<u>Geometric Satake</u> is an equiv of (2-)categories

$$\mathrm{Perv}\left(G^\vee(K)/G^\vee(\mathcal{O})\right) \xrightarrow{\sim} \mathrm{Rep}\, G$$

which is, well, hard! Ginzburg, Mirkovic-Vilonen, Lusztig

Today I'll reformulate as an equiv of 2-cats

$$\text{"maximal"} \mapsto M\mathbb{S}\mathbb{B}\mathrm{Bim}_{\tilde{\mathfrak{g}}} \xrightarrow{\sim} \mathrm{Rep}^{\Omega}\mathfrak{g}$$

which is, well, pretty easy!

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§1 Type A_1

Let's look at $\mathbb{H}_{\tilde{A}_1}$, i.e. ∞-dihedral group

Recall: $\underline{H}_s \underline{H}_t = \underline{H}_{st}$ $\longleftrightarrow$ $V_{std} \otimes V_0 = V_1$ in $\mathrm{Rep}\, \mathfrak{sl}_2$

$\underline{H}_s \underline{H}_{tsts} = \underline{H}_{ststs} + \underline{H}_{sts}$ $\qquad V_{std} \otimes V_n = V_{n+1} \oplus V_{n-1}$

analogous, but 2 colors on LHS, one on right? No identity in $\mathbb{H}$ on left...

Really, $\mathrm{Rep}\, \mathfrak{sl}_2 \cong \mathrm{Rep}^{even} \mathfrak{sl}_2 \oplus \mathrm{Rep}^{odd} \mathfrak{sl}_2$, $\otimes$ gives it a $\mathbb{Z}/2\mathbb{Z}$ grading, i.e. odd $\otimes$ odd = even, etc.

Think about $\underset{\text{monoidal cat}}{\mathrm{Rep}\, \mathfrak{sl}_2} \circlearrowright \underset{\text{regular repn}}{\mathrm{Rep}\, \mathfrak{sl}_2} = \begin{matrix} \text{even} \\ \oplus \\ \text{odd} \end{matrix}$, so the functor $V \otimes (\cdot)$ actually splits (even when V indec)

into $\left(V \otimes \cdot \big|_{even}\right)\Big|_{even}$ $\left(V \otimes \cdot \big|_{odd}\right)\Big|_{even}$

get a 2x2 matrix.

~~Better way to encode is~~ 2-category $\mathrm{Rep}^{\mathbb{Z}/2\mathbb{Z}} \mathfrak{sl}_2$. <u>Rmk:</u> Analogous to translation functors in $\mathcal{O}$.

<u>Ob:</u> $\mathbb{Z}/2\mathbb{Z}$ (or rather, a $\mathbb{Z}/2\mathbb{Z}$ torsor) like $\{s, t\}$

$\mathrm{Hom}(s,s) = \mathrm{Hom}(t,t) = \mathrm{Rep}^{even} \mathfrak{sl}_2$ $\qquad \mathrm{Hom}(s,t) = \mathrm{Hom}(t,s) = \mathrm{Rep}^{odd} \mathfrak{sl}_2$

composition = tensor.

$\mathrm{Rep}^{\mathbb{Z}/2\mathbb{Z}} \mathfrak{sl}_2$
$\cup$
$\mathrm{Fund}^{\mathbb{Z}/2\mathbb{Z}} \mathfrak{sl}_2$ Generated by V_{std}: even $\to$ odd and V_{std}: odd $\to$ even.

$t \cdot s \qquad s \cdot t$

$\mathrm{Fund}^{\mathbb{Z}/2\mathbb{Z}} \mathfrak{sl}_2 \cong 2^{\mathbb{Z}} TL.$

$\mathrm{Kar}(\mathrm{Fund}^{\mathbb{Z}/2\mathbb{Z}} \mathfrak{sl}_2) \cong \mathrm{Rep}^{\mathbb{Z}/2\mathbb{Z}} \mathfrak{sl}_2.$

Better way to encode: $(\Omega = \mathbb{Z}/2\mathbb{Z})$ a 2-category $\mathrm{Rep}^{\Omega}\mathfrak{sl}_2 \subset \bigoplus_{\substack{\text{even}\\ \text{odd}}} \mathrm{Rep}^{\cdot}\mathfrak{sl}_2$

Ob: $\{s,t\}$ indexing the 2x2 matrix (it's a $\mathbb{Z}/2\mathbb{Z}$ torsor, no element is special)

$\mathrm{Hom}(s,s) = \mathrm{Hom}(t,t) = \mathrm{Rep}^{\text{even}}\mathfrak{sl}_2$ $\qquad \mathrm{Hom}(s,t) = \mathrm{Hom}(t,s) = \mathrm{Rep}^{\text{odd}}\mathfrak{sl}_2$

Composition $= \otimes$.

Unraveling the recursion relation gives

Prop (Lusztig): $[\mathrm{Rep}^{\Omega}\mathfrak{sl}_2] \cong {}_M\mathcal{H}$, full subcat of $\mathcal{H}$ w/ objects $\{s,t\}$

i.e. in $[\mathrm{Hom}(s,t)]$ $\qquad$ in $\mathrm{Hom}(s,t) = \underline{H}_s \mathcal{H} \cap \mathcal{H}\underline{H}_t$ you have

$[V_1]$	$\underline{H}_{st}$
$[V_3]$	$\underline{H}_{stst}$
$\vdots$	$\vdots$

and so forth.

How should this categorify? Thm ("E"): $\mathrm{Rep}^{\Omega}\mathfrak{sl}_2 \cong {}_M\mathbb{S}\mathbb{S}\mathrm{Bim}$ (follows formally from GS)

In fact, we can prove this directly!

$\mathrm{Fund}^{\Omega}\mathfrak{sl}_2$ generated by $V\otimes : s \to t$, $V\otimes : t \to s$ $\qquad \cong 2\text{-}TL_{q=1}$

$\cap$

$\mathrm{Rep}^{\Omega}\mathfrak{sl}_2$

$V \otimes V$

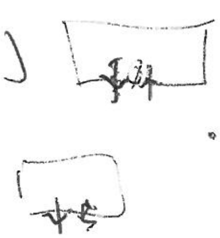
${}_M\mathbb{B}\mathbb{S}\mathrm{Bim}$ generated by

$\cap$

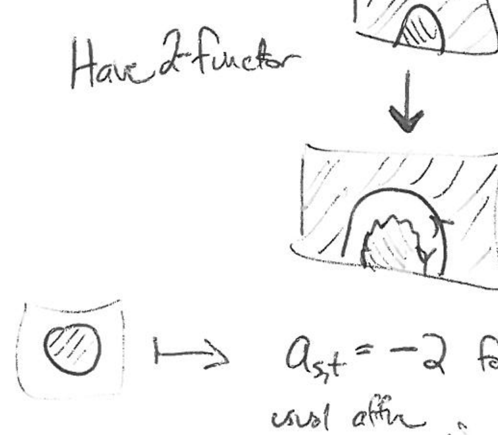
${}_M\mathbb{S}\mathbb{B}\mathrm{im}$

$\mathrm{Res}_t \, \mathrm{Ind}_s = {}_{R^t}R_{R^s}(1)$

$\mathrm{Ind}_s \, \mathrm{Res}_t$



Have 2-functor



(This is the more natural version of $2TL \to \mathbb{B}\mathbb{S}\mathrm{Bim}$ I had promised)

$\mapsto a_{s,t} = -2$ for usual affine Cartan matrix.

Thm (E): $\mathrm{Fund}^{\Omega}\mathfrak{sl}_2 \xrightarrow{\sim} {}_M\mathbb{B}\mathbb{S}\mathrm{Bim} \underset{\text{Kar}}{\Longrightarrow} \mathrm{Rep}^{\Omega}\mathfrak{sl}_2 \xrightarrow{\sim} {}_M\mathbb{S}\mathbb{B}\mathrm{im}$. New proof, indep. of GS

§2 General case | Consider $\widetilde{W}$, affine Cartan matrix, W Weyl group of $\mathfrak{g}$ fix s.s. lie alg.

Let $\Omega = \pi_1(\mathfrak{g}) \equiv \Lambda_{wt}/\Lambda_{rt} \cong Z(G_{s.c.}) \cong \pi_1(G_{adj})$ a finite abelian group

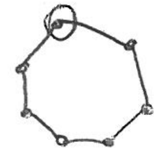
$\mathrm{Rep}\,\mathfrak{g}$ is monoidally Ω-graded, so for each $\omega \in \Omega$ get $\mathrm{Rep}^{\omega}\mathfrak{g}$.

$\mathrm{Rep}^{\Omega}\mathfrak{g}$ is 2-cat w/ Ob: a Ω-torsor

$\mathrm{Hom}(s,t) = \mathrm{Rep}^{t-s}\mathfrak{g}$. $t-s \in \Omega$.

What is your favorite Ω-torsor? Mine is $\widetilde{\Gamma}_{removable}$.

Ex: $A_n = \mathfrak{sl}_{n+1}$ $\Omega = \mathbb{Z}/_{n+1}\mathbb{Z}$ $\widetilde{\Gamma} =$



Assign a weight to each vertex in $\widetilde{\Gamma}$.

$s \mapsto \omega_s$

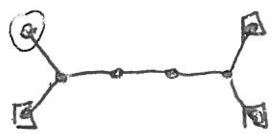
$\odot \mapsto 0$

$\widetilde{\Gamma} \setminus \odot = \Gamma$

Def: A vertex of $\widetilde{\Gamma}$ is removable if $\widetilde{\Gamma} \setminus v \cong \Gamma$

so all vertices in $\widetilde{A}_n$ are removable

Ex: D_n $\widetilde{\Gamma} =$



4 vertices are removable.

Claim: $\Omega \xhookrightarrow{\text{natural}} \mathrm{Aut}(\widetilde{\Gamma})$, acts simply trans. on removable vertices.

Moreover, the weights associated to removable vertices enumerate cosets in $\Lambda_{wt}/\Lambda_{rt}$.

So my favorite Ω-torsor is $\{ I \subset \underset{\text{for } \widetilde{W}}{\widetilde{S}} \text{ s.t. } W_I \cong W_{fin} \}$

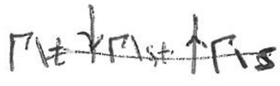
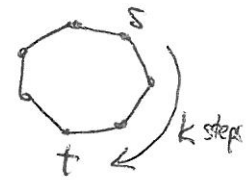
Prop (Lusztig): $[\mathrm{Rep}^{\Omega}\mathfrak{g}] \cong {}_M\mathcal{H}$ ← full subcat w/ objects

Thm ("E"): $\mathrm{Rep}^{\Omega}\mathfrak{g} \cong {}_M\mathbb{S}\mathrm{SBim}$ ← full SSBim-2-cat

follows formally from GS, actually equivalent.

In type A, using diagrams for SBSBim and $\mathfrak{sl}_n$-webs of CKM (analogous to TL diags, diags for Fund $\mathfrak{sl}_n$)

Thm(E): $\mathrm{Fund}^{\Omega}\mathfrak{g} \cong {}_M\mathbb{S}\mathrm{BSBim}$, gives new proof of GS, only in type A.

More precisely, $\Lambda^k V_{std} = V_{\omega_k} \rightsquigarrow$  where 

by definition, $\underline{mSBSBim}$ is generated by these

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§3 Mystery. Remember that TL, 2TL has q-deformation describing $\mathrm{Rep}\,U_q(\mathfrak{sl}_2)$. Exact same diagrammatic argument yields:

Find

$\underline{\text{Thm}}$(E): Let $\begin{pmatrix} 2 & -(q+q^{-1}) \\ -(q+q^{-1}) & 2 \end{pmatrix}$ be the $\underbrace{\text{q'M affine Cartan matrix}}_{\text{of } A_1}$. It gives a "reflection rep" of $\widetilde{A}_1$ over $\mathbb{Q}(q)$. Define SBSBim as usual, but using this action.

$$\text{SBSBim} = \text{SBSBim}_q$$

Then $mSBSBim_q \xleftrightarrow{\sim} \mathrm{Rep}^{\mathrm{Fund}}\, U_q(\mathfrak{sl}_2)$

Also, there is a q-deformation of $\mathfrak{sl}_n$-webs describing Fund $U_q(\mathfrak{sl}_n)$, and

$\underline{\text{Thm}}$(E): Let
$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & -q^{-1} \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -q \\ \hline -q & 0 & 0 & -q^{-1} & 2 \end{array}\right)$$
be the q'M aff Cartan matrix of A_n. Gives ref rep over $\widetilde{A}_n$ (exercise)

Then $mSBSBim_q \xleftrightarrow{\sim} \mathrm{Rep}^{\mathrm{Fund}}\, U_q(\mathfrak{sl}_n)$.

(Where did this matrix come from?? What is the geometric source? Only time will tell.

What happens at $q = \zeta_{2m}$ for a root of unity?

Ref. rep of $\widetilde{A}_1$ factors thru dihedral group $I_2(m)$, and new bimodule maps appear (the $2m$-valent vertex)

So no more equivalence.

meanwhile, $\mathrm{Rep}\,U_q(\mathfrak{sl}_2)$ is no longer semisimple

All you can do is $mSBSBim_{I_2(m)} \big/ 2m\text{-valent vertex} \xrightarrow{\sim} \mathrm{Rep}\,U_q(\mathfrak{sl}_2) \big/ \text{nilradical}$.

Is there more?

Ref rep of $\widetilde{A}_n$ factors thru a finite group — NOT a Coxeter group. What happens now???