Positivity theorems for hyperplane arrangements via intersection theory

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Arrangements and Flats

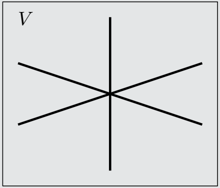
Let V be a finite dimensional vector space,

and $\mathcal{A}$ a finite set of hyperplanes in V with $\bigcap_{H \in \mathcal{A}} H = \{0\}$.

Definition

A **flat** $F \subset V$ is an intersection of some hyperplanes.

Example



- 1 flat of dimension 2 (V itself)
- 3 flats of dimension 1 (the lines)
- 1 flat of dimension 0 (the origin)

Arrangements and Flats

Example

Suppose $V = \mathbb{C}^6$ and $\mathcal{A}$ consists of 8 generic hyperplanes.

- $\binom{8}{0} = 1$ flat of dimension 6 (V itself)
- $\binom{8}{1} = 8$ flats of dimension 5 (the hyperplanes)
- $\binom{8}{2} = 28$ flats of dimension 4
- $\binom{8}{3} = 56$ flats of dimension 3
- $\binom{8}{4} = 70$ flats of dimension 2
- $\binom{8}{5} = 56$ flats of dimension 1
- 1 flat of dimension 0 (the origin)

Arrangements and Flats

Example

Suppose $V = \mathbb{C}^7/\mathbb{C}_\Delta = \{(z_1,\ldots,z_7) \in \mathbb{C}^7\}/\mathbb{C}\cdot(1,\ldots,1)$ and $\mathcal{A}$ consists of the $\binom{7}{2}$ hyperplanes $H_{ij} := \{z_i = z_j\}$.

$$H_{14} \cap H_{46} \cap H_{37} = \{(z_1,\ldots,z_7) \mid z_1 = z_4 = z_6, z_3 = z_7\}/\mathbb{C}_\Delta$$

is a flat of dimension 3.

$$\begin{aligned}
\text{flats} &\quad\leftrightarrow\quad \text{partitions of the set } \{1,\ldots,7\} \\
H_{14} \cap H_{46} \cap H_{37} &\quad\leftrightarrow\quad \{1,4,6\} \sqcup \{3,7\} \sqcup \{2\} \sqcup \{7\} \\
H_{14} &\quad\leftrightarrow\quad \{1,4\} \sqcup \{2\} \sqcup \{3\} \sqcup \{5\} \sqcup \{6\} \sqcup \{7\} \\
V &\quad\leftrightarrow\quad \{1\} \sqcup \{2\} \sqcup \{3\} \sqcup \{4\} \sqcup \{5\} \sqcup \{6\} \sqcup \{7\} \\
\{0\} &\quad\leftrightarrow\quad \{1,\ldots,7\} \\
\text{flats of dimension } k &\quad\leftrightarrow\quad \text{partitions into } k+1 \text{ parts}
\end{aligned}$$

The Top-Heavy conjecture

Theorem (Top-Heavy conjecture)

If $k \leq \frac{1}{2} \dim V$, then

$$\#\text{ of flats of codimension } k \quad \leq \quad \#\text{ of flats of dimension } k.$$

Furthermore, the same statement holds for matroids (combinatorial abstractions of hyperplane arrangements for which we can still make sense of flats).

- Conjectured by Dowling and Wilson, 1974
- True for arrangements: Huh and Wang, 2017
- True for matroids: Braden–Huh–Matherne–P–Wang, 2020

The Top-Heavy conjecture

Example

8 generic hyperplanes in $\mathbb{C}^6$

- $\binom{8}{1} = 8$ flats of codimension 1
  $\binom{8}{5} = 56$ flats of dimension 1
- $\binom{8}{2} = 28$ flats of codimension 2
  $\binom{8}{4} = 70$ flats of dimension 2

Example

$V = \mathbb{C}^7/\mathbb{C}_\Delta$, $\mathcal{A} = \{H_{ij}\}$

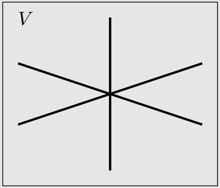
- $\#$ partitions into 6 parts $= \binom{7}{2} = 21$ flats of codimension 1
  $\#$ partitions into 2 parts $= \frac{2^7-2}{2} = 63$ flats of dimension 1
- $S(7,5) = 140$ flats of codimension 2
  $S(7,3) = 301$ flats of dimension 2

Characteristic polynomial

The **characteristic polynomial** $\chi_{\mathcal{A}}(t)$ is a polynomial, depending only on the poset of flats, with the following property:

If V is a vector space over $\mathbb{F}_q$, then $\chi_{\mathcal{A}}(q)$ is equal to the number of points on the complement of the hyperplanes.

Example



- $\chi_{\mathcal{A}}(q) = q^2 - 3q + 2$
- Since q could be any prime power, we must have $\chi_{\mathcal{A}}(t) = t^2 - 3t + 2$.

More generally, $\chi_{\mathcal{A}}(t) = \sum_{F} \mu(V, F) t^{\dim F}$,

where μ is the Möbius function for the poset of flats.

The characteristic polynomial

Example

$V = \mathbb{C}^7/\mathbb{C}_\Delta$, $\mathcal{A} = \{H_{ij}\}$

- $\chi_{\mathcal{A}}(q) = (q-1)(q-2)(q-3)(q-4)(q-5)(q-6)$
- Since q could be any prime power,
  we must have
  $\chi_{\mathcal{A}}(t) = (t-1)(t-2)(t-3)(t-4)(t-5)(t-6)$.

If V is a vector space over $\mathbb{F}_q$ and $\dim V > 0$, then $\mathbb{F}_q^\times$ acts freely on the set of the points in the complement of the hyperplanes. That means that $\chi_{\mathcal{A}}(q)$ is a multiple of $q-1$, and therefore $\chi_{\mathcal{A}}(t)$ is a multiple of $t-1$. The **reduced characteristic polynomial** is

$$\bar{\chi}_{\mathcal{A}}(t) := \chi_{\mathcal{A}}(t)/(t-1).$$

The characteristic polynomial

Example

$V = \mathbb{C}^7/\mathbb{C}_\Delta$, $\mathcal{A} = \{H_{ij}\}$

$$\begin{aligned} \chi_{\mathcal{A}}(t) &= (t-1)(t-2)(t-3)(t-4)(t-5)(t-6) \\ \bar{\chi}_{\mathcal{A}}(t) &= (t-2)(t-3)(t-4)(t-5)(t-6) \\ &= t^5 - 20t^4 + 155t^3 - 580t^2 + 1044t - 720 \end{aligned}$$

Easy Lemma: There exist positive integers $a_0, a_1, \ldots$ such that

$$\bar{\chi}_{\mathcal{A}}(t) = \sum_{i \geq 0} (-1)^i \, a_i \, t^{\dim V - 1 - i}.$$

Example

$a_0 = 1,\ a_1 = 20,\ a_2 = 155,\ a_3 = 580,\ a_4 = 1044,\ a_5 = 720.$

Log concavity

Theorem

The sequence $a_0, a_1, \ldots$ *is* **log concave**. *That is, for all* $i > 0$,

$$a_i^2 \geq a_{i-1} a_{i+1}.$$

Furthermore, the statement holds for matroids.

- Conjectured for graphs by Hoggar, 1974
- Conjectured for all arrangements, and in fact for all matroids, by Welsh, 1976
- True for arrangements over $\mathbb{C}$: Huh, 2012
- True for all arrangements: Huh and Katz, 2012
- True for matroids: Adiprasito–Huh–Katz, 2017

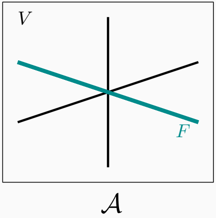
Contractions

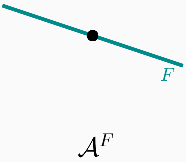
Definition

Let F be a flat. The **contraction of $\mathcal{A}$ at F** is the arrangement

$$\mathcal{A}^F := \{H \cap F \mid F \not\subset H \in \mathcal{A}\}$$

in the vector space F.





Note the special case $\mathcal{A}^V = \mathcal{A}$.

The Kazhdan–Lusztig polynomial

The **Kazhdan–Lusztig polynomial** $P_{\mathcal{A}}(t)$ is uniquely determined by the following three properties:

- If $\dim V = 0$, then $P_{\mathcal{A}}(t) = 1$
- If $\dim V > 0$, then $\deg P_{\mathcal{A}}(t) < \frac{1}{2} \dim V$
- The Z-polynomial

$$Z_{\mathcal{A}}(t) = \sum_{F} P_{\mathcal{A}^F}(t) t^{\operatorname{codim} F}$$

  is palindromic. That is, $t^{\dim V} Z_{\mathcal{A}}(t^{-1}) = Z_{\mathcal{A}}(t)$.

Easy Lemma: For any $\mathcal{A}$, $P_{\mathcal{A}}(0) = 1$. In particular, $P_{\mathcal{A}}(t) = 1$ whenever $\dim V \leq 2$.

The Kazhdan–Lusztig polynomial

Example

Suppose that $\dim V = 3$ and $\mathcal{A}$ consists of 5 hyperplanes in general position.

- We have 1 flat V of codimension 0, and $P_{\mathcal{A}^V}(t) = P_{\mathcal{A}}(t)$.
- We have 5 flats F of codimension 1, and $P_{\mathcal{A}^F}(t) = 1$.
- We have $\binom{5}{2} = 10$ flats F of codimension 2, and $P_{\mathcal{A}^F}(t) = 1$.
- We have 1 flat $\{0\}$ of codimension 3, and $P_{\mathcal{A}^{\{0\}}}(t) = 1$.

$$Z_{\mathcal{A}}(t) = P_{\mathcal{A}}(t) + 5t + 10t^2 + t^3.$$

Since $\deg P_{\mathcal{A}}(t) < \frac{3}{2}$, we must have

$$P_{\mathcal{A}}(t) = 1 + 5t.$$

Example

Suppose that $\dim V = 4$ and $\mathcal{A}$ consists of 6 hyperplanes in general position.

- We have 1 flat V of codimension 0, and $P_{\mathcal{A}^V}(t) = P_{\mathcal{A}}(t)$.
- We have 6 flats F of codimension 1, and $P_{\mathcal{A}^F}(t) = 1 + 5t$.
- We have $\binom{6}{2} = 15$ flats F of codimension 2, and $P_{\mathcal{A}^F}(t) = 1$.
- We have $\binom{6}{3} = 20$ flats F of codimension 3, and $P_{\mathcal{A}^F}(t) = 1$.
- We have 1 flat $\{0\}$ of codimension 3, and $P_{\mathcal{A}^{\{0\}}}(t) = 1$.

$$\begin{aligned} Z_{\mathcal{A}}(t) &= P_{\mathcal{A}}(t) + 6(1+5t)t + 15t^2 + 20t^3 + t^4 \\ &= P_{\mathcal{A}}(t) + 6t + 45t^2 + 20t^3 + t^4. \end{aligned}$$

Since $\deg P_{\mathcal{A}}(t) < \frac{4}{2} = 2$, we must have $P_{\mathcal{A}}(t) = 1 + 14t$.

KL positivity

Theorem

The coefficients of $P_{\mathcal{A}}(t)$ are non-negative. Furthermore, the statement holds for matroids.

- Proved for hyperplane arrangements and conjectured for matroids: Elias–P–Wakefield, 2016
- True for matroids: Braden–Huh–Matherne–P–Wang, 2020

The Hodge–Riemann bilinear relations

Let X be a (nonempty, connected) smooth projective variety of dimension $r > 0$ over $\mathbb{C}$, and let $\alpha \in H^2(X;\mathbb{Q})$ be the class of an ample line bundle.

Poincaré duality provides an isomorphism

$$\deg : H^{2r}(X;\mathbb{Q}) \to \mathbb{Q}.$$

Consider the symmetric bilinear pairing $\langle\, , \rangle_\alpha$ on $H^2(X;\mathbb{Q})$ given by the formula

$$\langle \eta, \xi \rangle_\alpha := \deg(\alpha^{r-2}\eta\xi).$$

Theorem (Hodge–Riemann bilinear relations in degree 2)

- *The form $\langle\, , \rangle_\alpha$ is positive definite on $\mathbb{Q}\alpha$, i.e. $\langle \alpha, \alpha \rangle_\alpha > 0$.*
- *Let $P_\alpha := (\mathbb{Q}\alpha)^\perp$. The form $\langle\, , \rangle_\alpha$ is negative definite on P_α.*

The Hodge–Riemann and log concavity

Corollary

Suppose that α and β are two ample classes, and let

$$a_i := \deg(\alpha^{r-i}\beta^i).$$

The sequence $a_0, \ldots, a_r$ is log concave.

Proof. We'll prove that $a_1^2 \geq a_0 a_2$; the other inequalities follow from this one by passing to a hyperplane section associated with β. We can assume that α and β are linearly independent; otherwise the statement is trivial.

Restrict the form to $L = \mathbb{Q}\{\alpha, \beta\}$. Since $\alpha \in L$, it has at least one positive eigenvalue. But it only had one positive eigenevalue on all of $H^2(X;\mathbb{Q})$, so its other eigenvalue must be negative.

Hodge–Riemann and log concavity

This means that

$$0 > \det \begin{pmatrix} \langle \alpha, \alpha \rangle_\alpha & \langle \alpha, \beta \rangle_\alpha \\ \langle \beta, \alpha \rangle_\alpha & \langle \beta, \beta \rangle_\alpha \end{pmatrix} = a_0 a_2 - a_1^2.$$

Corollary

Suppose that α and β are two nef classes, and let

$$a_i := \deg(\alpha^{r-i}\beta^i).$$

The sequence $a_0, \ldots, a_r$ is log concave.

Proof. Approximate by ample classes.

Hodge–Riemann and log concavity

Now we want to interpret the coefficients of the reduced characteristic polynomial as intersection numbers on some smooth projective variety.

Given an arrangement $\mathcal{A}$ in V, construct the **wonderful variety** $X_{\mathcal{A}}$ as follows:

- Start with $\mathbb{P}(V)$.
- Blow up $\mathbb{P}(F)$ for each 1-dimensional flat F.
- Blow up the proper transform of $\mathbb{P}(F)$ for each 2-dimensional flat F.
- Continue through all flats $F \subsetneq V$.

Hodge–Riemann and log concavity

Proposition (Adiprasito–Huh–Katz)

Let $r := \dim V - 1 = \dim X_{\mathcal{A}}$. *There exist nef classes* $\alpha, \beta \in H^2(X; \mathbb{Q})$ *such that*

$$\bar{\chi}_{\mathcal{A}}(t) = \sum_{i \geq 0} (-1)^i \deg(\alpha^{r-i}\beta^i)\, t^{r-i}.$$

For arbitrary matroids, there is no analogue of $X_{\mathcal{A}}$, but there is a combinatorially defined ring that stands in for $H^*(X_{\mathcal{A}}; \mathbb{Q})$. The hard work of AHK is showing that this ring satisfies the Hodge–Riemann bilinear relations.

The hard Lefschetz theorem

Let Y be a (singular) projective variety of dimension d. We have the cohomology ring $H^*(Y)$ and the **intersection cohomology** module $IH^*(Y)$. There is a $H^*(Y)$-module homomorphism

$$H^*(Y) \to IH^*(Y)$$

which is an isomorphism when Y is smooth, but is in general neither injective nor surjective.

Let $\alpha \in H^2(Y)$ be an ample class.

Theorem (Hard Lefschetz)

For all $k \leq d$, multiplication by α^{d-k} gives an isomorphism

$$IH^k(Y) \xrightarrow{\cong} IH^{2d-k}(Y).$$

The Schubert variety of $\mathcal{A}$

Let $\mathcal{A}$ be an arrangement in V. We have

$$V \hookrightarrow \bigoplus_{H \in \mathcal{A}} V/H \cong \bigoplus_{H \in \mathcal{A}} \mathbb{A}^1 \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1.$$

Definition

We define the **Schubert variety** $Y_{\mathcal{A}} := \bar{V} \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1.$

For any flat F, let

$$U_F = \{p \in Y_{\mathcal{A}} \mid p_H \neq \infty \Leftrightarrow F \subset H\}.$$

Example

We have $U_V = (\infty, \dots, \infty)$ and $U_{\{0\}} = V$.

The Schubert variety of $\mathcal{A}$

Proposition

- *For each F, $U_F \cong V/F$ (it's a V-orbit with stabilizer F).*
- *We have $Y_{\mathcal{A}} = \bigsqcup_F U_F$ (i.e. every orbit is of this form).*

The fact that $Y_{\mathcal{A}}$ admits a stratification by affine spaces has two consequences.

Corollary

- *We have $H^{2k}(Y_{\mathcal{A}}) \cong \mathbb{Q}\{\text{codimension } k \text{ flats}\}$.*
- *The natural map from $H^*(Y_{\mathcal{A}})$ to $IH^*(Y_{\mathcal{A}})$ is an inclusion.*

Hard Lefschetz and the Top-Heavy conjecture

$$\begin{array}{ccc}
IH^{2k}(Y_{\mathcal{A}}) & \xrightarrow[\alpha^{d-2k}]{\cong} & IH^{2(d-k)}(Y_{\mathcal{A}}) \\
\uparrow & & \uparrow \\
H^{2k}(Y_{\mathcal{A}}) & \xhookrightarrow[\alpha^{d-2k}]{} & H^{2(d-k)}(Y_{\mathcal{A}}) \\
\cong\uparrow & & \cong\uparrow \\
\mathbb{Q}\{\text{codim } k \text{ flats}\} & \hookrightarrow & \mathbb{Q}\{\text{dim } k \text{ flats}\}
\end{array}$$

This proves the Top-Heavy conjecture!

For arbitrary matroids, there is no analogue of $Y_{\mathcal{A}}$, but one can give a combinatorial definition of a ring and a module that stand in for $H^*(Y_{\mathcal{A}})$ and $IH^*(Y_{\mathcal{A}})$. The hard work of BHMPW is showing that this module satisfies the hard Lefschetz theorem.

A spectral sequence for $IH^*(Y_{\mathcal{A}})$

Let $IC_{Y_{\mathcal{A}}}$ be the intersection cohomology sheaf of $Y_{\mathcal{A}}$; this is a sheaf (actually an object in the derived category of sheaves) with $H^*(IC_{Y_{\mathcal{A}}}) = IH^*(Y_{\mathcal{A}})$.

Let $IC_{Y_{\mathcal{A}},F}$ be the stalk at a point in U_F.

Let $j_F : U_F \to Y_{\mathcal{A}}$ be the inclusion. There is a spectral sequence converging to $IH^*(Y_{\mathcal{A}})$ with

$$\begin{aligned} E_1^{p,q} &= \bigoplus_{\text{codim } F=p} H_c^{p+q}(j_F^* IC_{Y_{\mathcal{A}}}) \\ &\cong \dots \\ &\cong H^{q-p}(IC_{Y_{\mathcal{A}},F}). \end{aligned}$$

A spectral sequence for $IH^*(Y_\mathcal{A})$

Lemma

For any flat F, $H^*(IC_{Y_\mathcal{A},F}) \cong H^*(IC_{Y_{\mathcal{A}^F},\{0\}})$, and this cohomology vanishes in odd degree.

This implies that the spectral sequence degenerates at the E_1 page. This in turn means that $IH^*(Y_\mathcal{A})$ vanishes in odd degree, and we have a vector space isomorphism

$$IH^{2k}(Y_\mathcal{A}) \cong \bigoplus_{p+q=2k} E_1^{p,q} \cong \bigoplus_F H^{2(k-\operatorname{codim} F)}\left(IC_{Y_{\mathcal{A}^F},\{0\}}\right)$$

or equivalently

$$IH^*(Y_\mathcal{A}) \cong \bigoplus_F H^*\left(IC_{Y_{\mathcal{A}^F},\{0\}}\right)[-2\operatorname{codim} F].$$

KL positivity from the spectral sequence

Let

$$\tilde{Z}_{\mathcal{A}}(t) := \sum_{k \geq 0} \dim IH^{2k}(Y_{\mathcal{A}}) t^k \ \text{ and } \ \tilde{P}_{\mathcal{A}}(t) := \sum_{k \geq 0} \dim H^{2k}(IC_{Y_{\mathcal{A}},\{0\}}) t^k.$$

The previous equation implies that

$$\tilde{Z}_{\mathcal{A}}(t) = \sum_{F} \tilde{P}_{\mathcal{A}^F}(t) t^{\operatorname{codim} F}.$$

Hard Lefschetz implies that $\tilde{Z}_{\mathcal{A}}(t)$ is palindromic, and general nonsense implies that $\deg \tilde{P}_{\mathcal{A}}(t) < \frac{d}{2}$ unless $d = 0$. Thus we must have

$$\tilde{Z}_{\mathcal{A}}(t) = Z_{\mathcal{A}}(t) \ \text{ and } \ \tilde{P}_{\mathcal{A}}(t) = P_{\mathcal{A}}(t).$$

In particular, $P_{\mathcal{A}}(t)$ has non-negative coefficients.

KL positivity for matroids

For arbitrary matroids, one can give a combinatorial definition of a vector space that stands in for $H^*(IC_{\mathcal{A},\{0\}})$. The hard work of BHMPW (in addition to hard Lefschetz) is showing that this vector space vanishes in degrees greater than or equal to d, i.e. that $\deg \tilde{P}_{\mathcal{A}}(t) < \frac{d}{2}$.

Thanks!