Much talk of morphisms b/w BSBim. Now we return to viewing them as vector spaces/R-bim again.

Interested in elements. First construct 2 bases for BS(w). Has size 2^d ...

§1) Two bases)

Recall: B_s has basis $c_{bot} = 1\otimes 1 = c_1$ (temp) $\quad \deg c_{bot} = -1$

$c_s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \quad \deg c_s = +1$

as a right R-mod ← or left, but we've chosen right.

with $f c_s = c_s f \qquad f c_1 = c_1 s(f) + c_s \partial_s(f)$

Also, note that $c_s = (\text{dot-dot})(c_1)$

So choose $\underline{w}$, BS($\underline{w}$) has basis $\{c_{\underline{e}}\}$ for $\underline{e} \subset \underline{w}$ as right R-mod. $\quad c_{\underline{e}} = c_{s_1 e_1} c_{s_2 e_2} \cdots c_{s_d e_d} \in B_{s_1} \cdots B_{s_d}$

$c_{0100110} = (\,|\,\text{dot-dot}\,|\,|\,\text{dot-dot}\,\text{dot-dot}\,|\,)(c_{bot}) \qquad c_{bot} = 1\otimes 1 \otimes \ldots \otimes 1 = c_{0000000}$

Obvious Cor: Every elt of BS($\underline{w}$) is $\varphi(c_{bot})$ for some $\varphi \in \mathrm{End}(BS(\underline{w}))$

in fact $f_1 \otimes f_2 \otimes \ldots \otimes f_{d+1} = (\boxed{f_1}\,|\,\boxed{f_2}\,|\,|\,|\,|\,\boxed{f_{d+1}})\, c_{bot}$; but could also use above description.

But we have a basis for End(BS($\underline{w}$)), double leaves. What is (double leaf)(c_{bot})?

Claim! $\boxed{LL}(c_{bot}) = 0$ if $\underline{e}$ has any downs

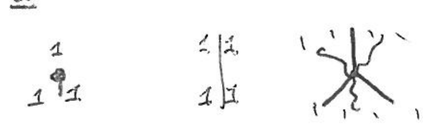
Pf:

If all ups, ∃! subexpression for each x, called the canonical exp

So $\begin{pmatrix} f \\ \underline{e} \end{pmatrix}(c_{bot}) = \begin{cases} 0 & \text{if } \underline{e} \neq \mathrm{can}_x \\ \boxed{f}\; c^x_{bot} & \text{if } \underline{e} = \mathrm{can}_x. \end{cases}$

Claim: $\underset{\underline{w}}{\overset{x}{\boxed{\ }}}(c^{\underline{w}}_{bot}) = c^{x}_{bot}$ if all ups.

Pf:



Basis 2: $\left\{ \underset{x}{\boxed{f}}(c^{x}_{bot}) \right\}_{\substack{\underline{f} \subset \underline{w},\ \underline{w}^{\underline{f}} = x \\ x \leq \underline{w}.}}$

Not canonical, because LL are not canonical.

Adapted well for certain things.

$\Longrightarrow$ key conclusion: $\boxed{f}(c_{bot})$ are all linearly independent!

§21 Ring Structure + GIF

$BS(\underline{w}) = R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_d}} R(d)$ and $R \otimes_{R^{s}} \cdots \otimes_{R^{s_d}} R$ is not just a bimodule, it is also a commutative ring!!

$$(f_1 \otimes \cdots \otimes f_{d+1})\cdot(g_1 \otimes \cdots \otimes g_{d+1}) = f_1 g_1 \otimes \cdots \otimes f_{d+1} g_{d+1}$$

So $BS(\underline{w})$ is a shifted ring ($\underset{c_{bot}}{1}$ is in degree $-d$)

Ex: $BS(st)$

$$c_{10} \circ c_{10} = (c_{bot}) \cdot (c_{bot}) \overset{*}{=} (c_{bot}) = \left(t(\alpha_s) + a_{t,s} \right)(c_{bot})$$

$$= c_{10}\, t(\alpha_s) + c_{11}\, a_{t,s}$$

Why $*$? Have nice commutative subring $\{ \ \} \subset \mathrm{End}(BS(\underline{w}))$

for which $\varphi(c_{bot}) \cdot \psi(c_{bot}) = \varphi \circ \psi (c_{bot})$

Things of the form ( ) are inside this subring $\rightsquigarrow$ $c_{\underline{\varepsilon}}$ basis is nicely adapted to multiplication.

Warning: $\varphi(c_{bot}) \cdot \psi(c_{bot}) \neq \varphi \circ \psi (c_{bot})$ in general, so harder with $\left\{ \begin{matrix} \Gamma_{\underline{\varepsilon}} \\ \mathrm{can}_x \end{matrix} (c_{bot}) \right\}$ basis

Finding mult, CoB is hard!

Def: Global intersection form $\quad c_{top} = c_{111\ldots1} = (\ )(c_{bot}) = \Gamma_1 BS(\underline{w})$

$Tr : BS(\underline{w}) \to R$ takes coeff of c_{top}, $*$ $\quad f c_{top} = c_{top} f$

$\langle\ ,\ \rangle : BS(\underline{w}) \times BS(\underline{w}) \to R$

$\langle a, b \rangle = \mathrm{Tr}(a \cdot b)$

in $\{c_{\underline{\varepsilon}}\}$ basis, ~~...~~ $\rightsquigarrow$ right Rmod

but it is really canonical, b/c of support stuff.

Ex:

$BS(st)$	c_{bot}	c_{01}	c_{10}	c_{top}
c_{bot}	0	0	0	1
c_{01}	0	0	1	α_t
c_{10}	0	1	$\partial_t(\alpha_s)$	α_s
c_{top}	1	α_t	α_s	$\alpha_s\alpha_t$

Prop: Easy "upper-triangularity" argument $\Rightarrow$ GIF is non-deg.

Exercise: wrt what order?

What kind of thing is GIF?

The pairing $\langle , \rangle$ is
- graded $\deg \langle a,b \rangle = \deg a + \deg b$
- (right) invariant $\langle af, b\rangle = \langle a, bf\rangle = \langle a,b\rangle f$ $f \in R$

  $\langle fa, b\rangle = \langle a, fb\rangle$ (nothing)
- non-deg. to degree 0

Abstraction: $\mathrm{Hom}_{\text{right } R\text{-mod}}(B,R) = \mathbb{D}B$. It's an R-bimod $\mathbb{D}(B(1)) = B(-1)$

$\mathrm{Hom}^{\circ}_{R\text{-bim}}(B, \mathbb{D}B) =$ space of invariant graded forms on B.

Isom $\longleftrightarrow$ nondegen to degree 0

Thus $BS(\underline{w})$ has nondegen form $\Longrightarrow$ it is self-dual.

Easy argument: $B_w \overset{\oplus}{\subset} BS(\underline{w})$ (rex) $\quad \mathbb{D}B_w \overset{\oplus}{\subset} BS(\underline{w})$, !-ness $\Longrightarrow B_w \cong \mathbb{D}B_w \Longrightarrow \exists$ nondegen inv't form

Meanwhile, $B_w \subset BS(\underline{w})$ inherits GIF as restriction. Exercise: $c_{bot}, c_{top} \in B_w$ so $\langle , \rangle|_{B_w}$ is nonzero.

Cor: If SConj is true, then $\mathrm{End}^{\circ}(B_w) = \mathbb{R}$ so any nonzero $\langle , \rangle$ is non-degenerate!

Expect $\langle , \rangle|_{B_w}$ to be nondegenerate. This is the GIF on B_w.

Rmk: Nondegen to degree 0 $\Longrightarrow$ descends to nondegen $\mathbb{R}$-valued form on $\overline{B}^{\circ} \times \overline{B} \longrightarrow \overline{R} = \mathbb{R}$. GIF on $\overline{B}_w$.