GRAND INDUCTION (+ other lemmas)

Fix $x, \underline{x}$ with $xs > x$.

Start,

~~$S(<x)$~~

$S(<xs)$

$hL(<xs)$

$HR(<xs)$

$RoHR(<x)$

$hL(y;t)_\zeta$ $y<x$

$HR(y;t)_\zeta$ $yt>y$ $\zeta \geq 0$

Then the rest we know already.

① $S(\leq x) \Rightarrow F_x$ has diagonal miracle. $DM(x)$ (Exercises, or if we have time)

② $DM(x)$, $RoHR(<x)$, $HR(y;t)$ $\Longrightarrow RoHR(x)$

③ Everything so far $\Longrightarrow hL(x;s)_\zeta \quad \forall \zeta \geq 0.$

④ We proved $HR(x;s)_\zeta$ for $\zeta >> 0$.

Limiting argument $+ hL(x,s)_\zeta \ \forall \zeta \geq 0$ gives $HR(x;s)_\zeta \ \forall \zeta \geq 0.$

⑤ Setting $\zeta = 0$ gives $HR(x;s)$. The embedding theorem then shows the nondegeneracy of all LIF on $B_x B_s \Longrightarrow S(xs)$.

⑥ Finally, $S(xs) + HR(x;s)$ yields $HR(xs)$, $hL(xs)$. The loop is complete.

Rmk: Along the way, we use ⑦ $HR(y)$, $ys<s$ $\Longrightarrow HR(y;s)_\zeta$, hL for $\zeta > 0$ (but NOT $\zeta=0$)

a simple, non-inductive argument. Exercise.

Let's do ②

Prop: $S(\leq x)$, $HR(z;t)$ $zt>t$, $z<x$, $RoHR(<x)$ $\Longrightarrow RoHR(x)$

Pf: $x = ys > y$. $F_x \overset{\oplus}{\subset} F_yF_s$, a homotopy equivalence. $F_y = \cdots \to F_y^{j-1} \to F_y^j \to \cdots$

$F_yF_s = \cdots \to F_y^j B_s \oplus \overline{F_y^{j-1}(1)} \to$

$F_yF_s^j(-j) = F_y^j(-j)\,B_s \oplus F_y^{j-1}(-(j-1))$. $RoHR(y) \Longrightarrow HR$ for $\overline{F_y^{j-1}(-(j-1))}$ $\overline{F_y^j(-j)}$

Write $F_y^j(-j) = \bigoplus B_z(0) \overset{\cdot}{=} B^\uparrow \oplus B^\downarrow$ when $B^\uparrow = \bigoplus_{zs>z} B_z$ $B^\downarrow = \bigoplus_{zs<z} B_z$.

$F_y^j(-j) B_s = B^\uparrow B_s \oplus B^\downarrow B_s$ is NOT semismall, can't have HR... triage suggested we restrict to the "0-shifted" "perverse" part.

$$= \underbrace{B^\uparrow B_s}_{\substack{\text{semismall} \\ \text{has HR by HR}(z,s)}} \oplus \underbrace{B^\downarrow(1) \oplus B^\downarrow(-1)}_{\text{bad stuff}}.$$

Claim: The map $F_x^j(-j) \longrightarrow (F_y F_s)^j(-j) \xrightarrow{\text{project}} B^\uparrow B_s \oplus F_y^{j-1}(-(j-1))$

is an L-stable split inclusion and an isometry for the Lefschetz form.

Pf: This is obvious for the first map, but the projection kills stuff...

① By $S(\leq x)$, there are no maps $F_x^j(-j) \longrightarrow B^\downarrow(-1)$ using SHF. (negative degree!!) so that term didn't contribute.

② Any map $F_x^j(-j) \longrightarrow B^\downarrow(+1)$ is pos deg $\rightsquigarrow$ in max'l ideal, so killing it won't affect the map being a split inclusion.

③ Exercise: $(\, , \,)_L \big|_{B^\downarrow(+1)} = 0$ so killing this term doesn't affect the Lefschetz form. ▨

"Unspoken" ⊕ L is left mult, so commutes w/ all these bimodule maps, decomps, etc.

As an L-stable summand of HR, $F_x^j(-j)$ has HR. ▨

Now we apply a similar trick to $F_x F_s$ in order to prove ③

On something of the form $B B_s$, let $L_\zeta = \rho_0 + id_B(\zeta \rho_0)$

Assume everything so far. $F_x F_s = B_x B_s \xrightarrow{\Phi} B_x(1) \oplus F_x^1 B_s \longrightarrow \ldots$

Think of Φ as a degree +1 map $\underset{\circlearrowright L_\zeta}{B_x B_s} \longrightarrow \underset{\substack{\circlearrowright \\ L}}{B_x} \oplus \underset{\substack{\circlearrowright \\ L_\zeta}}{F_x^1(-1) B_s}$.

Write $F_x^1(-1) = B^\uparrow \oplus B^\downarrow$ just as before. (or L_ζ but ζ_ρ acts on right, will die in $\overline{B_x}$)

Recall/Lemma: Φ is a factorization of L_ζ up to positive renormalization.

I.e. $L_\zeta \circlearrowright BS(\underline{x}s)$ is equal to $\sum \lambda_i \; |||| \overset{\bullet}{\underset{\bullet}{}} ||||$ for $\lambda_i > 0$.

Last time we suggested factoring L_ζ as a composition of maps

$$\sigma : BS(\underline{x}s) \to \bigoplus BS(\underline{x}s)_{\hat{i}}$$

$\sigma = \bigoplus \sqrt{\lambda_i} \; ||||\underset{\bullet}{|}|||$ ← adjoint

and $\sigma^* = \bigoplus \sqrt{\lambda_i} \; |||| \overset{\bullet}{|}|||$

but alternatively, we consider $\Phi = \bigoplus ||||\underset{\bullet}{|}|||$ and renormalise the intersection form on the target.

$$\langle v, L_\zeta v' \rangle = \sum_i \langle \Phi_i v, \Phi_i v' \rangle_{BS(\underline{x}s)_{\hat{i}}}.$$

$\langle v, Lv' \rangle$
$\shortparallel$
$\langle \sigma v, \sigma v' \rangle$
$L\sigma = \sigma L$

Restricting to the summand $B_x B_s \subset BS(\underline{x}s)$, the restricted map Φ still satisfies this property.

The positive scalars λ_i are irrelevant – they don't affect HR, etc. We will ignore.

Moreover, **Claim:** Φ is injective from negative degrees.

Pf: $\operatorname{Ker} \Phi = H^0(F_{xs}) = R_{xs}(-\ell(xs))$ ~~lives~~ in strictly positive degrees. ▨

Thm: $R \circ HR(x)$ etc $\Longrightarrow hL(x;s)_\zeta$, $xs > x$, $\zeta \geq 0$.

Pf: Case 1: $\zeta > 0$.

$$\overline{B_x B_s} \xrightarrow{\Phi} \overset{L_\zeta \circlearrowright}{\overline{B^\uparrow B_s}} \oplus \overset{L_\zeta \circlearrowright}{\overline{B^\downarrow B_s}} \oplus \overset{L \circlearrowright}{\overline{B_x}}$$

↑ has HR by $HR(z;s)_\zeta$ ↑ has HR by ⊕, trivial case ↖ has HR by $HR(x)$

Key remark: Can NOT split $B^\downarrow B_s$ into $B_s^\uparrow(1) \oplus B^\uparrow(-1)$ as before. Splitting does NOT commute with L_ζ, because middle mult doesn't commute (see example of $B_s B_s$)

Now the wL sub $\Longrightarrow \overline{B_x B_s}$ has hL.

Case 2: $\zeta = 0$. $\overline{B_x B_s} \xrightarrow{\Phi} \overline{B_x} \oplus \overline{B^\uparrow B_s} \oplus \overline{B^\downarrow(1)} \oplus \overline{B^\downarrow(-1)}$ can split now.

$L_\zeta = L$.

BUT can't repeat the same argument. $\deg \Phi = +1$, not 0, so there ARE maps to $B^\downarrow(-1)$.

In fact, there must be! $B_x B_s = B_{xs} \oplus$ lower terms, but $F_{xs}^0 = B_{xs}$, so lower terms must contract, and they contract against $B^\downarrow(-1)$!!

However, can ignore $B^{\downarrow}(1)$ for two reasons.

① $(\, , \,)_L |_{B^{\downarrow}(1)} = 0$ ② $B^{\downarrow}(1)$ must contract against something in hom. degree 2 in order for Fxs to have DM. (less convincing)

So two possibilities.

ⓐ The map $\overline{B_x B_s} \xrightarrow{\Phi} (\quad) \xrightarrow{\text{project}} \overline{B^{\downarrow}(-1)}$ is zero.

Then $\overline{B_x B_s} \longrightarrow \overline{B_x} \oplus \overline{B^{\downarrow} B_s}$ satisfies $\langle v, Lv' \rangle = \langle \Phi v, \Phi v' \rangle$

and HR of RHS $\Longrightarrow$ hL of LHS by WL sub.

ⓑ The map $\overline{B_x B_s} \longrightarrow \overline{B^{\downarrow}(-1)}$ is nonzero. Now we argue analogously to WL sub's proof. Fix $v \in \overline{B_x B_s}^{-k}$. $\underset{\overset{\uparrow}{(B^{\downarrow})^{-k}}}{\Phi(v)} \neq 0$, so hL on $B^{\downarrow}$ implies $L^k \Phi(v) \neq 0 \Rightarrow L^k v \neq 0$, as desired.

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