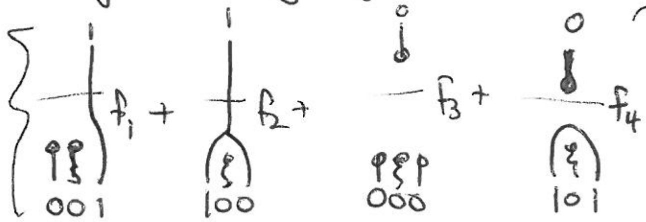
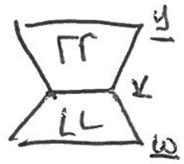
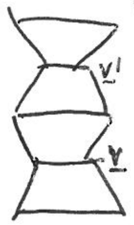
Last time: $\mathcal{D} \cong \mathbb{BS}\mathrm{Bim}$ has a right R-basis given by double leaves.

Ex: $\mathrm{Hom}^\bullet(\underline{sts}, s) = \Big\{$  $\Big\}$

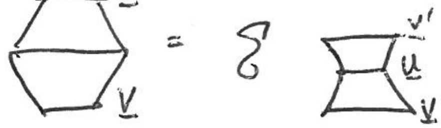
$\mathrm{Hom}_{\mathcal{D}}(\underline{w}, \underline{y}) = \sum$  R

Why trapezoids? Factors thru shorter + shorter sequences, never gets longer.

$\mathcal{D}$ is filtered by ideals $\mathcal{D}_I = \mathrm{span}\{\mathbb{LL}_I\}$ for various ideals $I \subseteq W$. I.e. compose



then resolve middle

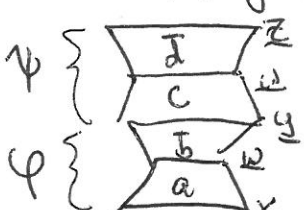


$u \leq v, v' \Rightarrow u \in I$

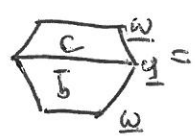
etc.

Let $\underline{w}$ be a rex. $\mathrm{End}(\underline{w}) / \mathcal{D}_{<w} =$  $\cdot R = \boxed{\beta} \cdot R = \mathbb{1}_{\underline{w}} \cdot R$ modulo $\mathcal{D}_{<w}$.

Thus 1D property gives control over $\mathcal{D}_{\leq w} / \mathcal{D}_{<w}$!

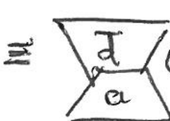


$= ?$



$= \mathbb{1}_{\underline{w}} \cdot f$ + lower terms

$f = \varphi_{\underline{y}}(c, \bar{b})$

So $\boxed{\frac{\psi}{\varphi}} \equiv$  $\varphi_{\underline{y}}(c, \bar{b})$

To compose, only need to understand the pairing $\varphi_{\underline{y}}$ on $M(\underline{y}, w) = \{ e \subset \underline{y} \mid \underline{y}^e = w \}$

local intersection form.

Formalism: $\mathcal{D}$ is an (object-adapted) cellular category.

Another ex: TL. 3 thru strands

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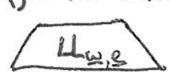
§2 Cellular Categories!

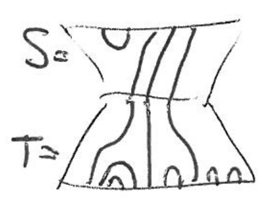
Def: A *cellular cat* $\mathcal{C}$ is a cat equipped with:

Examples:

	$\mathcal{D}$	TL
• a poset Λ	$W, \leq$	$\mathbb{N}, \leq$
• i antiinvolution	flipping diagrams upside down	
• For each $X \in \mathrm{Ob}(\mathcal{C})$, $\lambda \in \Lambda$, finite sets $M(\lambda, X) \overset{\sim}{\underset{i}{\longleftrightarrow}} E(X, \lambda)$	$E(\underline{w}, x) = \{ e \subset \underline{w} \mid \underline{w}^e = x \}$  $M(x, \underline{w}) =$ same = 	$E(n,k) = \{$  $\}$ $M(k,n) =$ upside down.

• For each quintuple

$X, Y \in \mathrm{Ob}\,\mathcal{C}$, $\lambda \in \Lambda$, $T \in E(X,\lambda)$, $S \in M(\lambda, Y)$ a morphism $C^\lambda_{S,T} \in \mathrm{Hom}(X,Y)$ s.t. $i(C^\lambda_{S,T}) = C^\lambda_{i(T),i(S)}$



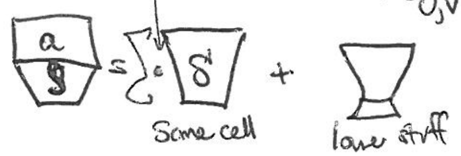
Satisfying ⓐ $\{C^\lambda_{S,T}\}_{\lambda,S,T}$ is a basis for $\mathrm{Hom}(X,Y)$

ⓑ $a C^\lambda_{S,T} = \sum \ell(a,S,S')\, C^\lambda_{S',T}$ + lower terms

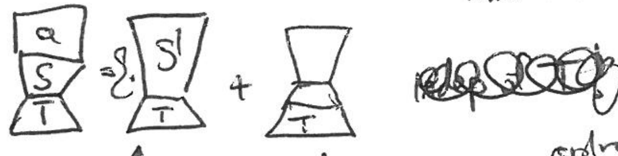
lower terms $\in \mathrm{Span}\{C^\mu_{U,V} \mid \mu < \lambda\}$

Most subtle

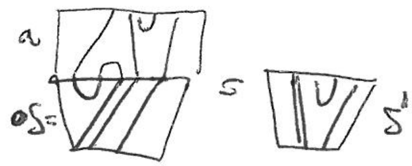
idea:



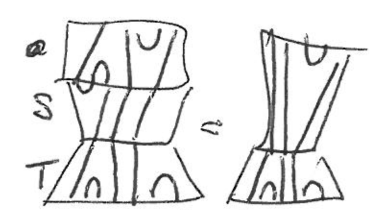
So



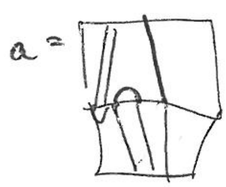
Ex:



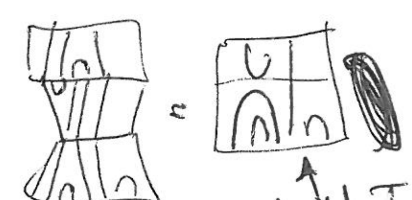
So



BUT



then



Rmks: ① Applying i to ⓑ, get condition on right mult.

② NO guarantee that $C^\lambda_{S,T} = C^\lambda_S \circ C^\lambda_T$, as in our example!!!

So ~~[scribble]~~ ⓑ is much less intuitive!!!

Ex: $\mathbb{H}_{S_n}$ $\Lambda = \mathrm{Part}(n)$, $M = E =$ std tableau, $C^\lambda_{S,T} = \underline{H}_{w(S,T)}$ RS algorithm

Def: $\mathcal{C}$ is (strictly) object adapted if

• $\Lambda \subset \mathrm{Ob}(\mathcal{C})$

• $\exists$ $C_S \in \mathrm{Hom}(\lambda, X)$, $C_T \in \mathrm{Hom}(X, \lambda)$ s.t. $i(C_S) = C_{i(S)}$

pick reps $\Lambda = \{\underline{\omega}\}$ $\quad \Lambda \cong W \subset \mathrm{Ob}(\mathcal{TL})$

$C^\lambda_{S,T} = C_S \circ C_T$

ⓑ $a c_S = \sum \ell(a,S,S')\, c_{S'}$ + l.t.

$\Longrightarrow$ $M(\lambda,\lambda) = \{*\} = E(\lambda,\lambda)$ and $c_* = \mathbb{1}_\lambda$ (+ l.t.)

Rmk ③ Can always adjust $C^\lambda_{S,T}$ or c_S by l.t. and get a new cellular basis.

Most of the SCT is just general nonsense for Obj Ad Cell Cat.

Exercise: Prove that indecomposables $\longleftrightarrow \Lambda$, each $\lambda \in \Lambda$ has ! prim. idempotent e_λ with $e_\lambda \equiv 1$ mod l.t.

Def: Local Intersection Form $\mathrm{LIF}_{X,\lambda}$

$$E(X,\lambda) \times M(\lambda, X) \longrightarrow R$$

$$T \quad S \longmapsto T \circ S \text{ mod l.t in } \mathrm{End}(\lambda)/_{l.t} \cong R$$

Prove that rank $\mathrm{LIF}_{X,\lambda}$ = multiplicity of $\mathrm{Im}\, e_\lambda$ as summand of X.

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§3 Practice in $\mathcal{D}$

Fix $\underline{\omega}, x$, $k \in \mathbb{Z}$. Pair

$$M(x, \underline{\omega}, k) \times E(\underline{\omega}, x, -k)$$

$\mathrm{LIF}_{\underline{\omega},x,k}$ $\quad \{ \underline{e} \subset \underline{\omega}, \underline{\omega}^{\underline{e}} = x, \mathrm{def}(\underline{e}) = k \} \longmapsto$

 $\in \mathrm{End}^{\circ}(\underline{X}) / \mathcal{D}_{<x} = R$

coeff of identity in LL bars

Ex: $\underline{\omega} = sts$ $\quad x = s$ $\quad k = 0$

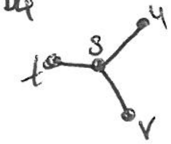
only guy

pair

$= \boxed{\partial_s \alpha_t} \; | = a_{st} \; |$

$M_{st} \neq 2 \Rightarrow a_{st} \neq 0 \Rightarrow \mathrm{LIF}$ rank 1 $\Rightarrow B_s(0) \overset{\oplus}{\subset} BS(sts)$

idempotent is $\frac{1}{a_{st}}$ 

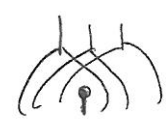
Ex: D_4 

$\underline{\omega} = tuvstuv$

$x = tuv$

8 maps

degree



-2

0

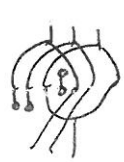
pairing

 . . .

$+2$

$+4$

$k=-2$ (or 2) get 3×1 matrix

 $= - |||$

$\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ rank = 1

$k=0$ $\quad 3\times 3$ matrix $\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ det $= -2$ so rank = 3

(in char 2, rank = 2 !!)

$\Rightarrow BS(tuvstuv) \overset{\oplus}{\supset} B_{tuv} \begin{pmatrix} 2 \\ 000 \\ 2 \end{pmatrix}$