§1) A_1

BSBim was built by taking the Frob Ext $R^s \subset R$ + making a Frob alg $R \otimes_{R^s} R(1)$ inside R-bim.

Why not look at the Frob Ext itself?

Def: (Type A_1) SBSBim $\subset$ BIM, a 2-cat. Obj: $\emptyset, \{s\} \longleftrightarrow$ rings R, R^s

1-mor: Generated by the bimodules $\mathrm{Ind}_s = {}_R R_{R^s}$, $\mathrm{Res}_s = {}_{R^s} R_R(1)$

2-mor: bimodule maps.

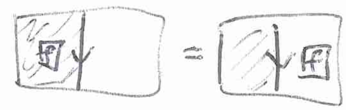
We know 4

mult $+1$, demazure -1, incl. -1, comult $+1$

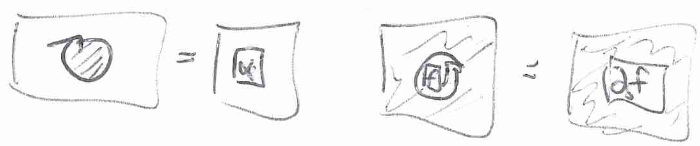
+ poly

$f \in R$ $f \in R^s$

Relations: Isotopy. Sliding



Eval + Coeval



Decomposition

${}_{R^s} R_{R^s} \cong R^s \oplus R^s(-2)$

Thm (E-W): These give a presentation of SBSBim. (à la previous theorems)

In fact, this gives a presentation of the analogous 2-cat for ANY Frob ext, provided the relations are written in general form: $\alpha_s = \mu \circ \Delta(1)$

use sum over dual bases, etc.

Def: SSBim = Kar(SBSBim), i.e. apply Kar to each category Hom$(\emptyset,\emptyset)$, Hom$(\emptyset,s)$, Hom$(s,\emptyset)$, Hom(s,s).

Rmks: 4 categories here... but most are boring. ${}_R R \cong R^s \oplus R^s(-2)$... in Hom$(?,s)$ or Hom$(s,?)$ all objects are just $\oplus R^s$.

What is this categorifying? Some algebroid $\longleftrightarrow$ category w/ 2 objects

$$\emptyset \underset{[\mathrm{Res}]}{\overset{[\mathrm{Ind}]}{\rightleftarrows}} s \qquad [\mathrm{Res} \circ \mathrm{Ind}] = v + v^{-1} \qquad [\mathrm{Ind} \circ \mathrm{Res}] = H_s$$

$$\mathrm{End}(\emptyset) = [\mathrm{End}_S(\emptyset)] = \mathbb{H} \qquad \mathrm{End}(s) = [\mathrm{End}(s)] = \mathbb{Z}[q,q^{-1}]$$

Best way to view this is as ideals inside $\mathbb{H}_W$

$\mathrm{Hom}(\emptyset,\emptyset) = \mathbb{H}$ $\underset{\mathrm{Hom}(s,\emptyset) = \mathbb{H}\underline{H}_s}{\overset{\mathrm{Hom}(\emptyset,s) = \underline{H}_s\mathbb{H}}{\rightleftarrows}}$ $\mathrm{Hom}(s,s) = \underline{H}_s\mathbb{H} \cap \mathbb{H}\underline{H}_s$ ← not $\underline{H}_s\mathbb{H}\underline{H}_s$ since that doesn't actually contain $\underline{H}_s$ itself.

$[\mathrm{Ind}] = \underline{H}_s$ $\qquad [R^s] = \underline{H}_s.$

$\underline{H}_s\underline{H}_s = (v+v^{-1})\underline{H}_s$ ↖ not invertible

quasi-idempotent

Composition: $a \underset{\emptyset}{*} b = ab \qquad a \underset{s}{*} b = \frac{ab}{v+v^{-1}}$

Check: This makes sense. Called the Hecke algebroid.

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§2 General Case Def: The Hecke Algebroid $\mathcal{H}$ has Ob $= I \overset{f}{\subset} S$.

Let $\ell(I) = \ell(w_I)$ and $\underline{H}_I = \underline{H}_{w_I}$ and $[I]$ = Poincaré poly of W_I

Ex: $[A_3] = [3][2] = (v+v^{-1})(v^2+1+v^{-2})$

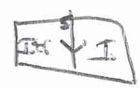
$\underline{H}_I^2 = [I]\,\underline{H}_I$ a quasi-idempotent

$\mathrm{Hom}(I,J) = \underline{H}_J\mathbb{H} \cap \mathbb{H}\underline{H}_I$ Composition: $a \underset{J}{*} b = \frac{ab}{[J]}$.

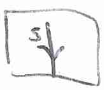
Def: $\mathbb{S}\mathrm{BSBim}$ has Ob: $I \overset{f}{\subset} S \leftrightarrow R^I$ ← because finitary, get Frob Exts.

1-mor: Generated by Ind^I_{I+s} $\qquad \mathrm{Res}^I_{I+s}$ $\qquad I, I+s \equiv I \sqcup \{s\}$ both finitary

2-mor: All bimod maps.

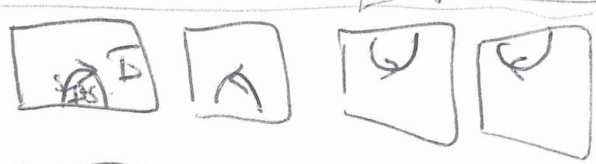
 $\mathrm{Ind}^I_{I+s} = {}_{R^I}R^I_{R^{I+s}}$ $\qquad \mathrm{Res} = {}_{R^{I+s}}R^I_{R^I}(\ell(I+s)-\ell(I))$ 

(remark: don't need Ind^I_J for $I \subset J$, since $J = I+stu$, $\mathrm{Ind}^I_J = \mathrm{Ind}^{I+tu}_{I+stu}\mathrm{Ind}^{I+t}_{I+tu}\mathrm{Ind}^I_{I+s}$)

(Rmk:  represents many functors, depending on consistent labeling.)

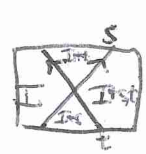
Def: $\mathbb{S}\mathrm{SBim} = \mathrm{Kar}(\mathbb{S}\mathrm{BSBim})$ $\qquad$ Thm(W): $[\mathbb{S}\mathrm{SBim}] = \mathcal{H}$

Exercise: Discover the Soergel-Williamson Hom Formula

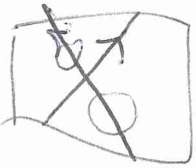
§3 Diagrammatics $R^{I+s} \subset R^I$ a Frob Ext, so have 

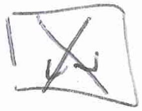
have  $f \in R^I$

These satisfy the generic Frob Ext relations, as above.

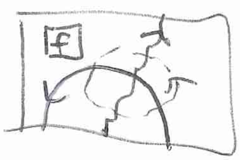
ALSO, $\mathrm{Ind}^I_{I+s} \circ \mathrm{Ind}^{I+s}_{I+s,t} \overset{\sim}{\rightleftarrows} \mathrm{Ind}^I_{I+t}\,\mathrm{Ind}^{I+t}_{I+s+t}$ Let  denote the bimod map reg $1\otimes 1 \mapsto 1 \mapsto 1\otimes 1$

$R^I \otimes_{R^{I+s}} R^{I+s}_{R^{I+s,t}} \cong R^I \cong R^I \otimes_{R^{I+t}} R^{I+t}_{R^{I+st}}$

Reiterate: In  the circled are two separate functors, as seen by region labels. But nice notation makes it look like a crossing of 1-manifolds!!

Claim:   are cyclic wrt the previously given cups + caps ⇝ Isotopy!

Theorem (E-W) These are all the generators. I.e., any bimodule map b/w iterated Ind, Res can be drawn as colored 1-mflds w/ bdry in a planar box.

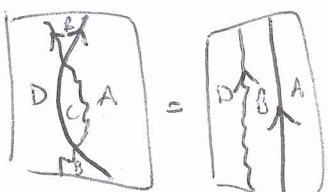


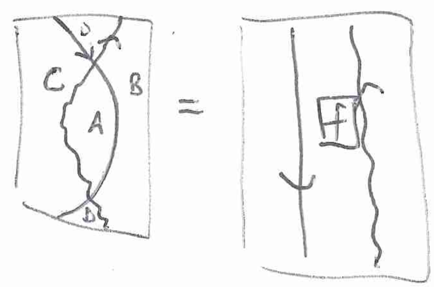
Pf: Not written up. The surjectivity part was always easy though — use localization arguments.

Relations: ① Generic relations for

$$\begin{tikzcd} & D & \\ B \arrow[ur, phantom, "\subset" sloped] & & C \arrow[ul, phantom, "\supset" sloped] \\ & A \arrow[ul, phantom, "\subset" sloped] \arrow[ur, phantom, "\supset" sloped] & \end{tikzcd}$$

"compatible" square of Frob ext.





for $f = \dfrac{\mu \circ \Delta_A^D}{\mu\circ\Delta_B^D \cdot \mu\circ\Delta_C^D}$

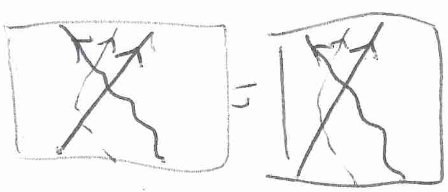
+ a couple more

i.e. for dihedral groups

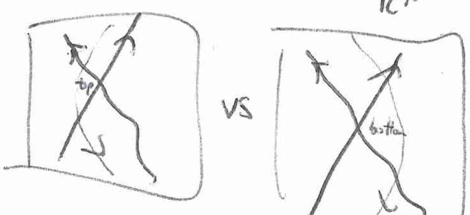
$$\begin{tikzcd} & R & \\ R^s \arrow[ur, phantom, "\subset" sloped] & & R^t \arrow[ul, phantom, "\supset" sloped] \\ & R^{s,t} \arrow[ul, phantom, "\subset" sloped] \arrow[ur, phantom, "\supset" sloped] & \end{tikzcd}$$

$f = \dfrac{\prod \text{pos roots}}{\alpha_s \alpha_t}$

② Generic relations for a cube of Frob ext



vs



More complicated

③ Very Specific SB_m Relations, only known in dihedral type, and almost proven in types $A, \tilde{A}$. REALLY NASTY. But you'll see why...

§4 Talking points

ⓐ Suppose W finite, so $I \in S$ is finitary. $\mathcal{H}$ has a maximal object, $H_S \mathcal{H}$ is 1-dimensional.

Similarly, $\mathrm{Hom}(\emptyset, S)$ is just the category of free graded R^W-modules. Yawn. But does it remind you of something?

Yes, of course... if $\mathbb{S}\mathrm{Bim} = \mathrm{Hom}(\emptyset,\emptyset) \longleftrightarrow \mathrm{Proj}\,\mathcal{O}_0$

then $\mathrm{Hom}(\emptyset, S) \longleftrightarrow \mathrm{Proj}\,\mathcal{O}_{-\rho}$ most singular block

and more generally, $\mathrm{Hom}(\emptyset, I) \longleftrightarrow \mathrm{Proj}\,\mathcal{O}_I$ ← a block for wt λ with $\mathrm{Stab}_{W_0}\lambda = W_I$.

More specifically, these categories have a right quotient (they are right R-modules, it's the same) and $\overline{\mathrm{Hom}(\emptyset, I)} \cong \mathrm{Proj}\,\mathcal{O}_I$.

Total bullshit! What was I thinking!

~~Meanwhile, what is $\mathrm{Hom}(I,\emptyset)$? It is "dual" in some sense. It is parabolic category $\mathcal{O}$~~

~~$\mathcal{O}_0^I = \{M \in \mathcal{O}_0 \mid \mathfrak{p}_I$ acts locally finitely$\}$ So $\mathcal{O}_0^\emptyset = \mathcal{O}_0$ since $\mathfrak{b}$ acts loc. finitely~~

~~Most vermas don't live here. Need to take $\mathrm{Ind}_{U(\mathfrak{p})}^{U(\mathfrak{g})}\mathbb{C}_\lambda$, much weirder.~~

~~$\mathrm{Hom}(I,J)$ = parabolic singular $\mathcal{O}$.~~

At least I didn't say it aloud!

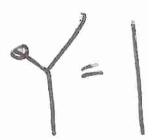
To get parabolic, need to use Koszul duality... not Verdier duality...

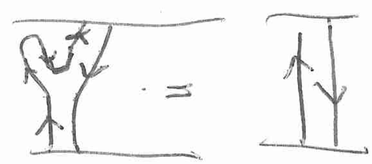
(2) So diagrammatics seems harder. But it is also *easier*, in some ways.

(a) Expressing hom in $\mathbb{BSB}\mathrm{im}$ as planar graphs makes you want to use planar graph theory — like an Euler characteristic argument — but I've tried + it isn't easy.

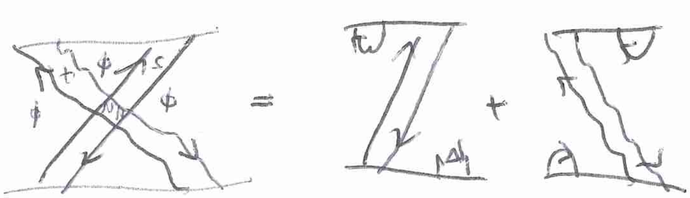
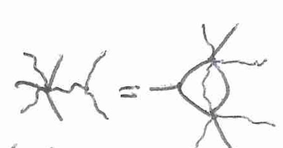
Expressing hom in $\mathbb{SBSB}\mathrm{im}$ as planar 1-mflds lets you use topology — Morse theory, Euler char arguments — and they work much better!!

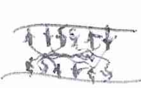
(b) Singular diagrams are more "local".

Ex: In $\mathbb{BSB}\mathrm{im}$  needed a relation

In $\mathrm{End}(\emptyset)$  is just isotopy.

This is a vast improvement when it comes to the 2-color + 3-color relations.

Ex: $m_{st}=3$  implies both  and 

Leads to only formula for 6-valent vertex in bimodules  once you know what  is.