§1| Outline/Recap|

Let $S(w) := [B_w] = \underline{H}_w$. We've seen $S(w) \Rightarrow \exists!$ nonzero invt form on $B_w \cong \mathbb{D}B_w$ up to scalar, it is nondegen.

$\langle , \rangle_{BS(\underline{w})}\big|_{B_w}$ is nonzero, so it is nondegen.

can state w/o $S(w)$:

- $hL(\underline{w}) :=$ $\overline{B_w}$ equipped w/ $\langle , \rangle_{BS(\underline{w})}\big|_{B_w}$ and L_ρ = left mult by ρ $(\partial_s(\rho) > 0)$ has hL.
- rex $\updownarrow$
- $HR(\underline{w}) :=$ has HR

$hL(w), HR(w) :=$ true for all rexes $\underline{w}$. $S(w) \Rightarrow (hL(\underline{w}) \Leftrightarrow hL(w))$

Rmk: $\langle , \rangle_{BS(\underline{w})}$ is already normalized s.t. $(c_{bot}, c_{bot})^{-l(w)}_{\rho} > 0$, via exercises

Really, we want to prove $S(W)$, and our inductive step will be to assume $S(<ws)$ and prove $S(ws)$ $ws > w$.

$hL(\underline{w}, s) := \overline{B_w B_s} \subset BS(\underline{ws})$ has hL

$HR(\underline{w}, s)$

Now $\underline{H}_w \underline{H}_s = \underline{H}_{ws} + \sum_y \mu(w,s,y) \underline{H}_y$ $\quad \mu(w,s,y) \overset{S(y), S(w), SCT}{=} \dim \operatorname{Hom}^0(B_y, B_w B_s) = \dim \operatorname{Hom}^0(B_w B_s, B_y)$

and no negative degree maps

The LI Pairing $\quad \operatorname{Hom}^0(B_y, B_w B_s) \times \operatorname{Hom}^0(B_w B_s, B_y) \longrightarrow \operatorname{End}^0(B_y) \overset{S(y)}{=} \mathbb{R}$

has rank = # of summands B_y inside $B_w B_s$. So LIP nondegen $\Rightarrow S(ws)$.

We've been identifying the two sides (by flipping diagrams upside down), let's do abstractly.

Both $B_y, B_w B_s$ have nondegen forms $\langle , \rangle_{B_y}$, $\langle , \rangle_{B_w B_s}$ ← induced form from exercises

For $\varphi \in \operatorname{Hom}(B_y, B_w B_s)$ define $\varphi^* : B_w B_s \to B_y$ via $\langle \varphi(b), b' \rangle_{B_w B_s} = \langle b, \varphi^*(b') \rangle_{B_y}$

This transfers the LIP to a LI Form on $\operatorname{Hom}^0(B_y, B_w B_s)$ $\quad (\varphi, \psi)^{w,s}_y$ = coeff of $\mathbb{1}$ in $\psi^* \circ \varphi$.

Embedding Thm: $\quad \operatorname{Hom}(B_y, B_w B_s) \longrightarrow \overline{B_w B_s}$, $\quad \varphi \longmapsto \overline{\varphi(c_{bot})}$ has image inside $P_\rho^{-l(y)}$, is injective, is isometry up to pos scalar.

Pf: $\rho^{l(y)+1} \underset{\overline{B_y}}{\overline{c_{bot}}} = 0$ for degree reasons. Thus $\rho^{l(y)+1} \overline{\varphi(c_{bot})} = 0$. Image is in $P_\rho^{-l(y)}$.

Injectivity come from the fact that (C_{bot}) was a basis, unraveling this is an exercise.

Now, $\langle C_{bot}, C_{top}\rangle_{B_y} = 1$ $\quad \langle C_{bot}, \rho^{\ell(y)} C_{bot}\rangle = N > 0$. Thus

$$(\varphi,\psi)_y^{w,s} = \text{coeff of } \mathbb{1} \text{ in } \psi^* \circ \varphi = \langle \psi^*\circ\varphi(C_{bot}), C_{top}\rangle = \frac{1}{N}\langle \psi^*\circ\varphi(C_{bot}), \rho^{\ell(y)} C_{bot}\rangle$$

$$= \frac{1}{N}\langle \varphi(C_{bot}), \psi(\rho^{\ell(y)} C_{bot})\rangle = \frac{1}{N}\left(\overline{\varphi(C_{bot})}, \overline{\psi(C_{bot})}\right)_\rho^{-\ell(y)}. \quad \blacksquare$$

Cor: $HR(\underline{w},s) \Rightarrow$ LIF is non-deg ($\forall y$ at once) $\Rightarrow S(ws)$.

Also, $B_{ws} \overset{\oplus}{\subset} B_w B_s$, preserved by L_ρ, restriction of HR to ~~...~~ L_ρ-invt subspace has HR (so long as restriction of $\langle , \rangle$ is nondegenerate) $\Rightarrow hL(\underline{ws}), HR(\underline{ws})$.

To finish the induction, us $S(<ws), hL(<ws), HR(<ws)$ to prove $HR(\underline{w},s)$.

Showing HR directly is hard. We use a limiting argument.

Let $L_\zeta \subset B_w B_s$ denote 

$\zeta \in \mathbb{R}$

write $hL(\underline{w},s)_\zeta$, $HR(\underline{w},s)_\zeta$ same but for L_ζ. $\quad L_0$ = our previous operator.

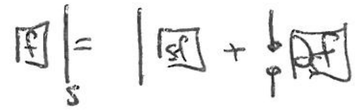
Rmk: L_ζ will NOT commute with all bimod maps, nor with idempotents, so will not restrict to B_{ws} or any other summand.

Rmk: Can even define them when $ws < w$!!

In fact, an easy exercise (using B_sB_s example as prototype) shows $hL(w) \Rightarrow hL(w,s)_\zeta$ when $ws<w$ and $\zeta \neq 0$.

Limit Thm: $HR(w) \Rightarrow HR(\underline{w},s)_\zeta$ for $\zeta \gg 0$ $\quad$ (either $ws>w$ or $ws<w$)

Pf: $L_\zeta = \rho + \zeta M$ where M is middle mult. $\quad L_\zeta^k$ has binomial expansion.

But $M^2 = 0$ on $\overline{B_w B_s}$ $\qquad$  $\quad$ if f bigger than linear, then $\partial_s f$, sf are at least linear.

$$\Rightarrow L_\zeta^k = \rho^k + \zeta k \rho^{k-1} M$$

the limiting term.

In exercises, got a basis for $\overline{B_w B_s}^{-k}$ in terms of $\overline{B_w}^{-k-1}$ and $\overline{B_w}^{-k+1}$.

let $\alpha_i \in B_w^{-k-1}$ project to an ONB of $\overline{B_w}^{-k-1}$

$\beta_i \in B_w^{-k+1}$ —— // —— $P^{-k+1} \subset \overline{B_w}^{-k+1}$

So $\{\rho\alpha_i, \beta_i\}$ an ONB for B_w^{-k+1}.

get basis $\left\{ \boxed{\alpha_i}\,{}^{b}_{p} = \alpha_i c_s \;,\; \boxed{\beta_i}\,| = \beta_i c_1 \;,\; \boxed{\rho\alpha_i}\,| = \rho\alpha_i c_1 \right\}$ for $\overline{B_w B_s}^{-k}$.

Exercise: Compute $\langle v, \rho^{k-1} M_w \rangle$ for this basis.

It has signature equal to signature of $(\ ,\ )^{-k+1}$ on P^{-k+1} (only β part matters, α_i and $\rho\alpha$ cancel each other)

Exercise ⟹ Correct signature for HR on $\overline{B_w B_s}$.

So, if we can show $hL(w,s)_\zeta$ for all $\zeta \geq 0$ (including 0) then continuity gives us $HR(w,s)_0$ and we win. How to get $hL(w,s)_\zeta$??

We've been roughly following dC+M's geometric proof.

ρ is relatively ample, $\rho + \zeta M$ is truly ample (rel. is on boundary of ample cone)

How to show (hL) in geometry? Have the weak lefschetz theorem.

$\mathcal{L}$ ample $\downarrow$ X smooth — for generic section of $\Gamma(\mathcal{L})$, the zeroes form a hyperplane $Y \overset{i}{\subset} X$. $\dim Y = \dim X - 1$

$$H^*(Y) \underset{i^*}{\overset{i_*}{\rightleftarrows}} H^*(X) \quad \text{and} \quad i_* i^* = c_1(\mathcal{L}).$$

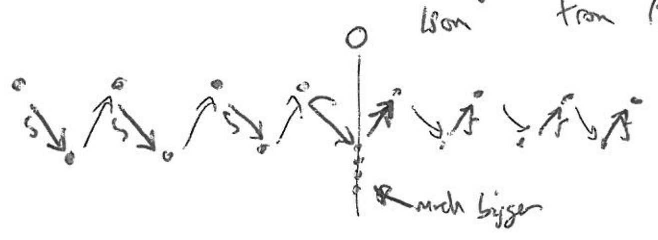
after recentering, i^* and i_* have degree $+1$. Commute w/ respective Lefschetz operators.

Weak Lefschetz Thm: i^* is injective from negative degrees, isom to negative degrees

i_* is surj to pos deg, isom from pos deg. and they are adjoint.

$\Rightarrow \langle v, Lw \rangle = \langle i^* v, i^* w \rangle$

i.e.



$H^*(X)$

$H^*(Y)$

Prop: Spose (V, L_V) and (W, L_W) are lefschetz spaces.

$\sigma : V \to W(1)$ satisfies • $\langle v, Lv' \rangle = \langle \sigma v, \sigma v' \rangle$ • $\sigma L = L \sigma$ • σ injective from neg degrees.

Then HR for $W \implies$ hL for V.

Pf: Spose $0 \neq v \in P^{-k} \subset V$, and $k > 0$ ($k=0$ trivial). Then $\sigma v \neq 0$.

① σv not primitive $\implies L^{k} \sigma v \neq 0 \implies \sigma L^k v \neq 0 \implies L^k v \neq 0$. ✓

② σv primitive $\implies 0 \neq (\sigma v, \sigma v)_L^{-k+1} = \langle \sigma v, L^{k-1} \sigma v \rangle_W = \langle v, L^k v \rangle_V \implies L^k v \neq 0$ ✓ ∎

So weak Lefschetz + HR induction $\rightsquigarrow$ hard lefschetz, + lind $\rightsquigarrow$ HR.

Unfortunately we have no algebraic notion of a generic hyperplane section (i.e. no combinatorics for $H^*(Y)$). We need to find some other map σ!

Factor L as $\sigma^* \circ \sigma$

Final step in setup: L_ζ on $BS(\underline{w}s)$ is $\rho|||||||+||||||\zeta\rho|$

$= \text{coeff } {}^{b}_{\rho}|||||+|s\rho|||||+|||||\zeta\rho|$

$= \text{coeff } {}^{b}_{\rho}|||||+\text{coeff } |{}^{b}_{\rho}||||+ \dots + \text{coeff } ||||||{}^{b}_{\rho}$

So $L_\zeta = \sum_{i=1}^{d+1} \lambda_i Br_i$ — break the i^{th} strand

Claim (Exercise): $\lambda_i > 0$ for all i!!! (when $\partial_j(\zeta) > 0 \ \forall j$, $\zeta > 0$)

So consider

$$BS(\underline{w}s) \underset{\sigma}{\overset{\sigma^*}{\rightleftarrows}} \bigoplus_i BS(\underline{w}s)_{\hat{i}}$$

with $\sigma^* = \sum \sqrt{\lambda_i}\ |||{}^{b}|||$ and $\sigma = \sum \sqrt{\lambda_i}\ |||\rho|||$. hooray for square roots!!

Then $\sigma^* \sigma = L_\zeta$, $\langle v, L_\zeta w \rangle = \sum_i \lambda_i \langle |||\rho|||\, v, |||\rho|||\, w \rangle$.

So want to show ① σ injective from neg degrees ② RHS has HR.

BUT! σ is the first differential in the rouquier complex $\bigotimes B_{s_i} \xrightarrow{\sqrt{\lambda_i}\,\rho} R(1)$

Unfortunately, ② is very false! Not semisimple!

BUT! We only care about $B_w B_s \subset BS(\underline{w}s)$. So why not restrict to a minimal complex! $F_w F_s$.

Diagonal miracle gets rid of all the trash!!! To be continued.