Ben Elias   Intro to Cat $\mathcal{O}$ Oregon 8/2014   Lecture 2.4 (1)

Fix $\mathfrak{g}$ a $\mathbb{C}$x fd s.s. l.a. Rep $\mathfrak{g}$ = fd $U(\mathfrak{g})$-repns is an "easy" semisimple category.

Irred V has a h.w. vector, its weight classifies V.

But what about ∞-dim repns?

<u>Ex</u>: $U(\mathfrak{g})$ itself. No h.w. Nasty.

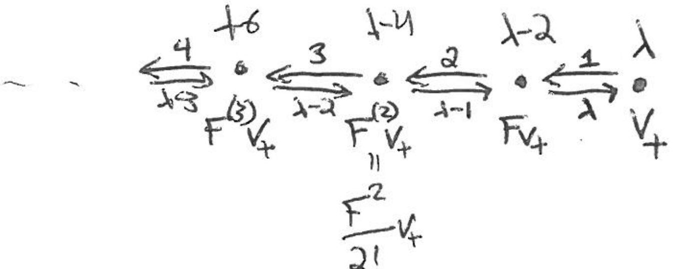
$\mathcal{O}$ is a class of "nice" ∞-dim repns that tries to preserve the useful notion of highest weights.

<u>Ex</u>: $\mathfrak{g} = \mathfrak{sl}_2 = \langle H, E, F \rangle$

$[H,E] = 2E$
$[H,F] = -2F$.
$[E,F] = H$

$v_+ \in V$ is <u>h.w. of weight λ</u> if $Hv_+ = \lambda v_+$, $Ev_+ = 0$

Generate a subrepn of V.



<u>Usual arg</u>: If fd, $\exists$ k s.t. $F^{(k)}v_+ = 0$ for first time. But then $\longrightarrow$ must have coeff 0.

$\Longrightarrow k = \lambda,\ \lambda \in \mathbb{N}$!!

But in ∞-dim, allowed for all $F^{(k)}v_+ \neq 0$, and λ can be arbitrary $\lambda \in \mathbb{C}$!

Get the <u>Verma module</u> $\Delta(\lambda)$.

<u>General defn</u>: $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}_\lambda$ where $\mathbb{C}_\lambda$ is 1D rep of $\mathfrak{b}^+$ (v_+)

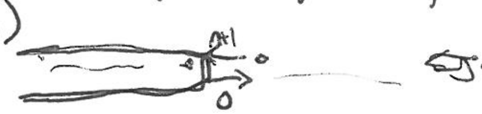
$= \mathrm{Ind}_{U(\mathfrak{b}^+)}^{U(\mathfrak{g})} \mathbb{C}_\lambda$.

<u>Properties</u>: (1) $\Delta(\lambda)$ is indecomp. <u>Pf</u>: Any submodule splits into wt spaces (action of H can project to wt spaces)

if $\Delta(\lambda) \cong M \oplus N$ then either $v^+ \in M$ or $v^+ \in N$ $\Rightarrow M = \Delta(\lambda)$
WLOG $N = 0$. ∎

(2) $\Delta(\lambda)$ is simple unless $\lambda \in \mathbb{Z}_{\geq 0}$. <u>Pf</u>: Otherwise, all $\leftrightarrows$ arrows are nonzero so any wt vector goes to any other

(3) If $\lambda \in \mathbb{Z}_{\geq 0}$, $\lambda = n$, then $\Delta(-n-2) \subset \Delta(n)$



$0 \to \Delta(-n-2) \to \Delta(n) \to L(n) \to 0$

$\nwarrow$ simple fd module

(4) $\Delta(\lambda)$ has weird self-extensions which are NOT weight modules!

$X = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}^2_\lambda$ $\nwarrow$ H acts via $\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$

filt as $U(\mathfrak{b}^+)$ module $\rightsquigarrow$ filt as $U(\mathfrak{g})$-module

$0 \to \mathbb{C}_\lambda \to \mathbb{C}^2_\lambda \to \mathbb{C}_\lambda \to 0$ $0 \to \Delta(\lambda) \to X \to \Delta(\lambda) \to 0$

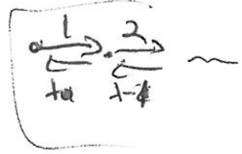
We don't like this.

Def: $\mathcal{O} \subset U(\mathfrak{g})$-rep is the full subcat whose objects are

- f. generated
- weight
- $U(\mathfrak{b}^+)$-finite $= \forall v,\ U(\mathfrak{b}^+)v$ is f.d.

$\Updownarrow$ wts bdd above, f.d. wt spaces.

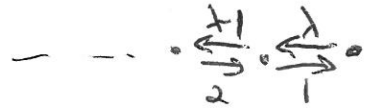
Ex: Includes $\Delta(\lambda)$ $L(n)$ but not X, not $U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} \mathbb{C}_\lambda$



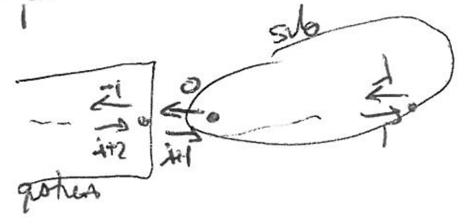
$\mathcal{O}$ has duality $M \longmapsto M^\vee$. Can't just take $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ because this reverses weights, bdd below... compose that with automorphism of $\mathfrak{g}$ $H \longleftrightarrow -H$, $E \longleftrightarrow F$.

Effectively reverse roles of arrows!

$\Delta(\lambda)^\vee = \nabla(\lambda) =$



If $\lambda \notin \mathbb{Z}_{\geq 0}$, $\nabla(\lambda) \cong \Delta(\lambda)$. But if $\lambda \in \mathbb{Z}_{\geq 0}$,



$0 \to L(n) \to \nabla(n) \to \nabla(-n-2) \to 0$ dual S.E.S.

$\nabla(-n-2) = \Delta(-n-2)$

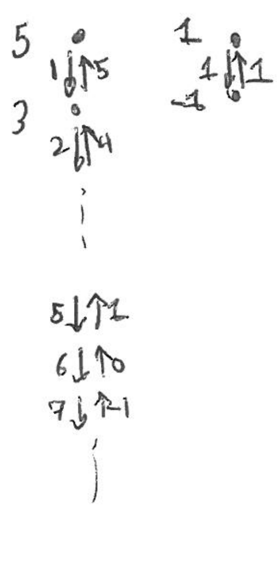
$\mathcal{O}$ is not monoidal. $V \otimes W$ not f.g.

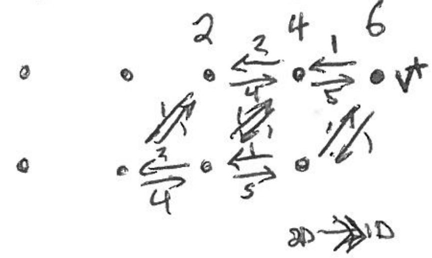
But $\mathrm{Rep}\,\mathfrak{g} \subset \mathcal{O}$ $V_{fd} \otimes W$ is f.g. etc. action of E is $E \otimes 1 + 1 \otimes E$

Lots of fun new examples!!

E = sum of two arrows

Ex 1: $\Delta(5) \otimes L(1)$





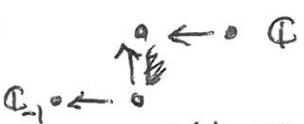
① v^+ = hw vector

$F^{(k)} v^+$ nasty linear combos of top + bottom

but gives copy of $\Delta(6)$

② Some $w^+ \in$ wt 4 with $E w^+ = 0$, gives copy of $\Delta(4)$

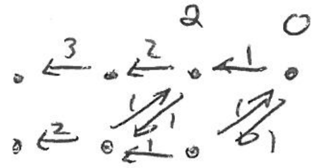
In fact, as $U(\mathfrak{b}^+)$-module,



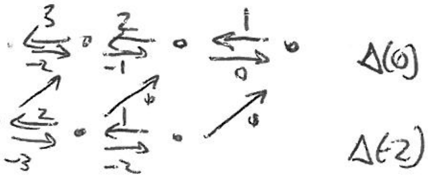
so $0 \to \Delta(6) \to \Delta(5) \otimes L(1) \rightleftarrows \Delta(4) \to 0$ Induction rules.

Actually, it splits! $\Delta(4)$ not inside $\Delta(6)$. $\neq 0$

Ex 2: $\Delta(0) \otimes L(1)$



change basis



w^+ still exists, but it lives inside $\Delta(0)$!!

So $0 \to \Delta(0) \to \Delta(-1) \otimes L(0) \xrightarrow{} \Delta(-2) \to 0$ NOT SPLIT.

$\overset{\leftarrow}{w_+}$ $\circlearrowleft \cdot 0$

~~great way to make new extensions of objects to play with.~~

What's going on.

$\exists$ better reason for sequence (1) to split.

$Z = Z(U(\mathfrak{g})) \subset U(\mathfrak{g})$ commute w/ H so acts on wt space by gen evalue

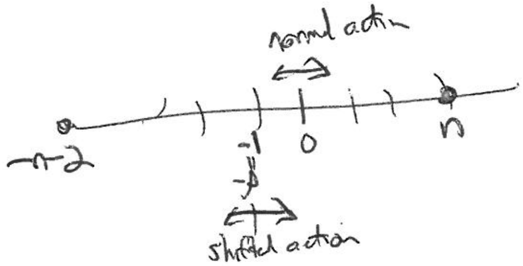
$Z \circlearrowright \Delta(\lambda)$ $\quad Z \subset v_\lambda$ by ~~evalue~~ scalar

Commutes w/ ~~F~~ so $Z \circlearrowright \Delta(\lambda)$ by scalar. $\lambda(\lambda+2)$

Since $\Delta(-n-2) \subset \Delta(n)$, scalar must be the same.

HC Isom: $Z \cong \mathrm{Sym}[\mathfrak{h}^*]^{W_\bullet}$ invariants under $W_\bullet$ action

$w \cdot \lambda = w(\lambda+\rho) - \rho$ ~~reflects the~~ shift origin to $-\rho$.



h.w. w/ same generalized Z-evalue $\Rightarrow$ must be in same $W_\bullet$ orbit.

M, N Different Z-evalue $\Rightarrow$ no morphisms, no extensions, etc.

Different "blocks"

$\Delta(6)$, $\Delta(4)$ have no extensions... different blocks!

Rmk: Z does not act by scalars, but gen. evalues. Scalar on $\Delta(\lambda)$

but gen'l on $\Delta(0) \otimes L(1)$

There is an alternate thing, $\mathcal{O}'$, where Z acts by scalars, but H acts by gen'l wts!!

So $\mathcal{O} \cong \bigoplus_{[\lambda] \in \mathfrak{h}^*/W_\bullet} \mathcal{O}_{[\lambda]}$ blocks. $\quad \mathcal{O}_0$ = block containing $\Delta(0)$.

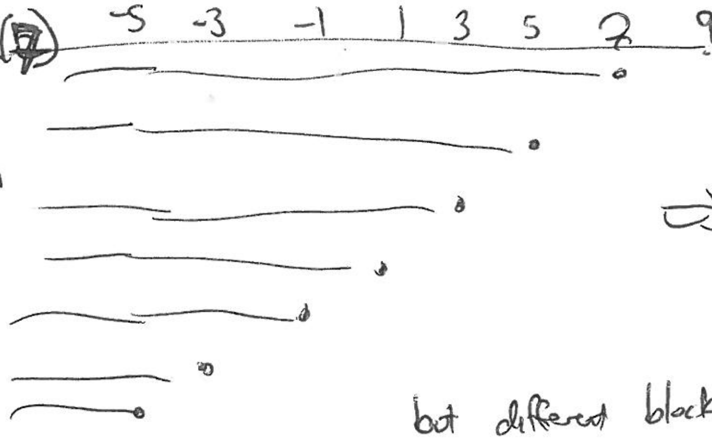
most complicated block!

Rmk: Actually, blocks are even smaller. Restrict to λ integral.

If λ, μ do not differ by a root, no possible maps b/w $\Delta(\lambda)$ $\Delta(\mu)$

$-3.1 \quad 1.1$ differ by 4.2, two separate blocks

Ex: $\Delta(2)\otimes L(7)$



$\Delta(9)$ sub
$\Delta(7)$ next
$\Delta(5)$ next
$\vdots$
$\Delta(-3)$ quotient

$L(5)$ has filt as $U(\mathfrak{b}^+)$-mod
$\mathbb{C}_7$ sub
$\mathbb{C}_5$ next
$\vdots$
$\mathbb{C}_{-5}$
$\mathbb{C}_{-7}$ quot

$\implies$ has filt

but different blocks must split.

$$\cong \Delta(9)\oplus\Delta(7)\oplus\Delta(5)\oplus\begin{pmatrix}\Delta(3)\\ \Delta(-5)\end{pmatrix}\oplus\begin{pmatrix}\Delta(1)\\ \Delta(-3)\end{pmatrix}\oplus\Delta(-1)$$

not split, similar to $\begin{pmatrix}\Delta(0)\\ \Delta(-2)\end{pmatrix}$

Prop: λ maxl in $W\cdot$ orbit $\implies$ $\Delta(\lambda)$ is projective in $\mathcal{O}$.

Pf: $\mathrm{Hom}(\Delta(\lambda), M)$ = space of hw vectors in M of wt λ (where does v^+ go?) $= \begin{cases} 0 & \text{if } M \text{ in wrong block} \\ \dim M_\lambda & \text{if } M \text{ in right block, b/c } \forall v,\ Ev=0 \text{ since that block has NOTHING of higher weight!!}\end{cases}$

assume in single block

Now $\dim M_\lambda$ adds in s.e.s., so $\mathrm{Hom}(\Delta(\lambda),\cdot)$ exact.

Rmk: $\mathcal{O}(\lambda)$ NOT exact in $U(\mathfrak{g})$-reps, we made self-extension X earlier.

Ex: $\Delta(-1)$ is both proj + simple.

$$\mathcal{O}_{[-1]} \cong \mathrm{Vect} = \left\{\bigoplus \Delta(-1)\right\}$$

Thm: $[\lambda]$ integral $\mathcal{O}_{[\lambda]}$ has simples / vermas / projectives classified by $w\cdot\lambda$ i.e. by $W/\mathrm{Stab}\,\lambda$

$$P(w\cdot\lambda)\twoheadrightarrow\Delta(w\cdot\lambda)\twoheadrightarrow L(w\cdot\lambda)$$

When λ dominant, write $P_w = P(w\cdot\lambda)$

BREAK

How to construct projectives? Using our functor $\otimes V_{\text{f.d.}}$

Def: A <u>projective functor</u> is a functor $\mathcal{O}_{[\lambda]} \to \mathcal{O}_{[\mu]}$ given by $M \mapsto p_\mu(M \otimes V)$ or a summand thereof.

Prop: Projective functors are exact, and preserve projectives + injectives.

Pf: Actually, they have biadjoints! $\otimes V$, $\otimes V^*$ biadjoint

$$\mathcal{O}_{[\lambda]} \underset{\otimes V}{\overset{\otimes V^*}{\rightleftarrows}} \mathcal{O}_{[\mu]}.$$

Def: A <u>translation functor</u> $\mathcal{O}_{[\lambda]} \xrightarrow{T_\lambda^\mu} \mathcal{O}_{[\mu]}$ when $\mu-\lambda$ integral, dominant is $p_\mu(\otimes V_{\mu-\lambda})$ the highest weight is just enough to get you there.

If antidominant, use $V^*_{\lambda-\mu}$ (lowest wt)

Ex:

$$T_2^5\, \Delta(2) = p_5(\Delta(2) \otimes V_3) = p_5\left(\Delta\begin{pmatrix}5\\3\\1\\-1\end{pmatrix}\right) = \Delta(5)$$

biadjoint

$$T_5^2\, \Delta(5) = p_2(\Delta(5) \otimes V_3) = p_2\left(\Delta\begin{pmatrix}8\\6\\4\\2\end{pmatrix}\right) = \Delta(2)$$

$$T_0^{-1}\, \Delta(0) = p_{-1}\left(\Delta\begin{pmatrix}2\\0\end{pmatrix}\right) = \Delta(0)$$

biadjoint

$$T_{-1}^{0}\, \Delta(-1) = p_0\left(\Delta\begin{pmatrix}0\\-2\end{pmatrix}\right) = \left|\begin{matrix}\Delta(0)\\ \Delta(-2)\end{matrix}\right.$$

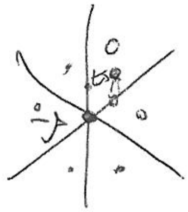
weird nonsplit guy.

Think: R-modules Res R^S-module Ind

Thm: λ, μ dominant integral regular then T_λ^μ is an equivalence. WLOG work w/ $\mathcal{O}_0$.

Can get interesting functors on $\mathcal{O}_0$ by translating to a nonregular element and back!

$$\Theta_s = T_\alpha^0 T_0^\alpha \quad \text{for } \alpha \text{ on the } s\text{-wall (irrelevant which)}$$



preserves projectives, acts on Groth gp as discussed.

Ex! $\mathcal{O}_0$ for $\mathfrak{sl}_2$ — simples, stds, projectives

injectives = $\nabla(0)$ costd, $\ker \nabla(0) \to \nabla(-2)$ ... $L(0) = \Delta(0)/\Delta(-2)$ $\quad \Delta(0) = \Delta(0)$ $\quad$ Koszul duality.

$\nabla(-2) = L(-2) = \Delta(-2) \quad P(-2) = \mathrm{Ext}(\Delta(0), \Delta(0))$

Soergel Theory

Thm (Endomorphismensatz): $\mathrm{End}(P_{w_0}) = C = R/R_+^W$.

$\quad H^*(Fl)$ — secretly graded

Def: The Soergel functor $\mathbb{V} = \mathrm{Hom}(P_{w_0}, \bullet)$ exact

a very nasty functor!

$\mathcal{O}_0 \longrightarrow \mathrm{End}(P_{w_0})^{op}\text{-mod} = C\text{-mod}$

$\mathbb{V}(L_{w_0}) = \mathbb{C}$

$\mathbb{V}(L_x) = 0 \quad x \neq w_0$

$\mathbb{V}(\Delta(w)) = \mathbb{C} \quad \forall w.$

$\mathbb{V}(\nabla(w)) = \mathbb{C}$

Thm: (Struktursatz) $\mathbb{V}$ is fully faithful on projectives

(i.e. every projective has some L_{w_0} inside it, and all morphism detected by this part)

$\Rightarrow \mathrm{Proj}\, \mathcal{O}_0 \xrightarrow{\sim}$ image in C-mod $\equiv$ Soergel Modules

what are they?

$\mathbb{V}(P_1) = \mathbb{C} = C/C_+$.

Prop: $\mathbb{V}(\Theta_s M) = C \otimes_{C^s} \mathbb{V}(M) \implies \mathbb{V}(BSProj(\underline{w})) = C \otimes_{C^s} \cdots \otimes C/C_+$

$\boxed{= R \otimes_{R^s} \cdots \otimes_{R^{s'}} R/R_+}$

b/c all R_+^W dies !!!

All proj are summands of BS Proj

$\Rightarrow \mathbb{V}$ is summand of $\mathbb{V}(BSProj)$.

Thm: $\mathbb{V}(P_w) = \overline{B_w} = B_w \otimes_R \mathbb{C} = B_w / R_+$ — choose right side arbitrarily

The bimodules themselves are some kind of "equivariant lift" coming from "deformed category $\mathcal{O}$"

Also, $\mathrm{Hom}_{SBim}(B, B') \otimes_R \mathbb{R}$

$\downarrow$

$\mathrm{Hom}_{SMod}(\overline{B}, \overline{B'})$

How to deal with non-projectives in $\mathcal{O}_0$?

Take a proj resolution and then apply $\mathbb{V}$!

Get a complex of SMod $\rightsquigarrow$ lifts to complex of SBim!

Which complexes of SBim come from $\mathcal{O}_0$? Thursday!