18, Definition 2.3], where $\mathcal{P}$ is the order filter of all subsets of $E \cup 0$ containing the new element 0, via the linear isomorphism

$$\mathbb{R}^E \longrightarrow \mathbb{R}^{E \cup 0}/\langle \mathbf{e}_E + \mathbf{e}_0 \rangle, \quad \mathbf{e}_j \longmapsto \mathbf{e}_j.$$

It is shown in [AHK18, Proposition 2.4] that $\Sigma_{\mathcal{P}}$ is an iterated stellar subdivision of the normal fan of the simplex.[9] □

A direct inspection shows that Π_{M} is a *unimodular fan*; that is, the set of primitive ray generators in any cone in Π_{M} is a subset of a basis of the free abelian group $\mathbb{Z}^E$. It follows that $\underline{\Pi}_{\mathrm{M}}$ is also a unimodular fan; that is, the set of primitive ray generators in any cone in Π_{M} is a subset of a basis of the free abelian group $\mathbb{Z}^E/\langle \mathbf{e}_E \rangle$.

2.2. **Stars.** For any element i of E, we write $\mathrm{cl}(i)$ for the closure of i in M, and write ι_i for the injective linear map

$$\iota_i : \mathbb{R}^{E \setminus \mathrm{cl}(i)} \longrightarrow \mathbb{R}^E/\langle \mathbf{e}_i \rangle, \qquad \mathbf{e}_j \longmapsto \mathbf{e}_j.$$

For any proper flat F of M, we write ι_F for the linear isomorphism

$$\iota_F : \mathbb{R}^{E \setminus F}/\langle \mathbf{e}_{E \setminus F} \rangle \oplus \mathbb{R}^F \longrightarrow \mathbb{R}^E/\langle \mathbf{e}_{E \setminus F} \rangle, \qquad \mathbf{e}_j \longmapsto \mathbf{e}_j.$$

For any nonempty proper flat F of M, we write $\underline{\iota}_F$ for the linear isomorphism

$$\underline{\iota}_F : \mathbb{R}^{E \setminus F}/\langle \mathbf{e}_{E \setminus F} \rangle \oplus \mathbb{R}^F/\langle \mathbf{e}_F \rangle \longrightarrow \mathbb{R}^E/\langle \mathbf{e}_E, \mathbf{e}_{E \setminus F} \rangle, \qquad \mathbf{e}_j \longmapsto \mathbf{e}_j.$$

[9] In fact, the augmented Bergman fan Π_{B} is the normal fan of the stellahedron in $\mathbb{R}^E$, the graph associahedron of the star graph with $|E|$ endpoints. We refer to [CD06] and [Dev09] for detailed discussions of graph associahedra and their realizations. An explicit description of their normal fans that motivated Definition 2.4 can be found in [FS05, Theorem 3.14] and [Pos09, Theorem 7.4].

Let M^F be the localization of M at F, and let M_F be the contraction of M by F.

Proposition 2.7. The following are descriptions of the stars of the rays in Π_{M} and $\underline{\Pi}_{\mathrm{M}}$ using the three linear maps above.

(1) For any element $i \in E$, the linear map ι_i identifies the augmented Bergman fan of $\mathrm{M}_{\mathrm{cl}(i)}$ with the star of the ray ρ_i in the augmented Bergman fan of M:
$$\Pi_{\mathrm{M}_{\mathrm{cl}(i)}} \cong \mathrm{star}_{\rho_i}\Pi_{\mathrm{M}}.$$
(2) For any proper flat F of M, the linear map ι_F identifies the product of the Bergman fan of M_F and the augmented Bergman fan of M^F with the star of the ray ρ_F in the augmented Bergman fan of M:
$$\underline{\Pi}_{\mathrm{M}_F} \times \Pi_{\mathrm{M}^F} \cong \mathrm{star}_{\rho_F}\Pi_{\mathrm{M}}.$$
(3) For any nonempty proper flat F of M, the linear map $\underline{\iota}_F$ identifies the product of the Bergman fan of M_F and the Bergman fan of M^F with the star of the ray $\underline{\rho}_F$ in the Bergman fan of M:
$$\underline{\Pi}_{\mathrm{M}_F} \times \underline{\Pi}_{\mathrm{M}^F} \cong \mathrm{star}_{\underline{\rho}_F}\underline{\Pi}_{\mathrm{M}}.$$

Repeated applications of the first statement show that, for any independent set I of M, the star of the cone σ_I in Π_{M} can be identified with the augmented Bergman fan of $\mathrm{M}_{\mathrm{cl}(I)}$, where $\mathrm{cl}(I)$ is the closure of I in M.

Proof. The first statement follows from the following facts: A flat of M contains i if and only if it contains $\mathrm{cl}(i)$, and an independent set of M containing i does not contain any other element in $\mathrm{cl}(i)$. The second and third statements follow directly from the definitions. □

2.3. **Weights.** For any simplicial fan Σ, we write Σ_k for the set of k-dimensional cones in Σ. If τ is a codimension 1 face of a cone σ, we write
$$\mathrm{e}_{\sigma/\tau} \coloneqq \text{the primitive generator of the unique ray in } \sigma \text{ that is not in } \tau.$$
A *k-dimensional balanced weight* on Σ is a $\mathbb{Q}$-valued function ω on Σ_k that satisfies the *balancing condition*: For every $(k-1)$-dimensional cone τ in Σ,
$$\sum_{\tau\subset\sigma} \omega(\sigma)\mathrm{e}_{\sigma/\tau} \text{ is contained in the subspace spanned by } \tau,$$
where the sum is over all k-dimensional cones σ containing τ. We write $\mathrm{MW}_k(\Sigma)$ for the group of k-dimensional balanced weights on Σ.

Proposition 2.8. The Bergman fan and the augmented Bergman fan of M have the following unique balancing property.

(1) A $(d-1)$-dimensional weight on $\underline{\Pi}_{\mathrm{M}}$ is balanced if and only if it is constant.

(2) A d-dimensional weight on Π_{M} is balanced if and only if it is constant.

Proof. The first statement is [AHK18, Proposition 5.2]. We prove the second statement.

Let $\sigma_{I\leqslant\mathcal{F}}$ be a codimension 1 cone of Π_{M}, and let F be the smallest flat in $\mathcal{F}\cup\{E\}$. We analyze the primitive generators of the rays in the star of the cone $\sigma_{I\leqslant\mathcal{F}}$ in Π_{M}. Let $\mathrm{cl}(I)$ be the closure of I in M. There are two cases.

When the closure of I is not F, the primitive ray generators in question are $-\mathbf{e}_{E\setminus\mathrm{cl}(I)}$ and $\mathbf{e}_i$, for elements i in F not in the closure of I. The primitive ray generators satisfy the relation

$$-\mathbf{e}_{E\setminus\mathrm{cl}(I)}+\sum_{i\in F\setminus\mathrm{cl}(I)}\mathbf{e}_i=-\mathbf{e}_{E\setminus F},$$

which is zero modulo the span of $\sigma_{I\leqslant\mathcal{F}}$. As the $\mathbf{e}_i$'s are independent modulo the span of $\sigma_{I\leqslant\mathcal{F}}$, any relation between the primitive generators must be a multiple of the displayed one.

When the closure of I is F, the fact that $\sigma_{I\leqslant\mathcal{F}}$ has codimension 1 implies that there is a unique integer k with $\mathrm{rk}\,F<k<\mathrm{rk}\,\mathrm{M}$ such that $\mathcal{F}$ does not include a flat of rank k. Let $F_\circ$ be the unique flat in $\mathcal{F}$ of rank $k-1$, and let F° be the unique flat in $\mathcal{F}\cup\{E\}$ of rank $k+1$. The primitive ray generators in question are $-\mathbf{e}_{E\setminus G}$ for the flats G in $\mathcal{G}$, where $\mathcal{G}$ is the set of flats of M covering $F_\circ$ and covered by F°. By the flat partition property of matroids [Oxl11, Section 1.4], the primitive ray generators satisfy the relation

$$\sum_{G\in\mathcal{G}}-\mathbf{e}_{E\setminus G}=-(|\mathcal{G}|-1)\mathbf{e}_{E\setminus F_\circ}-\mathbf{e}_{E\setminus F^\circ},$$

which is zero modulo the span of $\sigma_{I\leqslant\mathcal{F}}$. Since any proper subset of the primitive generators $-\mathbf{e}_{E\setminus G}$ for G in $\mathcal{G}$ is independent modulo the span of $\sigma_{I\leqslant\mathcal{F}}$, any relation between the primitive generators must be a multiple of the displayed one.

The local analysis above shows that any constant d-dimensional weight on Π_{M} is balanced. Since Π_{M} is connected in codimension 1 by Proposition 2.3, it also shows that any d-dimensional balanced weight on Π_{M} must be constant. □

Remark 2.9. The definition of Bergman fan and augmented Bergman fan generalizes to any atomic lattice. The above balancing condition is equivalent to the flat partition property: For any flat F, any element in $E\setminus F$ is contained in exactly one flat that is minimal among the flats strictly containing F.

2.4. **Chow rings.** Any unimodular fan Σ in $\mathbb{R}^E$ defines a graded commutative algebra $\mathrm{CH}(\Sigma)$, which is the Chow ring of the associated smooth toric variety X_Σ over $\mathbb{C}$ with rational coefficients. Equivalently, $\mathrm{CH}(\Sigma)$ is the ring of continuous piecewise polynomial functions on Σ with rational coefficients modulo the ideal generated by globally linear functions [Bri96, Section 3.1]. We write

$\mathrm{CH}^k(\Sigma)$ for the Chow group of codimension k cycles in X_Σ, so that

$$\mathrm{CH}(\Sigma) = \bigoplus_k \mathrm{CH}^k(\Sigma).$$

The group of k-dimensional balanced weights on Σ is related to $\mathrm{CH}^k(\Sigma)$ by the isomorphism

$$\mathrm{MW}_k(\Sigma) \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(\mathrm{CH}^k(\Sigma), \mathbb{Q}), \quad \omega \longmapsto (x_\sigma \longmapsto \omega(\sigma)),$$

where x_σ is the class of the torus orbit closure in X_Σ corresponding to a k-dimensional cone σ in Σ. See [AHK18, Section 5] for a detailed discussion. For general facts on toric varieties and Chow rings, and for any undefined terms, we refer to [CLS11] and [Ful98].

Remark 2.10. For any simplicial fan Σ, we will say that Σ satisfies the hard Lefschetz theorem or the Hodge–Riemann relations with respect to some piecewise linear function on Σ if the ring $\mathrm{CH}(\Sigma)$ satisfies the hard Lefschetz theorem or the Hodge–Riemann relations with respect to the corresponding element of $\mathrm{CH}^1(\Sigma)$.

In Proposition 2.12 below, we show that the Chow ring of M coincides with $\mathrm{CH}(\underline{\Pi}_{\mathrm{M}})$ and that the augmented Chow ring of M coincides with $\mathrm{CH}(\Pi_{\mathrm{M}})$.

Lemma 2.11. The following identities hold in the augmented Chow ring $\mathrm{CH}(\mathrm{M})$.

(1) For any element i of E, we have $y_i^2 = 0$.

(2) For any two bases I_1 and I_2 of a flat F of M, we have $\prod_{i\in I_1} y_i = \prod_{i\in I_2} y_i$.

(3) For any dependent set J of M, we have $\prod_{j\in J} y_j = 0$.

Proof. The first identity is a straightforward consequence of the relations in I_{M} and J_{M}:

$$y_i^2 = y_i\Big(\sum_{i\notin F} x_F\Big) = 0.$$

For the second identity, we may assume that $I_1 \backslash I_2 = \{i_1\}$ and $I_2 \backslash I_1 = \{i_2\}$, by the basis exchange property of matroids and an induction on the size of the symmetric difference between I_1 and I_2. Since a flat of M contains I_1 if and only if it contains I_2, we have

$$\Big(\sum_{i_1\in G} x_G\Big)\prod_{i\in I_1\cap I_2} y_i = \Big(\sum_{I_1\subseteq G} x_G\Big)\prod_{i\in I_1\cap I_2} y_i = \Big(\sum_{I_2\subseteq G} x_G\Big)\prod_{i\in I_1\cap I_2} y_i = \Big(\sum_{i_2\in G} x_G\Big)\prod_{i\in I_1\cap I_2} y_i.$$

This immediately implies that we also have

$$\Big(\sum_{i_1\notin G} x_G\Big)\prod_{i\in I_1\cap I_2} y_i = \Big(\sum_{i_2\notin G} x_G\Big)\prod_{i\in I_1\cap I_2} y_i,$$

which tells us that

$$\prod_{i\in I_1} y_i = y_{i_1}\prod_{i\in I_1\cap I_2} y_i = \Big(\sum_{i_1\notin G} x_G\Big)\prod_{i\in I_1\cap I_2} y_i = \Big(\sum_{i_2\notin G} x_G\Big)\prod_{i\in I_1\cap I_2} y_i = y_{i_2}\prod_{i\in I_1\cap I_2} y_i = \prod_{i\in I_2} y_i.$$

For the third identity, we may suppose that J is a circuit, that is, a minimal dependent set. Since M is a loopless matroid, we may choose distinct elements j_1 and j_2 from J. Note that the independent sets $J \backslash j_1$ and $J \backslash j_2$ have the same closure because J is a circuit. Therefore, by the second identity, we have

$$\prod_{j \in J \backslash j_1} y_j = \prod_{j \in J \backslash j_2} y_j.$$

Combining the above with the first identity, we get

$$\prod_{j \in J} y_j = y_{j_1} \prod_{j \in J \backslash j_1} y_j = y_{j_1} \prod_{j \in J \backslash j_2} y_j = y_{j_1}^2 \prod_{j \in J \backslash \{j_1, j_2\}} y_j = 0.$$ □

By the second identity in Lemma 2.11, we may define

$$y_F \coloneqq \prod_{i \in I} y_i \ \text{ in } \mathrm{CH}(\mathrm{M})$$

for any flat F of M and any basis I of F. The element y_E will play the role of the fundamental class for the augmented Chow ring of M.

Proposition 2.12. We have isomorphisms

$$\underline{\mathrm{CH}}(\mathrm{M}) \cong \mathrm{CH}(\underline{\Pi}_{\mathrm{M}}) \qquad \text{and} \qquad \mathrm{CH}(\mathrm{M}) \cong \mathrm{CH}(\Pi_{\mathrm{M}}).$$

Proof. The first isomorphism is proved in [FY04, Theorem 3]; see also [AHK18, Section 5.3].

Let K_{M} be the ideal of S_{M} generated by the monomials $\prod_{j \in J} y_j$ for every dependent set J of M. The ring of continuous piecewise polynomial functions on Π_{M} is isomorphic to the Stanley–Reisner ring of Δ_{M}, which is equal to

$$S_{\mathrm{M}}/(J_{\mathrm{M}} + K_{\mathrm{M}}).$$

The ring $\mathrm{CH}(\Pi_{\mathrm{M}})$ is obtained from this ring by killing the linear forms that generate the ideal I_{M}. In other words, we have a surjective homomorphism

$$\mathrm{CH}(\mathrm{M}) \coloneqq S_{\mathrm{M}}/(I_{\mathrm{M}} + J_{\mathrm{M}}) \longrightarrow S_{\mathrm{M}}/(I_{\mathrm{M}} + J_{\mathrm{M}} + K_{\mathrm{M}}) \cong \mathrm{CH}(\Pi_{\mathrm{M}}).$$

The fact that this is an isomorphism follows from the third part of Lemma 2.11. □

Remark 2.13. By Proposition 2.12, the graded dimension of the Chow ring of the rank d Boolean matroid $\underline{\mathrm{CH}}(\mathrm{B})$ is given by the h-vector of the permutohedron in $\mathbb{R}^E$. In other words, we have

$$\dim \underline{\mathrm{CH}}^k(\mathrm{B}) = \text{the Eulerian number } \left\langle {d \atop k} \right\rangle.$$

See [Pet15, Section 9.1] for more on permutohedra and Eulerian numbers.

Remark 2.14. If E is nonempty, we have the balanced weight

$$1 \in \mathrm{MW}_{d-1}(\underline{\Pi}_{\mathrm{M}}) \cong \mathrm{Hom}_{\mathbb{Q}}(\underline{\mathrm{CH}}^{d-1}(\mathrm{M}), \mathbb{Q}),$$

which can be used to define a degree map on the Chow ring of M. Similarly, for any E,

$$1 \in \mathrm{MW}_{d}(\Pi_{\mathrm{M}}) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathrm{CH}^{d}(\mathrm{M}), \mathbb{Q})$$

can be used to define a degree map on the augmented Chow ring of M.

Definition 2.15. Consider the following *degree maps* for the Chow ring and the augmented Chow ring of M.

(1) If E is nonempty, the degree map for $\underline{\mathrm{CH}}(\mathrm{M})$ is the linear map

$$\underline{\deg}_{\mathrm{M}} : \underline{\mathrm{CH}}^{d-1}(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad x_{\mathscr{F}} \longmapsto 1,$$

where $x_{\mathscr{F}}$ is any monomial corresponding to a maximal cone $\underline{\sigma}_{\mathscr{F}}$ of $\underline{\Pi}_{\mathrm{M}}$.

(2) For any E, the degree map for $\mathrm{CH}(\mathrm{M})$ is the linear map

$$\deg_{\mathrm{M}} : \mathrm{CH}^{d}(\mathrm{M}) \longrightarrow \mathbb{Q}, \quad x_{I \leqslant \mathscr{F}} \longmapsto 1,$$

where $x_{I \leqslant \mathscr{F}}$ is any monomial corresponding to a maximal cone $\sigma_{I \leqslant \mathscr{F}}$ of Π_{M}.

By Proposition 2.8, the degree maps are well-defined and are isomorphisms. It follows that, for any two maximal cones $\underline{\sigma}_{\mathscr{F}_1}$ and $\underline{\sigma}_{\mathscr{F}_2}$ of the Bergman fan of M,

$$x_{\mathscr{F}_1} = x_{\mathscr{F}_2} \ \text{ in } \underline{\mathrm{CH}}^{d-1}(\mathrm{M}).$$

Similarly, for any two maximal cones $\sigma_{I_1 \leqslant \mathscr{F}_1}$ and $\sigma_{I_2 \leqslant \mathscr{F}_2}$ of the augmented Bergman fan of M,

$$y_{F_1} x_{\mathscr{F}_1} = y_{F_2} x_{\mathscr{F}_2} \ \text{ in } \mathrm{CH}^{d}(\mathrm{M}),$$

where F_1 is the closure of I_1 in M and F_2 is the closure of I_2 in M. Proposition 2.12 shows that

$$\underline{\mathrm{CH}}^{k}(\mathrm{M}) = 0 \ \text{ for } k \geqslant d \ \text{ and } \ \mathrm{CH}^{k}(\mathrm{M}) = 0 \ \text{ for } k > d.$$

Remark 2.16. Let $\mathbb{F}$ be a field, and let V be a d-dimensional linear subspace of $\mathbb{F}^E$. We suppose that the subspace V is not contained in $\mathbb{F}^S \subseteq \mathbb{F}^E$ for any proper subset S of E. Let B be the Boolean matroid on E, and let M be the loopless matroid on E defined by

$$S \text{ is an independent set of } \mathrm{M} \Longleftrightarrow \text{the restriction to } V \text{ of the projection } \mathbb{F}^E \to \mathbb{F}^S \text{ is surjective.}$$

Let $\mathbb{P}(\mathbb{F}^E)$ be the projective space of lines in $\mathbb{F}^E$, and let $\mathbb{T}_E$ be its open torus. For any proper flat F of M, we write H_F for the projective subspace

$$H_F := \big\{ p \in \mathbb{P}(V) \,|\, p_i = 0 \text{ for all } i \in F \big\}.$$

The *wonderful variety* $\underline{X}_V$ is obtained from $\mathbb{P}(V)$ by first blowing up H_F for every corank 1 flat F, then blowing up the strict transforms of H_F for every corank 2 flat F, and so on. Equivalently,

$$\begin{aligned}\underline{X}_V &= \text{the closure of } \mathbb{P}(V) \cap \underline{\mathbb{T}}_E \text{ in the toric variety } \underline{X}_{\mathrm{M}} \text{ defined by } \underline{\Pi}_{\mathrm{M}} \\ &= \text{the closure of } \mathbb{P}(V) \cap \underline{\mathbb{T}}_E \text{ in the toric variety } \underline{X}_{\mathrm{B}} \text{ defined by } \underline{\Pi}_{\mathrm{B}}.\end{aligned}$$

When E is nonempty, the inclusion $\underline{X}_V \subseteq \underline{X}_{\mathrm{M}}$ induces an isomorphism between their Chow rings,[10] and hence the Chow ring of $\underline{X}_V$ is isomorphic to $\underline{\mathrm{CH}}(\mathrm{M})$ [FY04, Corollary 2].

Let $\mathbb{P}(\mathbb{F}^E \oplus \mathbb{F})$ be the projective completion of $\mathbb{F}^E$, and let $\mathbb{T}_E$ be its open torus. The projective completion $\mathbb{P}(V \oplus \mathbb{F})$ contains a copy of $\mathbb{P}(V)$ as the hyperplane at infinity, and it therefore contains a copy of H_F for every nonempty proper flat F. The *augmented wonderful variety* X_V is obtained from $\mathbb{P}(V \oplus \mathbb{F}^1)$ by first blowing up H_F for every corank 1 flat F, then blowing up the strict transforms of H_F for every corank 2 flat F, and so on. Equivalently,

$$\begin{aligned}X_V &= \text{the closure of } \mathbb{P}(V \oplus \mathbb{F}) \cap \mathbb{T}_E \text{ in the toric variety } X_{\mathrm{M}} \text{ defined by } \Pi_{\mathrm{M}} \\ &= \text{the closure of } \mathbb{P}(V \oplus \mathbb{F}) \cap \mathbb{T}_E \text{ in the toric variety } X_{\mathrm{B}} \text{ defined by } \Pi_{\mathrm{B}}.\end{aligned}$$

The inclusion $X_V \subseteq X_{\mathrm{M}}$ induces an isomorphism between their Chow rings, and hence the Chow ring of X_V is isomorphic to $\mathrm{CH}(\mathrm{M})$.[11]

2.5. **The graded Möbius algebra.** For any nonnegative integer k, we define a vector space

$$\mathrm{H}^k(\mathrm{M}) \coloneqq \bigoplus_{F \in \mathcal{L}^k(\mathrm{M})} \mathbb{Q} y_F,$$

where the direct sum is over the set $\mathcal{L}^k(\mathrm{M})$ of rank k flats of M.

Definition 2.17. The *graded Möbius algebra* of M is the graded vector space

$$\mathrm{H}(\mathrm{M}) \coloneqq \bigoplus_{k \geqslant 0} \mathrm{H}^k(\mathrm{M}).$$

The multiplication in $\mathrm{H}(\mathrm{M})$ is defined by the rule

$$y_{F_1} y_{F_2} = \begin{cases} y_{F_1 \vee F_2} & \text{if } \mathrm{rk}_{\mathrm{M}}(F_1) + \mathrm{rk}_{\mathrm{M}}(F_2) = \mathrm{rk}_{\mathrm{M}}(F_1 \vee F_2), \\ 0 & \text{if } \mathrm{rk}_{\mathrm{M}}(F_1) + \mathrm{rk}_{\mathrm{M}}(F_2) > \mathrm{rk}_{\mathrm{M}}(F_1 \vee F_2), \end{cases}$$

where $\vee$ stands for the join operation in the lattice of flats $\mathcal{L}(\mathrm{M})$ of M.

Our double use of the symbol y_F is justified by the following proposition, whose proof is essentially identical to that of [HW17, Proposition 9].

[10] In general, the inclusion $\underline{X}_V \subseteq \underline{X}_{\mathrm{M}}$ does not induce an isomorphism between their singular cohomology rings.

[11] This can be proved using the interpretation of $\mathrm{CH}(\mathrm{M})$ in the last sentence of Remark 4.1.

Proposition 2.18. The graded linear map

$$\mathrm{H}(\mathrm{M}) \longrightarrow \mathrm{CH}(\mathrm{M}), \quad y_F \longmapsto y_F$$

is an injective homomorphism of graded algebras.

Proof. We first show that the linear map is injective. It is enough to check that the subset

$$\{y_F\}_{F \in \mathcal{L}^k(\mathrm{M})} \subseteq \mathrm{CH}^k(\mathrm{M})$$

is linearly independent for every nonnegative integer $k < d$. Suppose that

$$\sum_{F \in \mathcal{L}^k(\mathrm{M})} c_F y_F = 0 \ \text{ for some } c_F \in \mathbb{Q}.$$

For any given rank k flat G, we choose a saturated flag of proper flats $\mathcal{G}$ whose smallest member is G. By the defining relations of $\mathrm{CH}(\mathrm{M})$, we have $y_F x_{\mathcal{G}} = 0$ for any rank k flat F other than G, therefore

$$c_G y_G x_{\mathcal{G}} = \Big(\sum_{F \in \mathcal{L}^k(\mathrm{M})} c_F y_F\Big) x_{\mathcal{G}} = 0.$$

Since the degree of $y_G x_{\mathcal{G}}$ is 1, this implies that c_G must be zero.

We next check that the linear map is an algebra homomorphism using Lemma 2.11. Let I_1 be a basis of a flat F_1, and let I_2 be a basis of a flat F_2. If the rank of $F_1 \vee F_2$ is the sum of the ranks of F_1 and F_2, then I_1 and I_2 are disjoint and their union is a basis of $F_1 \vee F_2$. Therefore, in the augmented Chow ring of M,

$$y_{F_1} y_{F_2} = \prod_{i \in I_1} y_i \prod_{i \in I_2} y_i = \prod_{i \in I_1 \cup I_2} y_i = y_{F_1 \vee F_2}.$$

If the rank of $F_1 \vee F_2$ is less than the sum of the ranks of F_1 and F_2, then either I_1 and I_2 intersect or the union of I_1 and I_2 is dependent in M. Therefore, in the augmented Chow ring of M,

$$y_{F_1} y_{F_2} = \prod_{i \in I_1} y_i \prod_{i \in I_2} y_i = 0. \qquad \square$$

Remark 2.19. Consider the torus $\mathbb{T}_E$, the toric variety X_{B}, and the augmented wonderful variety X_V in Remark 2.16. The identity of $\mathbb{T}_E$ uniquely extends to a toric map

$$\mathrm{p}_{\mathrm{B}} : X_{\mathrm{B}} \longrightarrow (\mathbb{P}^1)^E.$$

Let p_V be the restriction of p_{B} to the augmented wonderful variety X_V. If we identify the Chow ring of X_V with $\mathrm{CH}(\mathrm{M})$ as in Remark 2.16, the image of the pullback p_V^* is the graded Möbius algebra $\mathrm{H}(\mathrm{M}) \subseteq \mathrm{CH}(\mathrm{M})$.

2.6. **Pullback and pushforward maps.** Let Σ be a unimodular fan, and let σ be a k-dimensional cone in Σ. The torus orbit closure in the smooth toric variety X_Σ corresponding to σ can be identified with the toric variety of the fan $\mathrm{star}_\sigma \Sigma$. Its class in the Chow ring of X_Σ is the monomial x_σ, which is the product of the divisor classes x_ρ corresponding to the rays ρ in σ. The inclusion ι of the torus orbit closure in X_Σ defines the *pullback* ι^* and the *pushforward* ι_* between the Chow rings, whose composition is multiplication by the monomial x_σ:

$$\begin{tikzcd} \mathrm{CH}(\Sigma) \arrow[rr, "x_\sigma"] \arrow[dr, "\iota^*"'] & & \mathrm{CH}(\Sigma) \\ & \mathrm{CH}(\mathrm{star}_\sigma \Sigma) \arrow[ur, "\iota_*"'] & \end{tikzcd}$$

The pullback ι^* is a surjective graded algebra homomorphism, while the pushforward ι_* is a degree k homomorphism of $\mathrm{CH}(\Sigma)$-modules.

We give an explicit description of the pullback ι^* and the pushforward ι_* when Σ is the augmented Bergman fan Π_{M} and σ is the ray ρ_F of a proper flat F of M. Recall from Proposition 2.7 that the star of ρ_F admits the decomposition

$$\mathrm{star}_{\rho_F} \Pi_{\mathrm{M}} \cong \underline{\Pi}_{\mathrm{M}_F} \times \Pi_{\mathrm{M}^F}.$$

Thus we may identify the Chow ring of the star of ρ_F with $\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)$. We denote the pullback to the tensor product by φ^F_{M} and the pushforward from the tensor product by ψ^F_{M}:

$$\begin{tikzcd} \mathrm{CH}(\mathrm{M}) \arrow[rr, "x_F"] \arrow[dr, "\varphi^F_{\mathrm{M}}"'] & & \mathrm{CH}(\mathrm{M}) \\ & \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \arrow[ur, "\psi^F_{\mathrm{M}}"'] & \end{tikzcd}$$

To describe the pullback and the pushforward, we introduce Chow classes α_{M}, $\underline{\alpha}_{\mathrm{M}}$, and $\underline{\beta}_{\mathrm{M}}$. They are defined as the sums

$$\alpha_{\mathrm{M}} \coloneqq \sum_G x_G \in \mathrm{CH}^1(\mathrm{M}),$$

where the sum is over all proper flats G of M;

$$\underline{\alpha}_{\mathrm{M}} \coloneqq \sum_{i \in G} x_G \in \underline{\mathrm{CH}}^1(\mathrm{M}),$$

where the sum is over all nonempty proper flats G of M containing a given element i in E; and

$$\underline{\beta}_{\mathrm{M}} \coloneqq \sum_{i \notin G} x_G \in \underline{\mathrm{CH}}^1(\mathrm{M}),$$

where the sum is over all nonempty proper flats G of M not containing a given element i in E. The linear relations defining $\underline{\mathrm{CH}}(\mathrm{M})$ show that $\underline{\alpha}_{\mathrm{M}}$ and $\underline{\beta}_{\mathrm{M}}$ do not depend on the choice of i.

Proposition 2.20. The pullback φ^F_{M} is the unique graded algebra homomorphism

$$\mathrm{CH}(\mathrm{M}) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F)$$

that satisfies the following properties:

- If G is a proper flat of M incomparable to F, then $\varphi^F_{\mathrm{M}}(x_G) = 0$.
- If G is a proper flat of M properly contained in F, then $\varphi^F_{\mathrm{M}}(x_G) = 1 \otimes x_G$.
- If G is a proper flat of M properly containing F, then $\varphi^F_{\mathrm{M}}(x_G) = x_{G \setminus F} \otimes 1$.
- If i is an element of F, then $\varphi^F_{\mathrm{M}}(y_i) = 1 \otimes y_i$.
- If i is an element of $E \setminus F$, then $\varphi^F_{\mathrm{M}}(y_i) = 0$.

Moreover, the above five properties imply the following additional properties of φ^F_{M}:

- The equality $\varphi^F_{\mathrm{M}}(x_F) = -1 \otimes \alpha_{\mathrm{M}^F} - \underline{\beta}_{\mathrm{M}_F} \otimes 1$ holds.
- The equality $\varphi^F_{\mathrm{M}}(\alpha_{\mathrm{M}}) = \underline{\alpha}_{\mathrm{M}_F} \otimes 1$ holds.

Proof. The first five properties follow immediately from the pullback formula for toric varieties. To show the last two properties, we fix an element $i \in E \setminus F$. Recall that $y_i = \sum_{i \notin G} x_G$. Thus, by the first three and the fifth properties, we have

$$\begin{aligned}\varphi^F_{\mathrm{M}}(x_F) &= \varphi^F_{\mathrm{M}}\left(x_F + y_i - \sum_{i \notin G} x_G\right) = \varphi^F_{\mathrm{M}}\left(-\sum_{G \subsetneq F} x_G - \sum_{i \notin G, F \subsetneq G} x_G\right) \\ &= -\sum_{G \subsetneq F} 1 \otimes x_G - \sum_{i \notin G, F \subsetneq G} x_{G \setminus F} \otimes 1 = -1 \otimes \alpha_{\mathrm{M}^F} - \underline{\beta}_{\mathrm{M}_F} \otimes 1,\end{aligned}$$

which gives the second to last property. By the first and third properties, we have

$$\varphi^F_{\mathrm{M}}(\alpha_{\mathrm{M}}) = \varphi^F_{\mathrm{M}}\left(\sum_{i \in G} x_G\right) = \varphi^F_{\mathrm{M}}\left(\sum_{i \in G, F \subsetneq G} x_G\right) = \sum_{i \in G, F \subsetneq G} x_{G \setminus F} \otimes 1 = \underline{\alpha}_{\mathrm{M}_F} \otimes 1,$$

which gives the last property. □

The next proposition follows immediately from the pushforward formula for toric varieties. The projection formula shows that the pushforward ψ^F_{M} is a $\mathrm{CH}(\mathrm{M})$-module homomorphism.

Proposition 2.21. The pushforward ψ^F_{M} is the unique $\mathrm{CH}(\mathrm{M})$-module homomorphism[12]

$$\psi^F_{\mathrm{M}} : \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \mathrm{CH}(\mathrm{M}^F) \longrightarrow \mathrm{CH}(\mathrm{M})$$

[12] We make ψ^F_{M} into a $\mathrm{CH}(\mathrm{M})$-module homomorphism via the pullback φ^F_{M}.

that satisfies, for any collection $\mathcal{S}'$ of proper flats of M strictly containing F and any collection $\mathcal{S}''$ of proper flats of M strictly contained in F,

$$\psi^F_{\mathrm{M}}\left(\prod_{F'\in\mathcal{S}'} x_{F'\setminus F}\otimes\prod_{F''\in\mathcal{S}''} x_{F''}\right) = x_F\prod_{F'\in\mathcal{S}'} x_{F'}\prod_{F''\in\mathcal{S}''} x_{F''}.$$

The composition $\psi^F_{\mathrm{M}}\circ\varphi^F_{\mathrm{M}}$ is multiplication by the element x_F, and the composition $\varphi^F_{\mathrm{M}}\circ\psi^F_{\mathrm{M}}$ is multiplication by the element $\varphi^F_{\mathrm{M}}(x_F)$.

Remark 2.22. Proposition 2.21 shows that the pushforward ψ^F_{M} commutes with the degree maps:

$$\underline{\deg}_{\mathrm{M}_F}\otimes\deg_{\mathrm{M}^F} = \deg_{\mathrm{M}}\circ\psi^F_{\mathrm{M}}.$$

Proposition 2.23. If $\underline{\mathrm{CH}}(\mathrm{M}_F)$ and $\mathrm{CH}(\mathrm{M}^F)$ satisfy the Poincaré duality part of Theorem 1.6, then ψ^F_{M} is injective.

In other words, assuming Poincaré duality for the Chow rings, the graded $\mathrm{CH}(\mathrm{M})$-module $\underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\mathrm{CH}(\mathrm{M}^F)[-1]$ is isomorphic to the principal ideal of x_F in $\mathrm{CH}(\mathrm{M})$. In particular,

$$\underline{\mathrm{CH}}(\mathrm{M})[-1]\cong \mathrm{ideal}(x_\varnothing)\subseteq\mathrm{CH}(\mathrm{M}).$$

Proof. We will use the symbol $\deg_F$ to denote the degree function $\underline{\deg}_{\mathrm{M}_F}\otimes\deg_{\mathrm{M}^F}$. For contradiction, suppose that $\psi^F_{\mathrm{M}}(\eta)=0$ for $\eta\neq 0$. By the two Poincaré duality statements in Theorem 1.6, there is an element ν such that $\deg_F(\nu\eta)=1$. By surjectivity of the pullback φ^F_{M}, there is an element μ such that $\nu=\varphi^F_{\mathrm{M}}(\mu)$. Since ψ^F_{M} is a $\mathrm{CH}(\mathrm{M})$-module homomorphism that commutes with the degree maps, we have

$$1=\deg_F(\nu\eta)=\deg_{\mathrm{M}}(\psi^F_{\mathrm{M}}(\nu\eta))=\deg_{\mathrm{M}}(\psi^F_{\mathrm{M}}(\varphi^F_{\mathrm{M}}(\mu)\eta))=\deg_{\mathrm{M}}(\mu\psi^F_{\mathrm{M}}(\eta))=\deg_{\mathrm{M}}(0)=0,$$

which is a contradiction. □

We next give an explicit description of the pullback ι^* and the pushforward ι_* when Σ is the Bergman fan $\underline{\Pi}_{\mathrm{M}}$ and σ is the ray $\underline{\rho}_F$ of a nonempty proper flat F of M. Recall from Proposition 2.7 that the star of $\underline{\rho}_F$ admits the decomposition

$$\mathrm{star}_{\underline{\rho}_F}\underline{\Pi}_{\mathrm{M}}\cong\underline{\Pi}_{\mathrm{M}_F}\times\underline{\Pi}_{\mathrm{M}^F}.$$

Thus we may identify the Chow ring of the star of $\underline{\rho}_F$ with $\underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\underline{\mathrm{CH}}(\mathrm{M}^F)$. We denote the pullback to the tensor product by $\underline{\varphi}^F_{\mathrm{M}}$ and the pushforward from the tensor product by $\underline{\psi}^F_{\mathrm{M}}$:

$$\begin{tikzcd}
\underline{\mathrm{CH}}(\mathrm{M}) \arrow[rr, "x_F"] \arrow[dr, "\underline{\varphi}^F_{\mathrm{M}}"'] & & \underline{\mathrm{CH}}(\mathrm{M}) \\
& \underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\underline{\mathrm{CH}}(\mathrm{M}^F) \arrow[ur, "\underline{\psi}^F_{\mathrm{M}}"'] &
\end{tikzcd}$$

The following analogues of Propositions 2.20 and 2.21 can be proved by similar straightforward computations.

Proposition 2.24. The pullback $\underline{\varphi}^F_{\mathrm{M}}$ is the unique graded algebra homomorphism

$$\underline{\mathrm{CH}}(\mathrm{M}) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$$

that satisfies the following properties:

- If G is a nonempty proper flat of M incomparable to F, then $\underline{\varphi}^F_{\mathrm{M}}(x_G) = 0$.
- If G is a nonempty proper flat of M properly contained in F, then $\underline{\varphi}^F_{\mathrm{M}}(x_G) = 1 \otimes x_G$.
- If G is a nonempty proper flat of M properly containing F, then $\underline{\varphi}^F_{\mathrm{M}}(x_G) = x_{G \setminus F} \otimes 1$.

The above three properties imply the following additional properties of $\underline{\varphi}^F_{\mathrm{M}}$:

- The equality $\underline{\varphi}^F_{\mathrm{M}}(x_F) = -1 \otimes \underline{\alpha}_{\mathrm{M}^F} - \underline{\beta}_{\mathrm{M}_F} \otimes 1$ holds.
- The equality $\underline{\varphi}^F_{\mathrm{M}}(\underline{\alpha}_{\mathrm{M}}) = \underline{\alpha}_{\mathrm{M}_F} \otimes 1$ holds.
- The equality $\underline{\varphi}^F_{\mathrm{M}}(\underline{\beta}_{\mathrm{M}}) = 1 \otimes \underline{\beta}_{\mathrm{M}^F}$ holds.

Proposition 2.25. The pushforward $\underline{\psi}^F_{\mathrm{M}}$ is the unique $\underline{\mathrm{CH}}(\mathrm{M})$-module homomorphism

$$\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M})$$

that satisfies, for any collection $\mathcal{S}'$ of proper flats of M strictly containing F and any collection $\mathcal{S}''$ of nonempty proper flats of M strictly contained in F,

$$\underline{\psi}^F_{\mathrm{M}}\left(\prod_{F' \in \mathcal{S}'} x_{F' \setminus F} \otimes \prod_{F'' \in \mathcal{S}''} x_{F''} \right) = x_F \prod_{F' \in \mathcal{S}'} x_{F'} \prod_{F'' \in \mathcal{S}''} x_{F''}.$$

The composition $\underline{\psi}^F_{\mathrm{M}} \circ \underline{\varphi}^F_{\mathrm{M}}$ is multiplication by the element x_F, and the composition $\underline{\varphi}^F_{\mathrm{M}} \circ \underline{\psi}^F_{\mathrm{M}}$ is multiplication by the element $\underline{\varphi}^F_{\mathrm{M}}(x_F)$.

Remark 2.26. Proposition 2.25 shows that the pushforward $\underline{\psi}^F_{\mathrm{M}}$ commutes with the degree maps:

$$\underline{\deg}_{\mathrm{M}_F} \otimes \underline{\deg}_{\mathrm{M}^F} = \underline{\deg}_{\mathrm{M}} \circ \underline{\psi}^F_{\mathrm{M}}.$$

Proposition 2.27. If $\underline{\mathrm{CH}}(\mathrm{M}_F)$ and $\underline{\mathrm{CH}}(\mathrm{M}^F)$ satisfy the Poincaré duality part of Theorem 1.6, then $\underline{\psi}^F_{\mathrm{M}}$ is injective.

In other words, assuming Poincaré duality for the Chow rings, the graded $\underline{\mathrm{CH}}(\mathrm{M})$-module $\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)[-1]$ is isomorphic to the principal ideal of x_F in $\underline{\mathrm{CH}}(\mathrm{M})$.

Proof. The proof is essentially identical to that of Proposition 2.23. □

Last, we give an explicit description of the pullback ι^* and the pushforward ι_* when Σ is the augmented Bergman fan Π_{M} and σ is the cone σ_I of an independent set I of M. By Proposition 2.7, we have

$$\mathrm{star}_{\sigma_I}\Pi_{\mathrm{M}} \cong \Pi_{\mathrm{M}_F},$$

where F is the closure of I in M. Thus we may identify the Chow ring of the star of σ_I with $\mathrm{CH}(\mathrm{M}_F)$. We denote the corresponding pullback by φ^{M}_F and the pushforward by ψ^{M}_F:

$$\begin{tikzcd}
\mathrm{CH}(\mathrm{M}) \arrow[rr, "y_F"] \arrow[dr, "\varphi^{\mathrm{M}}_F"'] & & \mathrm{CH}(\mathrm{M}) \\
& \mathrm{CH}(\mathrm{M}_F) \arrow[ur, "\psi^{\mathrm{M}}_F"'] &
\end{tikzcd}$$

Note that the pullback and the pushforward only depend on F and not on I.

The following analogues of Propositions 2.20 and 2.21 are straightforward.

Proposition 2.28. The pullback φ^{M}_F is the unique graded algebra homomorphism

$$\mathrm{CH}(\mathrm{M}) \longrightarrow \mathrm{CH}(\mathrm{M}_F)$$

that satisfies the following properties:

- If G is a proper flat of M that contains F, then $\varphi^{\mathrm{M}}_F(x_G) = x_{G\setminus F}$.
- If G is a proper flat of M that does not contain F, then $\varphi^{\mathrm{M}}_F(x_G) = 0$.

The above two properties imply the following additional properties of φ^{M}_F:

- If i is an element of F, then $\varphi^{\mathrm{M}}_F(y_i) = 0$.
- If i is an element of $E\setminus F$, then $\varphi^{\mathrm{M}}_F(y_i) = y_i$.
- The equality $\varphi^{\mathrm{M}}_F(\alpha_{\mathrm{M}}) = \alpha_{\mathrm{M}_F}$ holds.

Proposition 2.29. The pushforward ψ^{M}_F is the unique $\mathrm{CH}(\mathrm{M})$-module homomorphism

$$\mathrm{CH}(\mathrm{M}_F) \longrightarrow \mathrm{CH}(\mathrm{M})$$

that satisfies, for any collection $\mathcal{S}'$ of proper flats of M containing F,

$$\psi^{\mathrm{M}}_F\left(\prod_{F'\in\mathcal{S}'} x_{F'\setminus F}\right) = y_F \prod_{F'\in\mathcal{S}'} x_{F'}.$$

The composition $\psi^{\mathrm{M}}_F \circ \varphi^{\mathrm{M}}_F$ is multiplication by the element y_F, and the composition $\varphi^{\mathrm{M}}_F \circ \psi^{\mathrm{M}}_F$ is zero.

Remark 2.30. Proposition 2.29 shows that the pushforward ψ^{M}_F commutes with the degree maps:

$$\deg_{\mathrm{M}_F} = \deg_{\mathrm{M}} \circ \psi^{\mathrm{M}}_F.$$

Proposition 2.31. If $\mathrm{CH}(\mathrm{M}_F)$ satisfies the Poincaré duality part of Theorem 1.6, then ψ_F^{M} is injective.

In other words, assuming Poincaré duality for the Chow rings, the graded $\mathrm{CH}(\mathrm{M})$-module $\mathrm{CH}(\mathrm{M}_F)[-\operatorname{rk}_{\mathrm{M}}(F)]$ is isomorphic to the principal ideal of y_F in $\mathrm{CH}(\mathrm{M})$.

Proof. The proof is essentially identical to that of Proposition 2.23. □

The basic properties of the pullback and the pushforward maps can be used to describe the fundamental classes of $\underline{\mathrm{CH}}(\mathrm{M})$ and $\mathrm{CH}(\mathrm{M})$ in terms of $\underline{\alpha}_{\mathrm{M}}$ and α_{M}.

Proposition 2.32. The degree of $\underline{\alpha}_{\mathrm{M}}^{d-1}$ is 1, and the degree of α_{M}^{d} is 1.

Proof. We prove the first statement by induction on $d \geqslant 1$. Note that, for any nonempty proper flat F of rank k, we have

$$x_F\, \underline{\alpha}_{\mathrm{M}}^{d-k} = \underline{\psi}_{\mathrm{M}}^{F}\big(\underline{\varphi}_{\mathrm{M}}^{F}(\underline{\alpha}_{\mathrm{M}}^{d-k})\big) = \underline{\psi}_{\mathrm{M}}^{F}\big(\underline{\alpha}_{\mathrm{M_F}}^{d-k} \otimes 1\big) = 0,$$

since $\underline{\mathrm{CH}}^{d-k}(\mathrm{M}_F) = 0$. Therefore, for any proper flat a of rank 1 and any element i in a, we have

$$\underline{\alpha}_{\mathrm{M}}^{d-1} = \Big(\sum_{i \in F} x_F\Big)\underline{\alpha}_{\mathrm{M}}^{d-2} = x_a\, \underline{\alpha}_{\mathrm{M}}^{d-2}.$$

By the induction hypothesis applied to the matroid M_a of rank $d-1$, we have $\underline{\deg}_{\mathrm{M}_a}(\underline{\alpha}_{\mathrm{M}_a}^{d-2}) = 1$, or equivalently, $\underline{\alpha}_{\mathrm{M}_a}^{d-2} = x_{\mathcal{F}}$ for any maximal flag $\mathcal{F}$ of nonempty proper flats of M_a. Thus, we have

$$\underline{\alpha}_{\mathrm{M}}^{d-1} = x_a\, \underline{\alpha}_{\mathrm{M}}^{d-2} = \underline{\psi}_{\mathrm{M}}^{a}\big(\underline{\varphi}_{\mathrm{M}}^{a}(\underline{\alpha}_{\mathrm{M}}^{d-2})\big) = \underline{\psi}_{\mathrm{M}}^{a}\big(\underline{\alpha}_{\mathrm{M}_a}^{d-2} \otimes 1\big) = x_{\mathcal{F}'},$$

where $\mathcal{F}'$ is any maximal flag of nonempty proper flats of M that starts from a.

For the second statement, note that, for any proper flat F of rank k,

$$x_F\, \alpha_{\mathrm{M}}^{d-k} = \psi_{\mathrm{M}}^{F}\big(\varphi_{\mathrm{M}}^{F}(\alpha_{\mathrm{M}}^{d-k})\big) = \psi_{\mathrm{M}}^{F}\big(\underline{\alpha}_{\mathrm{M_F}}^{d-k} \otimes 1\big) = 0.$$

Using the first statement, we get the conclusion from the identity

$$\alpha_{\mathrm{M}}^{d} = \Big(\sum_{F} x_F\Big)\alpha_{\mathrm{M}}^{d-1} = x_{\varnothing}\, \alpha_{\mathrm{M}}^{d-1} = \psi_{\mathrm{M}}^{\varnothing}\big(\varphi_{\mathrm{M}}^{\varnothing}(\alpha_{\mathrm{M}}^{d-1})\big) = \psi_{\mathrm{M}}^{\varnothing}\big(\underline{\alpha}_{\mathrm{M}}^{d-1}\big).$$ □

More generally, the degree of $\underline{\alpha}_{\mathrm{M}}^{d-k}\underline{\beta}_{\mathrm{M}}^{k}$ is the k-th coefficient of the reduced characteristic polynomial of M [AHK18, Proposition 9.5].

3. Proofs of the semi-small decompositions and the Poincaré duality theorems

In this section, we prove Theorems 1.2 and 1.5 together with the two Poincaré duality statements in Theorem 1.6. For an element i of E, we write π_i and $\underline{\pi}_i$ for the coordinate projections

$$\pi_i : \mathbb{R}^E \longrightarrow \mathbb{R}^{E\setminus i} \text{ and } \underline{\pi}_i : \mathbb{R}^E/\langle \mathbf{e}_E \rangle \longrightarrow \mathbb{R}^{E\setminus i}/\langle \mathbf{e}_{E\setminus i} \rangle.$$

Note that $\pi_i(\rho_i) = 0$ and $\underline{\pi}_i(\underline{\rho}_{\{i\}}) = 0$. In addition, $\pi_i(\rho_S) = \rho_{S\setminus i}$ and $\underline{\pi}_i(\underline{\rho}_S) = \underline{\rho}_{S\setminus i}$ for $S \subseteq E$. Here we recall from Definition 2.4 that ρ_i and ρ_S denote the rays generated by the vectors $\mathbf{e}_i$ and $-\mathbf{e}_{E\setminus S}$ in $\mathbb{R}^E$, respectively, and $\underline{\rho}_S$ denotes the image of ρ_S in $\mathbb{R}^E/\langle \mathbf{e}_E \rangle$.

Proposition 3.1. Let M be a loopless matroid on E, and let i be an element of E.

(1) The projection π_i maps any cone of Π_{M} onto a cone of $\Pi_{\mathrm{M}\setminus i}$.

(2) The projection $\underline{\pi}_i$ maps any cone of $\underline{\Pi}_{\mathrm{M}}$ onto a cone of $\underline{\Pi}_{\mathrm{M}\setminus i}$.

Recall that a linear map defines a morphism of fans $\Sigma_1 \to \Sigma_2$ if it maps any cone of Σ_1 into a cone of Σ_2 [CLS11, Chapter 3]. Thus the above proposition is stronger than the statement that π_i and $\underline{\pi}_i$ induce morphisms of fans.

Proof of Proposition 3.1. The projection π_i maps $\sigma_{I\leqslant\mathcal{F}}$ onto $\sigma_{I\setminus i\leqslant\mathcal{F}\setminus i}$, where $\mathcal{F}\setminus i$ is the flag of flats of $\mathrm{M}\setminus i$ obtained by removing i from the members of $\mathcal{F}$. Similarly, $\underline{\pi}_i$ maps $\underline{\sigma}_{\mathcal{F}}$ onto $\underline{\sigma}_{\mathcal{F}\setminus i}$. □

By Proposition 3.1, the projection π_i defines a map from the toric variety X_{M} of Π_{M} to the toric variety $X_{\mathrm{M}\setminus i}$ of $\Pi_{\mathrm{M}\setminus i}$, and hence the pullback homomorphism $\mathrm{CH}(\mathrm{M}\setminus i) \to \mathrm{CH}(\mathrm{M})$. Explicitly, the pullback is the graded algebra homomorphism

$$\theta_i = \theta_i^{\mathrm{M}} : \mathrm{CH}(\mathrm{M}\setminus i) \longrightarrow \mathrm{CH}(\mathrm{M}), \qquad x_F \longmapsto x_F + x_{F\cup i},$$

where a variable in the target is replaced with zero if its label is not a flat of M. Similarly, $\underline{\pi}_i$ defines a map from the toric variety $\underline{X}_{\mathrm{M}}$ of $\underline{\Pi}_{\mathrm{M}}$ to the toric variety $\underline{X}_{\mathrm{M}\setminus i}$ of $\underline{\Pi}_{\mathrm{M}\setminus i}$, and hence an algebra homomorphism

$$\underline{\theta}_i = \underline{\theta}_i^{\mathrm{M}} : \underline{\mathrm{CH}}(\mathrm{M}\setminus i) \longrightarrow \underline{\mathrm{CH}}(\mathrm{M}), \qquad x_F \longmapsto x_F + x_{F\cup i},$$

where a variable in the target is set to zero if its label is not a flat of M.

Remark 3.2. We use the notations introduced in Remark 2.16. Let $V\setminus i$ be the image of V under the i-th projection $\mathbb{F}^E \to \mathbb{F}^{E\setminus i}$. We have the commutative diagrams of wonderful varieties and their

Chow rings

$$\begin{tikzcd} \underline{X}_{\mathrm{B}} \arrow[r, hookleftarrow] \arrow[d, two heads, "\underline{p}_i^{\mathrm{B}}"'] & \underline{X}_V \arrow[d, two heads, "\underline{p}_i^V"] \\ \underline{X}_{\mathrm{B}\backslash i} \arrow[r, hookleftarrow] & \underline{X}_{V\backslash i}, \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathrm{CH}}(\mathrm{B}) \arrow[r, two heads] & \underline{\mathrm{CH}}(\mathrm{M}) \\ \underline{\mathrm{CH}}(\mathrm{B}\backslash i) \arrow[r, two heads] \arrow[u, hook] & \underline{\mathrm{CH}}(\mathrm{M}\backslash i). \arrow[u, hook] \end{tikzcd}$$

The map $\underline{p}_i^V$ is birational if and only if i is not a coloop of M. By Proposition 3.1, the fibers of $\underline{p}_i^{\mathrm{B}}$ are at most one-dimensional, and hence the fibers of $\underline{p}_i^V$ are at most one-dimensional. It follows that $\underline{p}_i^V$ is semi-small in the sense of Goresky–MacPherson when i is not a coloop of M.

Similarly, we have the diagrams of augmented wonderful varieties and their Chow rings

$$\begin{tikzcd} X_{\mathrm{B}} \arrow[r, hookleftarrow] \arrow[d, two heads, "p_i^{\mathrm{B}}"'] & X_V \arrow[d, two heads, "p_i^V"] \\ X_{\mathrm{B}\backslash i} \arrow[r, hookleftarrow] & X_{V\backslash i}, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CH}(\mathrm{B}) \arrow[r, two heads] & \mathrm{CH}(\mathrm{M}) \\ \mathrm{CH}(\mathrm{B}\backslash i) \arrow[r, two heads] \arrow[u, hook] & \mathrm{CH}(\mathrm{M}\backslash i). \arrow[u, hook] \end{tikzcd}$$

The map p_i^V is birational if and only if i is not a coloop of M. By Proposition 3.1, the fibers of p_i^{B} are at most one-dimensional, and hence p_i^V is semi-small when i is not a coloop of M.

Numerically, the semi-smallness of $\underline{p}_i^V$ is reflected in the identity

$$\dim x_{F\cup i}\underline{\mathrm{CH}}_{(i)}^{k-1} = \dim x_{F\cup i}\underline{\mathrm{CH}}_{(i)}^{d-k-2}.$$

Similarly, the semi-smallness of p_i^V is reflected in the identity[13]

$$\dim x_{F\cup i}\mathrm{CH}_{(i)}^{k-1} = \dim x_{F\cup i}\mathrm{CH}_{(i)}^{d-k-1}.$$

For a detailed discussion of semi-small maps in the context of Hodge theory and the decomposition theorem, see [dCM02] and [dCM09].

We show that the pullbacks θ_i and $\underline{\theta}_i$ are compatible with the degree maps of M and $\mathrm{M}\backslash i$.

Lemma 3.3. Suppose that $E\backslash i$ is nonempty.

(1) If i is not a coloop of M, then θ_i commutes with the degree maps:

$$\deg_{\mathrm{M}\backslash i} = \deg_{\mathrm{M}} \circ\, \theta_i.$$

(2) If i is not a coloop of M, then $\underline{\theta}_i$ commutes with the degree maps:

$$\underline{\deg}_{\mathrm{M}\backslash i} = \underline{\deg}_{\mathrm{M}} \circ\, \underline{\theta}_i.$$

[13]The displayed identities follow from Proposition 3.5 and the Poincaré duality parts of Theorem 1.6.

(3) If i is a coloop of M, we have

$$\deg_{\mathrm{M}\setminus i} = \deg_{\mathrm{M}} \circ\, x_{E\setminus i} \circ\, \theta_i = \deg_{\mathrm{M}} \circ\, \alpha_{\mathrm{M}} \circ\, \theta_i,$$

where the middle maps are multiplications by the elements $x_{E\setminus i}$ and α_{M}.

(4) If i is a coloop of M, we have

$$\underline{\deg}_{\mathrm{M}\setminus i} = \underline{\deg}_{\mathrm{M}} \circ\, x_{E\setminus i} \circ\, \underline{\theta}_i = \underline{\deg}_{\mathrm{M}} \circ\, \underline{\alpha}_{\mathrm{M}} \circ\, \underline{\theta}_i,$$

where the middle maps are multiplications by the elements $x_{E\setminus i}$ and $\underline{\alpha}_{\mathrm{M}}$.

Proof. If i is not a coloop of M, we may choose a basis B of $\mathrm{M}\setminus i$ that is also a basis of M. By Proposition 2.8 and Remark 2.14, the top degree components $\mathrm{CH}^d(\mathrm{M}\setminus i)$ and $\mathrm{CH}^d(\mathrm{M})$ are both one-dimensional, so we have

$$\mathrm{CH}^d(\mathrm{M}\setminus i) = \mathrm{span}(y_B) \;\text{ and }\; \mathrm{CH}^d(\mathrm{M}) = \mathrm{span}(y_B).$$

Since $\theta_i(y_j) = y_j$ for all j, the first identity follows. Similarly, by Proposition 2.32,

$$\underline{\mathrm{CH}}^{d-1}(\mathrm{M}\setminus i) = \mathrm{span}(\underline{\alpha}^{d-1}_{\mathrm{M}\setminus i}) \;\text{ and }\; \underline{\mathrm{CH}}^{d-1}(\mathrm{M}) = \mathrm{span}(\underline{\alpha}^{d-1}_{\mathrm{M}}).$$

Since $\underline{\theta}_i(\underline{\alpha}_{\mathrm{M}\setminus i}) = \underline{\alpha}_{\mathrm{M}}$ when i is not a coloop, the second identity follows.

Suppose now that i is a coloop of M. In this case, $E\setminus i$ is a flat and $\mathrm{M}\setminus i = \mathrm{M}^{E\setminus i}$. Hence

$$\varphi^{E\setminus i}_{\mathrm{M}} \circ\, \theta_i = \text{identity of } \mathrm{CH}(\mathrm{M}\setminus i) \;\text{ and }\; \underline{\varphi}^{E\setminus i}_{\mathrm{M}} \circ\, \underline{\theta}_i = \text{identity of } \underline{\mathrm{CH}}(\mathrm{M}\setminus i).$$

Using the compatibility of the pushforward $\psi^{E\setminus i}_{\mathrm{M}}$ with the degree maps, we have

$$\deg_{\mathrm{M}\setminus i} = \deg_{\mathrm{M}} \circ \psi^{E\setminus i}_{\mathrm{M}} = \deg_{\mathrm{M}} \circ \psi^{E\setminus i}_{\mathrm{M}} \circ\, \varphi^{E\setminus i}_{\mathrm{M}} \circ\, \theta_i = \deg_{\mathrm{M}} \circ\, x_{E\setminus i} \circ\, \theta_i.$$

Since $\theta_i(\alpha_{\mathrm{M}\setminus i}) = \alpha_{\mathrm{M}} - x_{E\setminus i}$, when i is a coloop of M, the above implies

$$\deg_{\mathrm{M}\setminus i} = \deg_{\mathrm{M}} \circ\, x_{E\setminus i} \circ\, \theta_i = \deg_{\mathrm{M}} \circ \big(\alpha_{\mathrm{M}} - \theta_i(\alpha_{\mathrm{M}\setminus i})\big) \circ\, \theta_i = \deg_{\mathrm{M}} \circ\, \alpha_{\mathrm{M}} \circ\, \theta_i,$$

where the last equality follows from the fact that the images of θ_i have degree at most $d-1$. The identities for $\underline{\deg}_{\mathrm{M}\setminus i}$ can be obtained in a similar way. □

Proposition 3.4. If $\mathrm{CH}(\mathrm{M}\setminus i)$ satisfies the Poincaré duality part of Theorem 1.6, then θ_i is injective. Also, if $\underline{\mathrm{CH}}(\mathrm{M}\setminus i)$ satisfies the Poincaré duality part of Theorem 1.6, then $\underline{\theta}_i$ is injective.

Proof. The proof is essentially identical to that of Proposition 2.23. □

For a flat F in $\mathcal{S}_i$, we write $\theta^{F\cup i}_i$ for the pullback map between the augmented Chow rings obtained from the deletion of i from the localization $\mathrm{M}^{F\cup i}$:

$$\theta^{F\cup i}_i : \mathrm{CH}(\mathrm{M}^F) \to \mathrm{CH}(\mathrm{M}^{F\cup i}).$$

Similarly, for a flat F in $\underline{\mathcal{S}}_i$, we write $\underline{\theta}_i^{F\cup i}$ for the pullback map between the Chow rings obtained from the deletion of i from the localization $\mathrm{M}^{F\cup i}$:

$$\underline{\theta}_i^{F\cup i} : \underline{\mathrm{CH}}(\mathrm{M}^F) \to \underline{\mathrm{CH}}(\mathrm{M}^{F\cup i}).$$

Note that i is a coloop of $\mathrm{M}^{F\cup i}$ in these cases.

Proposition 3.5. The summands appearing in Theorems 1.2 and 1.5 can be described as follows.

(1) If $F \in \mathcal{S}_i$, then $x_{F\cup i}\mathrm{CH}_{(i)} = \psi_{\mathrm{M}}^{F\cup i}\big(\underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \theta_i^{F\cup i}\mathrm{CH}(\mathrm{M}^F)\big)$.

(2) If $F \in \underline{\mathcal{S}}_i$, then $x_{F\cup i}\underline{\mathrm{CH}}_{(i)} = \underline{\psi}_{\mathrm{M}}^{F\cup i}\big(\underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \underline{\theta}_i^{F\cup i}\underline{\mathrm{CH}}(\mathrm{M}^F)\big)$.

(3) If i is a coloop of M, then $x_{E\setminus i}\,\mathrm{CH}_{(i)} = \psi_{\mathrm{M}}^{E\setminus i}\,\mathrm{CH}(\mathrm{M}\setminus i)$ and $x_{E\setminus i}\underline{\mathrm{CH}}_{(i)} = \underline{\psi}_{\mathrm{M}}^{E\setminus i}\underline{\mathrm{CH}}(\mathrm{M}\setminus i)$.

Remark 3.6. Assuming Poincaré duality for all of our Chow rings, Propositions 2.23, 2.27, and 3.4 imply that

$$x_{F\cup i}\mathrm{CH}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \mathrm{CH}(\mathrm{M}^F)[-1] \;\text{ and }\; x_{F\cup i}\underline{\mathrm{CH}}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F)[-1],$$

and therefore

$$\dim x_{F\cup i}\underline{\mathrm{CH}}_{(i)}^{k-1} = \dim x_{F\cup i}\underline{\mathrm{CH}}_{(i)}^{d-k-2} \;\text{ and }\; \dim x_{F\cup i}\mathrm{CH}_{(i)}^{k-1} = \dim x_{F\cup i}\mathrm{CH}_{(i)}^{d-k-1}.$$

Proof of Proposition 3.5. We prove the first statement. The proof of the second statement is essentially identical. The third statement is a straightforward consequence of the fact that $\varphi_{\mathrm{M}}^{E\setminus i} \circ \theta_i$ and $\underline{\varphi}_{\mathrm{M}}^{E\setminus i} \circ \underline{\theta}_i$ are the identity maps when i is a coloop.

Let F be a flat in $\mathcal{S}_i$. It is enough to show that

$$\varphi_{\mathrm{M}}^{F\cup i}\big(\mathrm{CH}_{(i)}\big) = \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \theta_i^{F\cup i}\mathrm{CH}(\mathrm{M}^F),$$

since the result will then follow by applying $\psi_{\mathrm{M}}^{F\cup i}$. The projection π_i maps the ray $\rho_{F\cup i}$ to the ray ρ_F, and hence π_i defines morphisms of fans

$$\begin{tikzcd}
\mathrm{star}_{\rho_{F\cup i}}\Pi_{\mathrm{M}} \arrow[d, "\pi_i'"'] & \underline{\Pi}_{\mathrm{M}_{F\cup i}} \times \Pi_{\mathrm{M}^{F\cup i}} \arrow[l, hook', "\iota_{F\cup i}"'] \arrow[d, "\pi_i''"'] \arrow[r, equal] & \underline{\Pi}_{(\mathrm{M}/i)_F} \times \Pi_{\mathrm{M}^{F\cup i}} \arrow[d, "\pi_i'''"'] \\
\mathrm{star}_{\rho_F}\Pi_{\mathrm{M}\setminus i} & \underline{\Pi}_{(\mathrm{M}\setminus i)_F} \times \Pi_{(\mathrm{M}\setminus i)^F} \arrow[l, hook', "\iota_F"'] \arrow[r, equal] & \underline{\Pi}_{(\mathrm{M}\setminus i)_F} \times \Pi_{\mathrm{M}^F},
\end{tikzcd}$$

where $\iota_{F\cup i}$ and ι_F are the isomorphisms in Proposition 2.7. The main point is that the matroid $(\mathrm{M}/i)_F$ is a quotient of $(\mathrm{M}\setminus i)_F$. In other words, we have the inclusion of Bergman fans

$$\underline{\Pi}_{(\mathrm{M}/i)_F} \subseteq \underline{\Pi}_{(\mathrm{M}\setminus i)_F}.$$

Therefore, the morphism π_i''' admits the factorization

$$\underline{\Pi}_{(\mathrm{M}/i)_F} \times \Pi_{\mathrm{M}^{F\cup i}} \twoheadrightarrow \underline{\Pi}_{(\mathrm{M}/i)_F} \times \Pi_{\mathrm{M}^F} \hookrightarrow \underline{\Pi}_{(\mathrm{M}\setminus i)_F} \times \Pi_{\mathrm{M}^F},$$

where the second map induces a surjective pullback map q between the Chow rings. By the equality $(\mathrm{M}/i)_F = \mathrm{M}_{F\cup i}$, we have the commutative diagram of pullback maps between the Chow rings

$$\begin{tikzcd}
\mathrm{CH}(\mathrm{M}\backslash i) \arrow[rr, "\theta_i"] \arrow[d, two heads, "\varphi^F_{\mathrm{M}\backslash i}"'] & & \mathrm{CH}(\mathrm{M}) \arrow[d, "\varphi^{F\cup i}_{\mathrm{M}}"] \\
\underline{\mathrm{CH}}((\mathrm{M}\backslash i)_F)\otimes \mathrm{CH}((\mathrm{M}\backslash i)^F) \arrow[r, two heads, "q"] & \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i})\otimes \mathrm{CH}(\mathrm{M}^F) \arrow[r, "1\otimes\theta_i^{F\cup i}"] & \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i})\otimes \mathrm{CH}(\mathrm{M}^{F\cup i}).
\end{tikzcd}$$

The conclusion follows from the surjectivity of the pullback maps $\varphi^F_{\mathrm{M}\backslash i}$ and q. □

Remark 3.7. Since i is a coloop in $\mathrm{M}^{F\cup i}$ when $F\in \mathcal{S}_i$ or $F \in \underline{\mathcal{S}}_i$, Proposition 3.5 implies that

$$x_{F\cup i}\mathrm{CH}^{d-1}_{(i)} = 0 \text{ for } F\in\mathcal{S}_i \quad\text{and}\quad x_{F\cup i}\underline{\mathrm{CH}}^{d-2}_{(i)} = 0 \text{ for } F\in\underline{\mathcal{S}}_i.$$

Proposition 3.8. The Poincaré pairing on the summands appearing in Theorems 1.2 and 1.5 can be described as follows.

(1) If $F\in\mathcal{S}_i$, then for any $\mu_1,\mu_2\in \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i})\otimes \mathrm{CH}(\mathrm{M}^F)$ of complementary degrees,

$$\deg_{\mathrm{M}}\big(\psi^{F\cup i}_{\mathrm{M}}(1\otimes\theta_i^{F\cup i}(\mu_1))\cdot\psi^{F\cup i}_{\mathrm{M}}(1\otimes\theta_i^{F\cup i}(\mu_2))\big) = -\underline{\deg}_{\mathrm{M}_{F\cup i}}\otimes\deg_{\mathrm{M}^F}(\mu_1\mu_2).$$

(2) If $F\in\underline{\mathcal{S}}_i$, then for any $\nu_1,\nu_2\in \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i})\otimes \underline{\mathrm{CH}}(\mathrm{M}^F)$ of complementary degrees,

$$\underline{\deg}_{\mathrm{M}}\big(\underline{\psi}^{F\cup i}_{\mathrm{M}}(1\otimes\underline{\theta}_i^{F\cup i}(\nu_1))\cdot\underline{\psi}^{F\cup i}_{\mathrm{M}}(1\otimes\underline{\theta}_i^{F\cup i}(\nu_2))\big) = -\underline{\deg}_{\mathrm{M}_{F\cup i}}\otimes\underline{\deg}_{\mathrm{M}^F}(\nu_1\nu_2).$$

It follows, assuming Poincaré duality for the Chow rings, that the restriction of the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$ to the subspace $x_{F\cup i}\,\mathrm{CH}_{(i)}$ is nondegenerate, and the restriction of the Poincaré pairing of $\underline{\mathrm{CH}}(\mathrm{M})$ to the subspace $x_{F\cup i}\underline{\mathrm{CH}}_{(i)}$ is nondegenerate.

Proof. We prove the first equality. The second equality can be proved in the same way.

Since the pushforward $\psi^{F\cup i}_{\mathrm{M}}$ is a $\mathrm{CH}(\mathrm{M})$-module homomorphism (Proposition 2.21), the left-hand side is

$$\deg_{\mathrm{M}}\big(\psi^{F\cup i}_{\mathrm{M}}\big(\varphi^{F\cup i}_{\mathrm{M}}\psi^{F\cup i}_{\mathrm{M}}(1\otimes\theta_i^{F\cup i}(\mu_1))\cdot(1\otimes\theta_i^{F\cup i}(\mu_2))\big)\big).$$

The pushforward commutes with the degree maps (Remark 2.22), so the above is equal to

$$\underline{\deg}_{\mathrm{M}_{F\cup i}}\otimes\deg_{\mathrm{M}^{F\cup i}}\big(\varphi^{F\cup i}_{\mathrm{M}}\psi^{F\cup i}_{\mathrm{M}}(1\otimes\theta_i^{F\cup i}(\mu_1))\cdot(1\otimes\theta_i^{F\cup i}(\mu_2))\big).$$

Using that the composition $\varphi^{F\cup i}_{\mathrm{M}}\psi^{F\cup i}_{\mathrm{M}}$ is multiplication by $\varphi^{F\cup i}_{\mathrm{M}}(x_{F\cup i})$ (Proposition 2.21), we get

$$-\underline{\deg}_{\mathrm{M}_{F\cup i}}\otimes\deg_{\mathrm{M}^{F\cup i}}\big((1\otimes\alpha_{\mathrm{M}^{F\cup i}}+\underline{\beta}_{\mathrm{M}_{F\cup i}}\otimes 1)\cdot(1\otimes\theta_i^{F\cup i}(\mu_1))\cdot(1\otimes\theta_i^{F\cup i}(\mu_2))\big).$$

Since i is a coloop of $\mathrm{M}^{F\cup i}$ and $\theta_i^{F\cup i}$ is a graded ring homomorphism, the product

$$(1\otimes\theta_i^{F\cup i}(\mu_1))\cdot(1\otimes\theta_i^{F\cup i}(\mu_2)) = 1\otimes\theta_i^{F\cup i}(\mu_1\cdot\mu_2)\in\underline{\mathrm{CH}}(\mathrm{M}_{F\cup i})\otimes\mathrm{CH}(\mathrm{M}^{F\cup i})$$

does not involve the top degree component of $\mathrm{CH}(\mathrm{M}^{F\cup i})$. Thus,

$$-\underline{\deg}_{\mathrm{M}_{F\cup i}} \otimes \deg_{\mathrm{M}^{F\cup i}} \left(\left(\underline{\beta}_{\mathrm{M}_{F\cup i}} \otimes 1 \right) \cdot \left(1 \otimes \theta_i^{F\cup i}(\mu_1) \right) \cdot \left(1 \otimes \theta_i^{F\cup i}(\mu_2) \right) \right) = 0,$$

and the left-hand side of the desired equality further simplifies to

$$-\underline{\deg}_{\mathrm{M}_{F\cup i}} \otimes \deg_{\mathrm{M}^{F\cup i}} \left(\left(1 \otimes \alpha_{\mathrm{M}^{F\cup i}} \right) \cdot \left(1 \otimes \theta_i^{F\cup i}(\mu_1) \right) \cdot \left(1 \otimes \theta_i^{F\cup i}(\mu_2) \right) \right).$$

Now the third part of Lemma 3.3 shows that the above quantity is the right-hand side of the desired equality. □

Lemma 3.9. If flats F_1, F_2 are in $\mathcal{S}_i$ and F_1 is a proper subset of F_2, then

$$x_{F_1\cup i}\, x_{F_2\cup i} \in x_{F_1\cup i} \mathrm{CH}_{(i)}.$$

Similarly, if F_1, F_2 are in $\underline{\mathcal{S}}_i$ and F_1 is a proper subset of F_2, then

$$x_{F_1\cup i}\, x_{F_2\cup i} \in x_{F_1\cup i} \underline{\mathrm{CH}}_{(i)}.$$

Proof. Since $F_1 \cup i$ is not comparable to F_2, we have

$$x_{F_1\cup i}\, x_{F_2\cup i} = x_{F_1\cup i}(x_{F_2} + x_{F_2\cup i}) = x_{F_1\cup i}\theta_i(x_{F_2}).$$

The second part follows from the same argument. □

Proof of Theorem 1.2, Theorem 1.5, and parts (1) and (4) of Theorem 1.6. All the summands in the proposed decompositions are cyclic, and therefore indecomposable in the category of graded modules.[14] We prove the decompositions by induction on the cardinality of the ground set E. If E is empty, then Theorem 1.2, Theorem 1.5, and part (1) of Theorem 1.6 are vacuous, while part (4) of Theorem 1.6 is trivial. Furthermore, all of these results are trivial when E is a singleton. Thus, we may assume that i is an element of E, that $E\backslash i$ is nonempty, and that all the results hold for loopless matroids whose ground set is a proper subset of E.

First we assume that i is not a coloop. Let us show that the terms in the right-hand side of the decomposition (D_1) are orthogonal. Multiplying $\mathrm{CH}_{(i)}$ and $x_{F\cup i}\mathrm{CH}_{(i)}$ lands in $x_{F\cup i}\mathrm{CH}_{(i)}$, and this ideal vanishes in degree d by Remark 3.7, so they are orthogonal. On the other hand, the product of $x_{F_1\cup i}\mathrm{CH}_{(i)}$ and $x_{F_2\cup i}\mathrm{CH}_{(i)}$ vanishes if $F_1, F_2 \in \mathcal{S}_i$ are not comparable, while if $F_1 < F_2$ or $F_2 < F_1$, the product is contained in $x_{F_1\cup i}\mathrm{CH}_{(i)}$ or $x_{F_2\cup i}\mathrm{CH}_{(i)}$ respectively, by Lemma 3.9. So these terms are also orthogonal.

It follows from the induction hypothesis and Lemma 3.3 that the restriction of the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$ to $\mathrm{CH}_{(i)}$ is nondegenerate. By Proposition 3.5, Proposition 3.8, and the induction hypothesis, the restriction of the Poincaré pairing of $\mathrm{CH}(\mathrm{M})$ to any other summand $x_{F\cup i}\mathrm{CH}_{(i)}$ is

[14] By [CF82, Corollary 2] or [GG82, Theorem 3.2], the indecomposability of the summands in the category of graded modules implies the indecomposability of the summands in the category of modules.

also nondegenerate. Therefore, we can conclude that the sum on the right-hand side of (D_1) is a direct sum with a nondegenerate Poincaré pairing.

To complete the proof of the decomposition (D_1) and the Poincaré duality theorem for $\mathrm{CH}(\mathrm{M})$, we must show that the direct sum

$$\mathrm{CH}_{(i)} \oplus \bigoplus_{F \in \mathcal{S}_i} x_{F \cup i} \mathrm{CH}_{(i)}$$

is equal to all of $\mathrm{CH}(\mathrm{M})$. This is obvious in degree 0. To see that it holds in degree 1, it is enough to check that x_G is contained in the direct sum for any proper flat G of M. If $G \backslash i \notin \mathcal{S}_i$, then $x_G = \theta_i(x_{G\backslash i})$ is in $\mathrm{CH}_{(i)}$. If $G\backslash i \in \mathcal{S}_i$, then either $i \in G$ or $i \notin G$. In the first case, $x_G = x_{(G\backslash i)\cup i}$ is an element of the summand indexed by $F = G\backslash i$. In the second case, we have $x_G = \theta_i(x_G) - x_{G\cup i}$, which lies in $\mathrm{CH}_{(i)} + x_{G \cup i}\mathrm{CH}_{(i)}$.

Since our direct sum is a sum of $\mathrm{CH}(\mathrm{M}\backslash i)$-modules and it includes the degree 0 and 1 parts of $\mathrm{CH}(\mathrm{M})$, it will suffice to show that $\mathrm{CH}(\mathrm{M})$ is generated in degrees 0 and 1 as a graded $\mathrm{CH}(\mathrm{M}\backslash i)$-module. In other words, we need to show that

$$\mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^k(\mathrm{M}) = \mathrm{CH}^{k+1}(\mathrm{M}) \ \text{ for any } k \geqslant 1.$$

We first prove the equality when $k = 1$. Since we have proved that the decomposition (D_1) holds in degree 1, we know that

$$\mathrm{CH}^2(\mathrm{M}) = \mathrm{CH}^1(\mathrm{M}) \cdot \mathrm{CH}^1(\mathrm{M}) = \Bigg(\mathrm{CH}^1_{(i)} \oplus \bigoplus_{F \in \mathcal{S}_i} \mathbb{Q} x_{F\cup i}\Bigg) \cdot \Bigg(\mathrm{CH}^1_{(i)} \oplus \bigoplus_{F \in \mathcal{S}_i} \mathbb{Q} x_{F\cup i}\Bigg).$$

Using Lemma 3.9, we may reduce the problem to showing that

$$x^2_{F\cup i} \in \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(\mathrm{M}) \ \text{ for any } F \in \mathcal{S}_i.$$

We can rewrite the relation $0 = x_F y_i$ in the augmented Chow ring of M as

$$\begin{aligned}
0 &= (\theta_i(x_F) - x_{F\cup i}) \sum_{i \notin G} x_G \\
&= \theta_i(x_F)\Big(\sum_{i\notin G} x_G\Big) - x_{F\cup i}\Big(\sum_{G \leqslant F} x_G\Big), \\
&= \theta_i(x_F)\Big(\sum_{i\notin G} x_G\Big) - (\theta_i(x_F) - x_F)\Big(\sum_{G < F} x_G\Big) - x_{F\cup i}x_F \\
&= \theta_i(x_F)\Big(\sum_{i\notin G} x_G - \sum_{G<F} x_G\Big) + x_F\Big(\sum_{G<F} x_G\Big) - x_{F\cup i}\theta_i(x_F) + x^2_{F\cup i},
\end{aligned}$$

thus reducing the problem to showing that

$$x_F x_G \in \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(\mathrm{M}) \ \text{ for any } G < F \in \mathcal{S}_i.$$

The collection $\mathcal{S}_i$ is downward closed, meaning that if $G < F \in \mathcal{S}_i$, then $G \in \mathcal{S}_i$; therefore,

$$x_F x_G = (\theta_i(x_F) - x_{F\cup i})(\theta_i(x_G) - x_{G\cup i}).$$

Lemma 3.9 tells us that $x_{F\cup i}x_{G\cup i} \in \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(\mathrm{M})$, thus so is $x_F x_G$.

We next prove the equality when $k \geqslant 2$. In this case, we use the result for $k = 1$ along with the fact that the algebra $\mathrm{CH}(\mathrm{M})$ is generated in degree 1 to conclude that

$$\mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^k(\mathrm{M}) = \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(\mathrm{M}) \cdot \mathrm{CH}^{k-1}(\mathrm{M}) = \mathrm{CH}^2(\mathrm{M}) \cdot \mathrm{CH}^{k-1}(\mathrm{M}) = \mathrm{CH}^{k+1}(\mathrm{M}).$$

This completes the proof of the decomposition (D_1) and the Poincaré duality theorem for $\mathrm{CH}(\mathrm{M})$ when there is an element i that is not a coloop of M.

The proof when i is a coloop is almost the same; we explain the places where something different must be said. The orthogonality of $x_{E\setminus i}\mathrm{CH}_{(i)}$ and $x_{F\cup i}\mathrm{CH}_{(i)}$ for $F \in \mathcal{S}_i$ follows because $E\setminus i$ and $F \cup i$ are incomparable. To show that the right-hand side of (D_2) spans $\mathrm{CH}(\mathrm{M})$, one extra statement we need to check is that

$$x^2_{E\setminus i} \in \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(\mathrm{M}).$$

Since i is a coloop, $\mathcal{S}_i$ is the set of all flats properly contained in $E\setminus i$, and we have

$$0 = x_{E\setminus i} y_i = \sum_{i \notin F} x_F x_{E\setminus i} = x^2_{E\setminus i} + \sum_{F \in \mathcal{S}_i} x_{E\setminus i} x_F = x^2_{E\setminus i} + \sum_{F\in\mathcal{S}_i} x_{E\setminus i}\theta_i(x_F),$$

where the last equality follows because $E\setminus i$ and $F \cup i$ are not comparable. Thus

$$x^2_{E\setminus i} = -\sum_{F\in\mathcal{S}_i} x_{E\setminus i}\theta_i(x_F) \in \mathrm{CH}^1_{(i)} \cdot \mathrm{CH}^1(\mathrm{M}).$$

By the induction hypothesis, we know $\mathrm{CH}(\mathrm{M}\setminus i)$ satisfies the Poincaré duality theorem. By the coloop case of Lemma 3.3, the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$ restricts to a perfect pairing between $\mathrm{CH}_{(i)}$ and $x_{E\setminus i}\mathrm{CH}_{(i)}$. Since $\mathrm{CH}_{(i)}$ is a subring of $\mathrm{CH}(\mathrm{M})$ and is zero in degree d, the restriction of the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$ to $\mathrm{CH}_{(i)}$ is zero. Therefore, the subspaces $\mathrm{CH}_{(i)}$ and $x_{E\setminus i}\mathrm{CH}_{(i)}$ intersect trivially, and the restriction of the Poincaré pairing on $\mathrm{CH}(\mathrm{M})$ to $\mathrm{CH}_{(i)} \oplus x_{E\setminus i}\mathrm{CH}_{(i)}$ is nondegenerate. This completes the proof of the theorems about $\mathrm{CH}(\mathrm{M})$ when i is a coloop.

Now, we show the statements about the decomposition ($\underline{\mathrm{D}}_1$). By an argument identical to the one used for (D_1), we know that the sum on the right-hand side of ($\underline{\mathrm{D}}_1$) is a direct sum with a nondegenerate Poincaré pairing. Next, we observe that the surjectivity of the pullback $\varphi^{\varnothing}_{\mathrm{M}}$ gives the equality

$$\underline{\mathrm{CH}}^1_{(i)} \cdot \underline{\mathrm{CH}}^k(\mathrm{M}) = \underline{\mathrm{CH}}^{k+1}(\mathrm{M}) \ \text{ for any } k \geqslant 1.$$

Thus, the direct sum decomposition ($\underline{\mathrm{D}}_1$) and the Poincaré duality theorem for $\underline{\mathrm{CH}}(\mathrm{M})$ follow when i is not a coloop. When i is a coloop, we can prove ($\underline{\mathrm{D}}_2$) and the Poincaré duality theorem by making the same adjustments as the ones in the proof about $\mathrm{CH}(\mathrm{M})$. □

4. PROOFS OF THE HARD LEFSCHETZ THEOREMS AND THE HODGE–RIEMANN RELATIONS

In this section, we prove Theorem 1.6. Parts (1) and (4) have already been proved in the previous section. We will first prove parts (2) and (3) by induction on the cardinality of E. The proof of parts (5) and (6) is nearly identical to the proof of parts (2) and (3), with the added nuance that we use parts (2) and (3) for the matroid M in the proof of parts (5) and (6) for the matroid M.

Proof of Theorem 1.6, parts (2) and (3). The statements are trivial when the cardinality of E is 0 or 1, so we will assume throughout the proof that the cardinality of E is at least 2.

Let B be the Boolean matroid on E. By the induction hypothesis, we know that for every nonempty proper flat F of M, the fans $\underline{\Pi}_{\mathrm{M}_F}$ and $\underline{\Pi}_{\mathrm{M}^F}$ satisfy the hard Lefschetz theorem and the Hodge–Riemann relations with respect to any strictly convex piecewise linear functions on $\underline{\Pi}_{\mathrm{B}_F}$ and $\underline{\Pi}_{\mathrm{B}^F}$, respectively. By [AHK18, Proposition 7.7], this implies that for every nonempty proper flat F of M, the product $\underline{\Pi}_{\mathrm{M}_F} \times \underline{\Pi}_{\mathrm{M}^F}$ satisfies the hard Lefschetz theorem and the Hodge–Riemann relations with respect to any strictly convex piecewise linear function on $\underline{\Pi}_{\mathrm{B}_F} \times \underline{\Pi}_{\mathrm{B}^F}$. In other words, $\underline{\Pi}_{\mathrm{M}}$ satisfies the *local Hodge–Riemann relations* [AHK18, Definition 7.14]:

The star of any ray in $\underline{\Pi}_{\mathrm{M}}$ satisfies the Hodge–Riemann relations.

This in turn implies that $\underline{\Pi}_{\mathrm{M}}$ satisfies the hard Lefschetz theorem with respect to any strictly convex piecewise linear function on $\underline{\Pi}_{\mathrm{B}}$ [AHK18, Proposition 7.15]. It remains to prove only that $\underline{\Pi}_{\mathrm{M}}$ satisfies the Hodge–Riemann relations with respect to any strictly convex piecewise linear function on $\underline{\Pi}_{\mathrm{B}}$.

Let ℓ be a piecewise linear function on $\underline{\Pi}_{\mathrm{B}}$, and let $\underline{\mathrm{HR}}^k_\ell(\mathrm{M})$ be the *Hodge–Riemann form*

$$\underline{\mathrm{HR}}^k_\ell(\mathrm{M}) : \underline{\mathrm{CH}}^k(\mathrm{M}) \times \underline{\mathrm{CH}}^k(\mathrm{M}) \longrightarrow \mathbb{Q}, \qquad (\eta_1, \eta_2) \longmapsto (-1)^k \underline{\deg}_{\mathrm{M}}(\ell^{d-2k-1}\eta_1\eta_2).$$

By [AHK18, Proposition 7.6], the fan $\underline{\Pi}_{\mathrm{M}}$ satisfies the Hodge–Riemann relations with respect to ℓ if and only if, for all $k < \frac{d}{2}$, the Hodge–Riemann form $\underline{\mathrm{HR}}^k_\ell(\mathrm{M})$ is nondegenerate and has the signature

$$\sum_{j=0}^{k} (-1)^{k-j} \Big(\dim \underline{\mathrm{CH}}^j(\mathrm{M}) - \dim \underline{\mathrm{CH}}^{j-1}(\mathrm{M}) \Big).$$

Since $\underline{\Pi}_{\mathrm{M}}$ satisfies the hard Lefschetz theorem with respect to any strictly convex piecewise linear function on $\underline{\Pi}_{\mathrm{B}}$ and signature is a locally constant function on the space of nonsingular forms, the following statements are equivalent:

(i) The fan $\underline{\Pi}_{\mathrm{M}}$ satisfies the Hodge–Riemann relations with respect to any strictly convex piecewise linear function on $\underline{\Pi}_{\mathrm{B}}$.

(ii) The fan $\underline{\Pi}_{\mathrm{M}}$ satisfies the Hodge–Riemann relations with respect to some strictly convex piecewise linear function on $\underline{\Pi}_{\mathrm{B}}$.

Furthermore, since satisfying the Hodge–Riemann relations with respect to a given piecewise linear function is an open condition on the function, statement (ii) is equivalent to the following:

(iii) The fan $\underline{\Pi}_{\mathrm{M}}$ satisfies the Hodge–Riemann relations with respect to some convex piecewise linear function on $\underline{\Pi}_{\mathrm{B}}$.

We show that statement (iii) holds using the semi-small decomposition in Theorem 1.2.[15]

If M is the Boolean matroid B, then $\underline{\mathrm{CH}}(\mathrm{M})$ can be identified with the cohomology ring of the smooth complex projective toric variety $X_{\underline{\Pi}_{\mathrm{B}}}$. Therefore, in this case, Theorem 1.6 is a special case of the usual hard Lefschetz theorem and the Hodge–Riemann relations for smooth complex projective varieties.[16]

If M is not the Boolean matroid B, then it has some element $i \in E$ that is not a coloop. Consider the morphism of fans

$$\underline{\pi}_i : \underline{\Pi}_{\mathrm{M}} \longrightarrow \underline{\Pi}_{\mathrm{M}\setminus i}.$$

By induction, we know that $\underline{\Pi}_{\mathrm{M}\setminus i}$ satisfies the Hodge–Riemann relations with respect to any strictly convex piecewise linear function ℓ on $\underline{\Pi}_{\mathrm{B}\setminus i}$. We will show that $\underline{\Pi}_{\mathrm{M}}$ satisfies the Hodge–Riemann relations with respect to the pullback $\ell_i \coloneqq \ell \circ \pi_i$, which is a piecewise linear function on $\underline{\Pi}_{\mathrm{B}}$ that is convex but not necessarily strictly convex.

By Theorem 1.2, we have the orthogonal decomposition of $\underline{\mathrm{CH}}(\mathrm{M})$ into $\underline{\mathrm{CH}}(\mathrm{M}\setminus i)$-modules

$$\underline{\mathrm{CH}}(\mathrm{M}) = \underline{\mathrm{CH}}_{(i)} \oplus \bigoplus_{F \in \underline{\mathcal{S}}_i} x_{F\cup i}\underline{\mathrm{CH}}_{(i)}.$$

By orthogonality, it is enough to show that each summand of $\underline{\mathrm{CH}}(\mathrm{M})$ satisfies the Hodge–Riemann relations with respect to ℓ_i:

(iv) For every nonnegative integer $k < \frac{d}{2}$, the bilinear form

$$\underline{\mathrm{CH}}^k_{(i)} \times \underline{\mathrm{CH}}^k_{(i)} \longrightarrow \mathbb{Q}, \qquad (\eta_1,\eta_2) \longmapsto (-1)^k \underline{\deg}_{\mathrm{M}}(\ell_i^{d-2k-1}\eta_1\eta_2)$$

is positive definite on the kernel of multiplication by ℓ_i^{d-2k}.

(v) For every nonnegative integer $k < \frac{d}{2}$, the bilinear form

$$x_{F\cup i}\underline{\mathrm{CH}}^{k-1}_{(i)} \times x_{F\cup i}\underline{\mathrm{CH}}^{k-1}_{(i)} \longrightarrow \mathbb{Q}, \qquad (\eta_1,\eta_2) \longmapsto (-1)^k \underline{\deg}_{\mathrm{M}}(\ell_i^{d-2k-1}\eta_1\eta_2)$$

is positive definite on the kernel of multiplication by ℓ_i^{d-2k}.

[15]This inductive paradigm of [AHK18] goes back to [McM93], where it was used to prove the hard Lefschetz theorem for simple polytopes.

[16]It is not difficult to directly prove the hard Lefschetz theorem and the Hodge–Riemann relations for $\underline{\mathrm{CH}}(\mathrm{B})$ using the coloop case of Theorem 1.2. Alternatively, we may apply McMullen's hard Lefschetz theorem and Hodge–Riemann relations for polytope algebras [McM93] to the standard permutohedron in $\mathbb{R}^E$.

By Proposition 3.4, the homomorphism $\underline{\theta}_i$ restricts to an isomorphism of $\underline{\mathrm{CH}}(\mathrm{M}\backslash i)$-modules

$$\underline{\mathrm{CH}}(\mathrm{M}\backslash i) \cong \underline{\mathrm{CH}}_{(i)}.$$

Thus, statement (iv) follows from Lemma 3.3 and the induction hypothesis applied to $\mathrm{M}\backslash i$. By Propositions 2.27, 3.4, and 3.5, the homomorphisms $\underline{\theta}_i^{F\cup i}$ and $\underline{\psi}_{\mathrm{M}}^{F\cup i}$ give a $\underline{\mathrm{CH}}(\mathrm{M}\backslash i)$-module isomorphism

$$\underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \cong \underline{\mathrm{CH}}(\mathrm{M}_{F\cup i}) \otimes \underline{\theta}_i^{F\cup i}\underline{\mathrm{CH}}(\mathrm{M}^F) \cong x_{F\cup i}\underline{\mathrm{CH}}_{(i)}[1].$$

Note that the pullback of a strictly convex piecewise linear function on $\underline{\Pi}_{\mathrm{B}\backslash i}$ to the star

$$\underline{\Pi}_{(\mathrm{B}\backslash i)_F} \times \underline{\Pi}_{(\mathrm{B}\backslash i)^F} = \underline{\Pi}_{\mathrm{B}_{F\cup i}} \times \underline{\Pi}_{\mathrm{B}^F}$$

is the class of a strictly convex piecewise linear function. Therefore, statement (v) follows from Proposition 3.8 and the induction applied to $\mathrm{M}_{F\cup i}$ and M^F. □

Proof of Theorem 1.6, parts (5) and (6). This proof is nearly identical to the proof of parts (2) and (3). In that argument, we used the fact that rays of $\underline{\Pi}_{\mathrm{M}}$ are indexed by nonempty proper flats of M and the star of the ray $\underline{\rho}_F$ is isomorphic to $\underline{\Pi}_{\mathrm{M}_F} \times \underline{\Pi}_{\mathrm{M}^F}$, which we can show satisfies the hard Lefschetz theorem and the Hodge–Riemann relations using the induction hypothesis. When dealing instead with the augmented Bergman fan Π_{M}, we have rays indexed by elements of E and rays indexed by proper flats of M, with

$$\mathrm{star}_{\rho_i}\Pi_{\mathrm{M}} \cong \Pi_{\mathrm{M}_{\mathrm{cl}(i)}} \qquad \text{and} \qquad \mathrm{star}_{\rho_F}\Pi_{\mathrm{M}} \cong \underline{\Pi}_{\mathrm{M}_F} \times \Pi_{\mathrm{M}^F}.$$

Thus the stars of ρ_i and ρ_F for nonempty F can be shown to satisfy the hard Lefschetz theorem and the Hodge–Riemann relations using the induction hypothesis. However, the star of $\rho_\varnothing$ is isomorphic to $\underline{\Pi}_{\mathrm{M}}$, so we need to use parts (2) and (3) of Theorem 1.6 for M itself. □

Remark 4.1. It is possible to deduce Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations for $\mathrm{CH}(\mathrm{M})$ using [AHK18, Theorem 6.19 and Theorem 8.8], where the three properties are proved for generalized Bergman fans $\Sigma_{\mathrm{N},\mathcal{P}}$ in [AHK18, Definition 3.2]. We sketch the argument here, leaving details to the interested readers. Consider the direct sum $\mathrm{M}\oplus 0$ of M and the rank 1 matroid on the singleton $\{0\}$ and the order filter $\mathcal{P}(\mathrm{M})$ of all proper flats of $\mathrm{M}\oplus 0$ that contain 0. The symbols $\mathrm{B}\oplus 0$ and $\mathcal{P}(\mathrm{B})$ are defined in the same way for the Boolean matroid B on E. It is straightforward to check that the linear isomorphism

$$\mathbb{R}^E \longrightarrow \mathbb{R}^{E\cup 0}/\langle \mathbf{e}_E + \mathbf{e}_0\rangle, \quad \mathbf{e}_j \longmapsto \mathbf{e}_j$$

identifies the complete fan Π_{B} with the complete fan $\Sigma_{\mathrm{B}\oplus 0,\mathcal{P}(\mathrm{B})}$, and the augmented Bergman fan Π_{M} with a subfan of $\Sigma_{\mathrm{M}\oplus 0,\mathcal{P}(\mathrm{M})}$. The third identity in Lemma 2.11 shows that the inclusion of the augmented Bergman fan Π_{M} into the generalized Bergman fan $\Sigma_{\mathrm{M}\oplus 0,\mathcal{P}(\mathrm{M})}$ induces an isomorphism between their Chow rings.

5. Proof of Theorem 1.8

In this section, we prove the decomposition ($\underline{\mathrm{D}}_3$) by induction on the cardinality of E. The decomposition (D_3) can be proved using the same argument. The results are trivial when E has at most one element. Thus, we may assume that i is an element of E, that $E \backslash i$ is nonempty, and that all the results hold for loopless matroids whose ground set is a proper subset of E.

We first prove that the summands appearing in the right-hand side of ($\underline{\mathrm{D}}_3$) are orthogonal to each other.

Lemma 5.1. Let F and G be distinct nonempty proper flats of M.

(1) The spaces $\underline{\psi}^F_{\mathrm{M}}\, \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F)$ and $\underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M})$ are orthogonal in $\underline{\mathrm{CH}}(\mathrm{M})$.

(2) The spaces $\underline{\psi}^F_{\mathrm{M}}\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F)$ and $\underline{\psi}^G_{\mathrm{M}}\underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^G)$ are orthogonal in $\underline{\mathrm{CH}}(\mathrm{M})$.

Proof. The fifth bullet point in Proposition 2.24, together with the fact that $\underline{\psi}^F_{\mathrm{M}}$ is a $\underline{\mathrm{CH}}(\mathrm{M})$-module homomorphism via $\underline{\varphi}^F_{\mathrm{M}}$ (Proposition 2.25), implies that both $\underline{\psi}^F_{\mathrm{M}}\, \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F)$ and $\underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M})$ are $\underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M})$-submodules of $\underline{\mathrm{CH}}(\mathrm{M})$. Thus, the product of $\mu \in \underline{\psi}^F_{\mathrm{M}}\, \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F)$ and $\nu \in \underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M})$ of complimentary degree lands in the degree $d-1$ component of $\underline{\psi}^F_{\mathrm{M}}\, \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F)$, which is zero. The first orthogonality follows.

For the second orthogonality, we may suppose that F is a proper subset of G. Since $\underline{\psi}^G_{\mathrm{M}}$ is a $\underline{\mathrm{CH}}(\mathrm{M})$-module homomorphism commuting with the degree maps (Proposition 2.29 and Remark 2.30), it is enough to show that

$$\underline{\varphi}^G_{\mathrm{M}}\underline{\psi}^F_{\mathrm{M}}\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F) \text{ and } \underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^G) \text{ are orthogonal in } \underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G).$$

For this, we use the commutative diagram of pullback and pushforward maps

$$\begin{tikzcd}
\underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \arrow[r, "\underline{\psi}^F_{\mathrm{M}}"] \arrow[d, "\underline{\varphi}^{G\backslash F}_{\mathrm{M}_F} \otimes 1"] & \underline{\mathrm{CH}}(\mathrm{M}) \arrow[d, "\underline{\varphi}^G_{\mathrm{M}}"] \\
\underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G_F) \otimes \underline{\mathrm{CH}}(\mathrm{M}^F) \arrow[r, "1 \otimes \underline{\psi}^F_{\mathrm{M}^G}"] & \underline{\mathrm{CH}}(\mathrm{M}_G) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G),
\end{tikzcd}$$

which further reduces to the assertion that

$$\underline{\psi}^F_{\mathrm{M}^G}\underline{\mathrm{CH}}(\mathrm{M}^G_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F) \text{ and } \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^G) \text{ are orthogonal in } \underline{\mathrm{CH}}(\mathrm{M}^G).$$

Since $\underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^G) \subseteq \underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M}^G)$, the above follows from the first orthogonality for M^G. □

We next show that the restriction of the Poincaré pairing of $\underline{\mathrm{CH}}(\mathrm{M})$ to each summand appearing in the right-hand side of ($\underline{\mathrm{D}}_3$) is nondegenerate.

Lemma 5.2. Let F be a nonempty proper flat of M, and let $k = \mathrm{rk}_{\mathrm{M}}(F)$.

(1) The restriction of the Poincaré pairing of $\underline{\mathrm{CH}}(\mathrm{M})$ to $\underline{\mathrm{H}}_\alpha(\mathrm{M})$ is nondegenerate.

(2) The restriction of the Poincaré pairing of $\underline{\mathrm{CH}}(\mathrm{M})$ to $\underline{\psi}^F_{\mathrm{M}}\underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)$ is nondegenerate.

Proof. The first statement follows from Proposition 2.32. We prove the second statement.

Let $N=\underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)$ and $N_i=\underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\underline{\mathrm{J}}^i_\alpha(\mathrm{M}^F)$. Notice that

$$\underline{\mathrm{J}}^i_\alpha(\mathrm{M}^F)\cong\begin{cases}\mathbb{Q}, & 0\leqslant i\leqslant k-2;\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, the total dimensions of N_i are the same for $0\leqslant i\leqslant k-2$. For the second statement, we need to show the nondegeneracy of the bilinear form on $N=\underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)$ defined by

$$\langle\mu_1\otimes\nu_1,\mu_2\otimes\nu_2\rangle=\underline{\deg}_{\mathrm{M}}\big(\underline{\psi}^F_{\mathrm{M}}(\mu_1\otimes\nu_1)\cdot\underline{\psi}^F_{\mathrm{M}}(\mu_2\otimes\nu_2)\big).$$

Since the pushfoward $\underline{\psi}^F_{\mathrm{M}}$ is a $\underline{\mathrm{CH}}(\mathrm{M})$-module homomorphism (Proposition 2.24) and commutes with the degree maps (Remark 2.26), we have

$$\begin{aligned}\underline{\deg}_{\mathrm{M}}\big(\underline{\psi}^F_{\mathrm{M}}(\mu_1\otimes\nu_1)\cdot\underline{\psi}^F_{\mathrm{M}}(\mu_2\otimes\nu_2)\big)&=\underline{\deg}_{\mathrm{M}}\big(\underline{\psi}^F_{\mathrm{M}}\big(\underline{\varphi}^F_{\mathrm{M}}\underline{\psi}^F_{\mathrm{M}}(\mu_1\otimes\nu_1)\cdot(\mu_2\otimes\nu_2)\big)\big)\\&=\underline{\deg}_{\mathrm{M}_F}\otimes\underline{\deg}_{\mathrm{M}^F}\big(\underline{\varphi}^F_{\mathrm{M}}\underline{\psi}^F_{\mathrm{M}}(\mu_1\otimes\nu_1)\cdot(\mu_2\otimes\nu_2)\big).\end{aligned}$$

Since the composition $\underline{\varphi}^F_{\mathrm{M}}\underline{\psi}^F_{\mathrm{M}}$ is multiplication by $\underline{\varphi}^F_{\mathrm{M}}(x_F)$ (Proposition 2.25), the above becomes

$$-\underline{\deg}_{\mathrm{M}_F}\otimes\underline{\deg}_{\mathrm{M}^F}\big((1\otimes\underline{\alpha}_{\mathrm{M}^F}+\underline{\beta}_{\mathrm{M}_F}\otimes 1)\cdot(\mu_1\otimes\nu_1)\cdot(\mu_2\otimes\nu_2)\big).$$

Assuming that $\nu_1\in\underline{\mathrm{J}}^{k_1}_\alpha(\mathrm{M}^F)$ and $\nu_2\in\underline{\mathrm{J}}^{k_2}_\alpha(\mathrm{M}^F)$, the above expression vanishes unless $k_1+k_2=k-1$ or $k-2$. In other words, the subspaces N_i and N_{k-2-j} are orthogonal with respect to the bilinear form on N, unless $i=j$ or $i=j+1$. So the bilinear form can be represented by a block lower-triangular matrix, and its nondegeneracy is equivalent to the nondegeneracy of each diagonal block. Thus, it suffices to show that the induced pairing between N_i and N_{k-2-i} is nondegenerate.

For this, assume that $\nu_1\in\underline{\mathrm{J}}^i_\alpha(\mathrm{M}^F)$ and $\nu_2\in\underline{\mathrm{J}}^{k-2-i}_\alpha(\mathrm{M}^F)$. By the above arguments, we have

$$\langle\mu_1\otimes\nu_1,\mu_2\otimes\nu_2\rangle=-\underline{\deg}_{\mathrm{M}_F}\otimes\underline{\deg}_{\mathrm{M}^F}\big((1\otimes\underline{\alpha}_{\mathrm{M}^F}+\underline{\beta}_{\mathrm{M}_F}\otimes 1)\cdot(\mu_1\mu_2\otimes\nu_1\nu_2)\big).$$

Since $\nu_1\nu_2\in\underline{\mathrm{CH}}^{k-2}(\mathrm{M}^F)$, we have $\underline{\deg}_{\mathrm{M}_F}\otimes\underline{\deg}_{\mathrm{M}^F}\big(\underline{\beta}_{\mathrm{M}_F}\mu_1\mu_2\otimes\nu_1\nu_2\big)=0$, and hence

$$\langle\mu_1\otimes\nu_1,\mu_2\otimes\nu_2\rangle=-\underline{\deg}_{\mathrm{M}_F}\otimes\underline{\deg}_{\mathrm{M}^F}\big(\mu_1\mu_2\otimes\underline{\alpha}_{\mathrm{M}^F}\nu_1\nu_2\big)=-\underline{\deg}_{\mathrm{M}_F}(\mu_1\mu_2)\underline{\deg}_{\mathrm{M}^F}(\underline{\alpha}_{\mathrm{M}^F}\nu_1\nu_2).$$

Notice that for any nonzero $\nu_1\in\underline{\mathrm{J}}^i_\alpha(\mathrm{M}^F)$ and nonzero $\nu_2\in\underline{\mathrm{J}}^{k-2-i}_\alpha(\mathrm{M}^F)$, we have $\underline{\deg}_{\mathrm{M}^F}(\underline{\alpha}_{\mathrm{M}^F}\nu_1\nu_2)\neq 0$. The nondegeneracy of the pairing between N_i and N_{k-2-i} follows from the nondegeneracy of the Poincaré pairing of $\underline{\mathrm{CH}}(\mathrm{M}_F)$. $\square$

To complete the proof, we only need to show that the graded vector spaces on both sides of (D_3) have the same dimension, which is the next proposition.

Proposition 5.3. There exists an isomorphism of graded vector spaces

$$\underline{\mathrm{CH}}(\mathrm{M}) \cong \underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M}) \oplus \bigoplus_{F \in \underline{\mathcal{Q}}(\mathrm{M})} \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F)[-1], \qquad (\underline{\mathrm{D}}_3')$$

where the sum is over the set $\underline{\mathcal{Q}}(\mathrm{M})$ of proper flats of M with rank at least two.

Proof. We prove the proposition using induction on the cardinality of E. Suppose the proposition holds for any matroid whose ground set is a proper subset of E. Suppose that there exists an element $i \in E$ that is not a coloop.

By Remark 3.6, for any G in $\underline{\mathcal{S}}_i$, we have $x_{G\cup i}\underline{\mathrm{CH}}_{(i)} \cong \underline{\mathrm{CH}}(\mathrm{M}_{G\cup i}) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G)[-1]$ as graded vector spaces. Thus, the decomposition ($\underline{\mathrm{D}}_1$) implies

$$\underline{\mathrm{CH}}(\mathrm{M}) \cong \underline{\mathrm{CH}}(\mathrm{M}\backslash i) \oplus \bigoplus_{G \in \underline{\mathcal{S}}_i(\mathrm{M})} \underline{\mathrm{CH}}(\mathrm{M}_{G\cup i}) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G)[-1].$$

By applying the induction hypothesis to the matroids $\mathrm{M}\backslash i$ and M^G, we see that the left-hand side of ($\underline{\mathrm{D}}_3'$) is isomorphic to the graded vector space

$$\begin{aligned}
&\underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M}\backslash i) \oplus \bigoplus_{G \in \underline{\mathcal{Q}}(\mathrm{M}\backslash i)} \underline{\mathrm{CH}}\big((\mathrm{M}\backslash i)_G\big) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}\big((\mathrm{M}\backslash i)^G\big)[-1] \\
&\quad \oplus \bigoplus_{G \in \underline{\mathcal{S}}_i(\mathrm{M})} \underline{\mathrm{CH}}(\mathrm{M}_{G\cup i}) \otimes \underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M}^G)[-1] \\
&\quad \oplus \bigoplus_{G \in \underline{\mathcal{S}}_i(\mathrm{M})} \bigoplus_{F \in \underline{\mathcal{Q}}(\mathrm{M}^G)} \underline{\mathrm{CH}}(\mathrm{M}_{G\cup i}) \otimes \underline{\mathrm{CH}}(\mathrm{M}^G_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F)[-2].
\end{aligned}$$

Since i is not a coloop, we may replace $\underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M}\backslash i)$ by $\underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M})$.

Now, we further decompose the right-hand side of ($\underline{\mathrm{D}}_3'$) to match the displayed expression. For this, we split the index set $\underline{\mathcal{Q}}(\mathrm{M})$ into three groups:

(1) $F \in \underline{\mathcal{Q}}(\mathrm{M}), i \in F, F\backslash i \in \underline{\mathcal{S}}_i(\mathrm{M})$,

(2) $F \in \underline{\mathcal{Q}}(\mathrm{M}), i \in F, F\backslash i \notin \underline{\mathcal{S}}_i(\mathrm{M})$, and

(3) $F \in \underline{\mathcal{Q}}(\mathrm{M}), i \notin F$.

Suppose F belongs to the first group. In this case, we have $\underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F) \cong \underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M}^{F\backslash i})$ as graded vector spaces, because they both have one-dimensional component from degree 0 to $\mathrm{rk}(F) - 2$ by Proposition 2.32. Therefore, we have

$$\bigoplus_{\substack{F \in \underline{\mathcal{Q}}(\mathrm{M}) \\ i \in F, F\backslash i \in \underline{\mathcal{S}}_i(\mathrm{M})}} \underline{\mathrm{CH}}(\mathrm{M}_F) \otimes \underline{\mathrm{J}}_{\underline{\alpha}}(\mathrm{M}^F)[-1] \cong \bigoplus_{G \in \underline{\mathcal{S}}_i(\mathrm{M})} \underline{\mathrm{CH}}(\mathrm{M}_{G\cup i}) \otimes \underline{\mathrm{H}}_{\underline{\alpha}}(\mathrm{M}^G)[-1].$$

Suppose F belongs to the second group. In this case, $\mathrm{M}_F = (\mathrm{M}\backslash i)_{F\backslash i}$, and the matroids M^F and $(\mathrm{M}\backslash i)^{F\backslash i}$ have the same rank. Therefore, we have

$$\bigoplus_{\substack{F\in\underline{\mathcal{Q}}(\mathrm{M})\\ i\in F, F\backslash i\notin\underline{\mathcal{S}}_i(\mathrm{M})}} \underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-1]\cong\bigoplus_{G\in\underline{\mathcal{Q}}(\mathrm{M}\backslash i)\backslash\underline{\mathcal{Q}}(\mathrm{M})}\underline{\mathrm{CH}}\big((\mathrm{M}\backslash i)_G\big)\otimes\underline{\mathrm{J}}_\alpha\big((\mathrm{M}\backslash i)^G\big)[-1].$$

Suppose F belongs to the third group. In this case, we apply ($\underline{\mathrm{D}}_1$) to M_F and get

$$\begin{aligned}
&\bigoplus_{F\in\underline{\mathcal{Q}}(\mathrm{M}), i\notin F}\underline{\mathrm{CH}}(\mathrm{M}_F)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-1]\\
\cong&\bigoplus_{F\in\underline{\mathcal{Q}}(\mathrm{M}), i\notin F}\Big(\underline{\mathrm{CH}}(\mathrm{M}_F\backslash i)\oplus\bigoplus_{G\in\underline{\mathcal{S}}_i(\mathrm{M}_F)}\underline{\mathrm{CH}}(\mathrm{M}_{G\cup i})\otimes\underline{\mathrm{CH}}(\mathrm{M}_F^G)[-1]\Big)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-1]\\
\cong&\bigoplus_{F\in\underline{\mathcal{Q}}(\mathrm{M}), i\notin F}\underline{\mathrm{CH}}\big(\mathrm{M}_F\backslash i\big)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-1]\oplus\bigoplus_{\substack{G\in\underline{\mathcal{S}}_i(\mathrm{M})\\ F\in\underline{\mathcal{Q}}(\mathrm{M}^G)}}\underline{\mathrm{CH}}(\mathrm{M}_{G\cup i})\otimes\underline{\mathrm{CH}}(\mathrm{M}_F^G)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-2]\\
\cong&\bigoplus_{G\in\underline{\mathcal{Q}}(\mathrm{M}\backslash i)\cap\underline{\mathcal{Q}}(\mathrm{M})}\underline{\mathrm{CH}}\big((\mathrm{M}\backslash i)_G\big)\otimes\underline{\mathrm{J}}_\alpha\big((\mathrm{M}\backslash i)^G\big)[-1]\oplus\bigoplus_{\substack{G\in\underline{\mathcal{S}}_i(\mathrm{M})\\ F\in\underline{\mathcal{Q}}(\mathrm{M}^G)}}\underline{\mathrm{CH}}(\mathrm{M}_{G\cup i})\otimes\underline{\mathrm{CH}}(\mathrm{M}_F^G)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-2].
\end{aligned}$$

The decomposition ($\underline{\mathrm{D}}_3'$) follows.

Suppose now that every element of E is a coloop of M, that is, M is a Boolean matroid. We fix an element $i\in E$. The decomposition ($\underline{\mathrm{D}}_2$) and Remark 3.6 imply

$$\underline{\mathrm{CH}}(\mathrm{M})\cong\underline{\mathrm{CH}}(\mathrm{M}\backslash i)\oplus\underline{\mathrm{CH}}(\mathrm{M}\backslash i)[-1]\oplus\bigoplus_{G\in\underline{\mathcal{S}}_i(\mathrm{M})}\underline{\mathrm{CH}}(\mathrm{M}_{G\cup i})\otimes\underline{\mathrm{CH}}(\mathrm{M}^G)[-1].$$

The assumption that i is a coloop implies that $\underline{\mathcal{S}}_i(\mathrm{M})\cap\underline{\mathcal{Q}}(\mathrm{M})=\underline{\mathcal{Q}}(\mathrm{M}\backslash i)$. The induction hypothesis applies to the matroids $\mathrm{M}\backslash i$ and M^G, and hence the left-hand side of ($\underline{\mathrm{D}}_3'$) is isomorphic to

$$\begin{aligned}
&\underline{\mathrm{H}}_\alpha(\mathrm{M}\backslash i)\oplus\bigoplus_{G\in\underline{\mathcal{Q}}(\mathrm{M}\backslash i)}\underline{\mathrm{CH}}\big((\mathrm{M}\backslash i)_G\big)\otimes\underline{\mathrm{J}}_\alpha\big((\mathrm{M}\backslash i)^G\big)[-1]\\
&\oplus\underline{\mathrm{H}}_\alpha(\mathrm{M}\backslash i)[-1]\oplus\bigoplus_{G\in\underline{\mathcal{Q}}(\mathrm{M}\backslash i)}\underline{\mathrm{CH}}\big((\mathrm{M}\backslash i)_G\big)\otimes\underline{\mathrm{J}}_\alpha\big((\mathrm{M}\backslash i)^G\big)[-2]\\
&\oplus\bigoplus_{G\in\underline{\mathcal{S}}_i(\mathrm{M})}\underline{\mathrm{CH}}(\mathrm{M}_{G\cup i})\otimes\Big(\underline{\mathrm{H}}_\alpha(\mathrm{M}^G)\oplus\bigoplus_{F\in\underline{\mathcal{Q}}(\mathrm{M}^G)}\underline{\mathrm{CH}}(\mathrm{M}_F^G)\otimes\underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-1]\Big)[-1].
\end{aligned}$$

Now, we further decompose the right-hand side of ($\underline{\mathrm{D}}_3'$) to match the displayed expression. For this, we split the index set $\underline{\mathcal{Q}}(\mathrm{M})$ into three groups:

(1) $F\in\underline{\mathcal{Q}}(\mathrm{M}), i\in F$,

(2) $F\in\underline{\mathcal{Q}}(\mathrm{M}), F=E\backslash i$, and

(3) $F\in\underline{\mathcal{Q}}(\mathrm{M}), F\in\underline{\mathcal{S}}_i(\mathrm{M})$.

If F belongs to the first group, then $\underline{\mathrm{J}}_\alpha(\mathrm{M}^F) \cong \underline{\mathrm{H}}_\alpha(\mathrm{M}^{F\setminus i})$, and hence

$$\bigoplus_{F\in\underline{\mathcal{Q}}(\mathrm{M}), i\in F} \underline{\mathrm{CH}}(\mathrm{M}_F)\otimes \underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-1] \cong \bigoplus_{G\in\underline{\mathcal{S}}_i(\mathrm{M})} \underline{\mathrm{CH}}(\mathrm{M}_{G\cup i})\otimes \underline{\mathrm{H}}_\alpha(\mathrm{M}^G)[-1].$$

If F is the flat $E\setminus i$, $\underline{\mathrm{H}}_\alpha(\mathrm{M})$ has one-dimensional component from degree 0 to $d-1$; $\underline{\mathrm{H}}_\alpha(\mathrm{M}\setminus i)$ has one-dimensional component from degree 0 to $d-2$; and $\underline{\mathrm{J}}_\alpha(\mathrm{M}^{E\setminus i})$ has one-dimensional component from degree 0 to $d-3$. Thus, we have

$$\underline{\mathrm{H}}_\alpha(\mathrm{M})\oplus \underline{\mathrm{CH}}(\mathrm{M}_{E\setminus i})\otimes \underline{\mathrm{J}}_\alpha(\mathrm{M}^{E\setminus i})[-1] \cong \underline{\mathrm{H}}_\alpha(\mathrm{M}\setminus i)\oplus \underline{\mathrm{H}}_\alpha(\mathrm{M}\setminus i)[-1].$$

If F belongs to the third group, we apply ($\underline{\mathrm{D}}_2$) to M_F and get

$$\begin{aligned}
&\bigoplus_{\substack{F\in\underline{\mathcal{Q}}(\mathrm{M})\\ F\in\underline{\mathcal{S}}_i(\mathrm{M})}} \underline{\mathrm{CH}}(\mathrm{M}_F)\otimes \underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-1]\\
\cong &\bigoplus_{\substack{F\in\underline{\mathcal{Q}}(\mathrm{M})\\ F\in\underline{\mathcal{S}}_i(\mathrm{M})}} \Big(\underline{\mathrm{CH}}(\mathrm{M}_F\setminus i)\oplus \underline{\mathrm{CH}}(\mathrm{M}_F\setminus i)[-1]\oplus \bigoplus_{G\in\underline{\mathcal{S}}_i(\mathrm{M}_F)} \underline{\mathrm{CH}}(\mathrm{M}_{G\cup i})\otimes \underline{\mathrm{CH}}(\mathrm{M}^G_F)[-1]\Big)\otimes \underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-1]\\
\cong &\bigoplus_{\substack{G\in\underline{\mathcal{Q}}(\mathrm{M})\\ G\in\underline{\mathcal{S}}_i(\mathrm{M})}} \underline{\mathrm{CH}}(\mathrm{M}_G\setminus i)\otimes \underline{\mathrm{J}}_\alpha(\mathrm{M}^G)[-1]\oplus \bigoplus_{\substack{G\in\underline{\mathcal{Q}}(\mathrm{M})\\ G\in\underline{\mathcal{S}}_i(\mathrm{M})}} \underline{\mathrm{CH}}(\mathrm{M}_G\setminus i)\otimes \underline{\mathrm{J}}_\alpha(\mathrm{M}^G)[-2]\oplus\\
&\bigoplus_{\substack{G\in\underline{\mathcal{S}}_i(\mathrm{M})\\ F\in\underline{\mathcal{Q}}(\mathrm{M}^G)}} \underline{\mathrm{CH}}(\mathrm{M}_{G\cup i})\otimes \underline{\mathrm{CH}}(\mathrm{M}^G_F)\otimes \underline{\mathrm{J}}_\alpha(\mathrm{M}^F)[-2].
\end{aligned}$$

The decomposition ($\underline{\mathrm{D}}'_3$) follows. ☐

Remark 5.4. The decomposition of graded vector spaces appearing in [AHK18, Theorem 6.18] specializes to decompositions of $\underline{\mathrm{CH}}(\mathrm{M})$ and of $\mathrm{CH}(\mathrm{M})$, where the latter goes through Remark 4.1. At the level of Poincaré polynomials, these decompositions coincide with those of Theorem 1.8. However, the subspaces appearing in the decompositions are not the same. In particular, the decompositions in [AHK18, Theorem 6.18] are not orthogonal, and they are not compatible with the $\underline{\mathrm{H}}_\alpha(\mathrm{M})$-module structure on $\underline{\mathrm{CH}}(\mathrm{M})$ or the $\mathrm{H}_\alpha(\mathrm{M})$-module structure on $\mathrm{CH}(\mathrm{M})$.

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