Interested in: Showing LIF are nondegenerate. $\mathrm{Hom}(B_y, B)$. Specifically, $B = B_w B_s$.
Suddenly switched to: studying GIF on $\overline{B}$. How do they relate?
Punchline: LIF embeds in the Lefschetz form for GIF wrt dominant weight, so Hodge-theoretic properties of GIF will imply LIF is nondegenerate. Let's figure it out.

Analogous to/borrowed from: In geometry, Decomposition Theorem (a statement about how things decompose into direct sums) $\Leftrightarrow$ nondegeneracy of certain LIFs. de Cataldo + Migliorini came up with proof using Hodge theory of the usual intersection form on $H^*(X)$ for X a smooth $\mathbb{C}$ projective vty.

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§1 Lefschetz Lin Alg

Let H be a f.d. gdd v.s $/\mathbb{R}$ w/ sym. bil form $\langle , \rangle : H \times H \to \mathbb{R}$ which is

1. graded $\langle a,b \rangle = 0$ unless $\deg a + \deg b = 0$ $\Longrightarrow \dim H^i = \dim H^{-i}$.
2. nondegenerate

Def: $L : H \to H(2)$ is a Lefschetz operator if $\forall v \in H^{-i}$, $w \in H^{i-2}$ $\langle v, Lw \rangle^{i} = \langle Lv, w \rangle^{i-2}$

i.e. $H^i \to H^{i+2}$

Ex: $H = \overline{BS(\underline{w})}$, $\langle , \rangle =$ GIF valued in $\mathbb{R}$. $L =$ left mult by any linear poly (since ring is commutative)
or anywhere mult. But right mult is zero, zero is fine

Ex: X a sm. proj vty of $\mathbb{C}$ dim n $\rightsquigarrow$ $H^{n+i}(X) \cong H^{n-i}(X)^*$ by PD, so let $H^i = H^{n+i}(X)$ to recenter.

$\langle \alpha, \beta \rangle = \mathrm{Tr}(\alpha\beta) = \int_X \alpha \wedge \beta$. $L =$ mult by any $\alpha \in H^2(X)$.

commutative if $\deg \alpha$ or β is even

Analogy is no accident. When $W =$ Weyl (crystallographic), $\overline{BS(\underline{w})} = H^*(BS(\underline{w}))$ the Bott-Samelson variety.

As usual, if not crystallographic, there is no geometry.

Def: L induces a form on each H^{-i}, $i \geq 0$, called the Lefschetz form, via $(v,w)^{-i}_L = \langle v, L^i w \rangle^{i}$

It depends on L, except $i=0$. Ex: $L=0$.

Def: L satisfies hard Lefschetz (HL) if $\forall i \geq 0$, $L^i : H^{-i} \to H^{+i}$ is injective

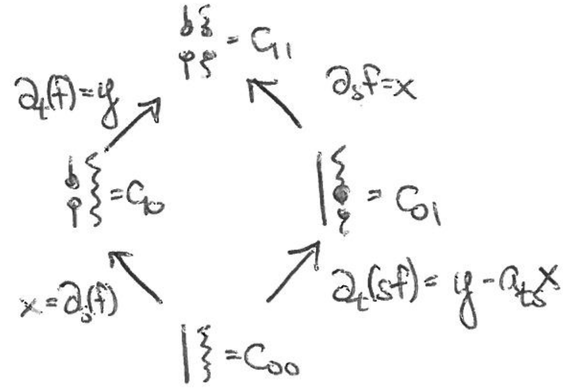
$\Longleftrightarrow$ isom $\Longleftrightarrow$ $(,)^{-i}_L$ is non-degenerate

(vacuous for $i=0$)

Ex 1: $H = \overline{BS}(st)$

L_f = left mult by f

where $\partial_s(f) = x$, $\partial_t(f) = y$



$L^Q_f(c_{bot}) = x(2y - a_{ts}x)\, c_{top}$

satisfies (hL) if $x \neq 0$, $2y \neq a_{ts}x$

Classic non-Ex:

(not quite the same as $x=0$ above, b/c pairing is different, but after c.o.b. yes.)

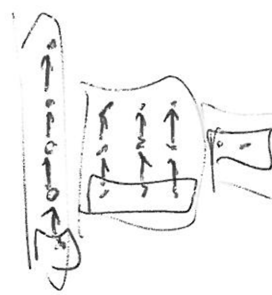
Exercise: L_f on $\overline{B_sB_s}$ **Never** has (hL). When does $L_f + M_g$ have hL?

↑ it's because it is not semismall!! ↑ middle mult by g.

Observe: $(v,w)^{-i}_L = (Lv, Lw)^{-i+2}_L$. Can use to define $(,)_L$ on positive degree too.

Def: Assume (hL). Think of as $\mathfrak{sl}_2$-repn (Not shifted $\mathfrak{sl}_2$, as in non-ex above)

Ex:



no arrow = Ker L.

$P^{-i}_L = \{ v \in H^{-i} \mid L^{i+1} v = 0 \}$ primitives, "lowest wt vectors"

$H^k = \bigoplus_{i \geq 0} L^i P^{k-2i}_L$ is Lefschetz decomp, "isotypic"

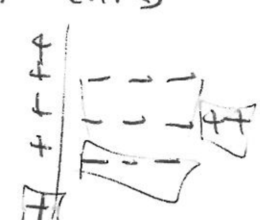
Claim: $(,)^{-i}_L$ is $\perp$ wrt Lef decomp. **Pf:** By example, exercise.

Def: Assume (hL). Assume H is even or odd.

L has the Hodge–Riemann bilinear relations w/ std sign (HR) if $(,)^{-i}_L$ is alternating definite on primitives.

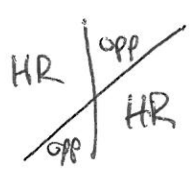
$\iff$ signature determined from graded rank

↑ depends on L ↑ indep of L



Ex 1 cont.: $P^{-2}_L = \langle c_{bot} \rangle$, $(c_{bot}, c_{bot})^{-2}_L = x(2y - a_{ts}x)$

$P^0_L = \mathrm{Span}(\cancel{x}c_{10} - y c_{01})$ and $\langle \cancel{x}c_{10} - y c_{01}, x c_{10} - y c_{01} \rangle = -x(2y - a_{ts}x)$



however, situation in other cases can be extremely complex! regions are cone, but not a linear one!

Claim: L_ζ a cont. family of operators w/ hL $\Longrightarrow$ if L_0 has HR then L_1 has HR.
$\zeta \in [0,1]$

Pf: Signature constant in family of nondegen forms.

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§21 HTSB: Fix $\rho \in \mathfrak{h}^*$, $\partial_s(\rho) > 0$ $\forall s \in S$. You may need to extend $\mathfrak{h}^*$ a bit to ensure one exists.

Draw an analogy b/w SBim + Geometry, to be explained better tomorrow.

SBim

Geometry (only when W is Weyl/cryst.)

$\overline{BS(\underline{w})}$ — $H^*(BS(\underline{w}))$ $BS(\underline{w}) \xrightarrow{\pi} G/B = Fl$

Smooth proj alg vtys

linear comb of mult by $f \in \mathfrak{h}^*$ in various slots — mult by $c_1(\mathcal{L}) \in H^2(BS(\underline{w}))$ for some line bundle $\mathcal{L}$

$\sum$ mult by ρ in each slot

doesn't

$\mathcal{L}$ a specific ample bundle

$$\begin{tikzcd} BS(\underline{w}) \arrow[r, hook] & \mathbb{P}^N \\ \mathcal{L} \arrow[u] \arrow[r] & \mathcal{O}(1) \arrow[u] \end{tikzcd}$$

mult by ρ on left

commutes w/ all bimod maps, respects $\oplus$ decomps.

$\mathcal{L}$ relatively ample for π (NOT ample)

$$\begin{tikzcd} BS(\underline{w}) \arrow[r, "\pi"] & G/B \arrow[r, hook] & \mathbb{P}^N \\ \mathcal{L} = \pi^*\mathcal{L}' \arrow[u] \arrow[r] & \mathcal{L}' \arrow[u] \arrow[r] & \mathcal{O}(1) \arrow[u] \end{tikzcd}$$

Thm: (hard lefschetz thm) X sm proj $\mathbb{C}$ alg vty, $\mathcal{L}$ ample

Then $H^*(X)$, $c(\mathcal{L})$ has hL, HR.

Thm: (Improved hL) X not ness smooth $IH^*(X)$ intersection cohomology

Then $IH^*(X)$, $c(\mathcal{L})$ has hL, HR when $\mathcal{L}$ ample.

Invented for this purpose, to fix PD etc when X not smooth

$\overline{B_w}$ $IH^*(\overline{BwB}/B)$ Schubert variety

Expect $(\overline{B_w}, L_\rho)$ to have hL, HR. Also, direct sums
(ample on G/B)

$(\overline{B_s B_t B_s}, L_\rho)$ since $B_s B_t B_s \cong B_{sts} \oplus B_s$ $m \geq 3$

but not $\overline{B_s B_s} \cong \overline{B_s(1)} \oplus \overline{B_s(-1)}$, shifted hL does not have hL.

$\uparrow$ has hL for $L_\rho + M_\rho$, but that doesn't respect $\oplus$ decomp.

hL is <u>hard</u>, old proof uses Weil conjectures, etc. dC+M use better proof, via

<u>Thm 2:</u> Suppose $\underset{\text{smooth}}{X} \xrightarrow{\pi} Y$ is proper + semismall.

fibers compact

fibers not too big.

Then $H^*(X)$ has hL for $\mathcal{L}$ only relatively ample, i.e. $\mathcal{L} = \pi^*\mathcal{L}'$, $\mathcal{L}'$ ample

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BS($\underline{w}$) semismall, i.e.

no LL of negative degree

($\underset{\text{SConj}}{\Longleftrightarrow}$ no shifted summands)

$BS(\underline{w}) \to G/B$ semismall

This is rare, but same principal should apply to other semismall things,

i.e. $B_w B_s$ $ws > w$

Expect $(\overline{B_w B_s}, L_\rho)$ has hL, HR.

Why is all this useful? We've seen: LIF on $\mathrm{Hom}^0(B_y, B_w B_s)$ nondegen $\Longrightarrow$ Soergel Conjecture.

<u>Embedding Thm (next time)</u> LIF $\underset{\text{isometry}}{\hookrightarrow}$ GIF on $\overline{B_w B_s}$, living inside primitives in degree $-\ell(y)$.

Restriction of GIF to primitive subspace is $\pm$ definite $\Longrightarrow$ nondegen!