Ben Elias — Oregon 8/2014 — Lectures 2,3 — ①

§1 (Rex) Path Morphisms

Recall from Alex's lecture: From a path in the expression graph of $w \in W$, one could construct a morphism in $\mathcal{S}\mathcal{L}_W$:

Every morphism in $\mathcal{S}\mathcal{L}_W$ is an isomorphism. So $\underline{w}' \rightleftarrows \underline{w}$ is identity. ($m=3$)

Relations $\Leftrightarrow$ loops in expression graph

$\Rightarrow$ any two path morphisms $\underline{w} \rightsquigarrow \underline{w}'$ are equal.

st
$\uparrow$
stts
$\uparrow$
tsts
$\uparrow$
ttst
$\uparrow$
st

Std diagrams!

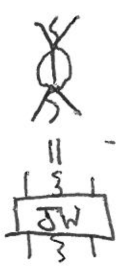
Now consider $\mathbb{BSB}$im instead. Could do the same thing using Soergel diagrams. Issues -

① $\cup$ is nowhere close to being an isomorphism! We shouldn't use it. Stick to reduced expressions.

Exercise: $\boxed{\bigcirc} = 0$ ⇒ diagram above $= 0$.

Def: Given a path in the rex graph for $w \in W$, the corresponding morphism in $\mathbb{BSB}$im is called the <u>path morphism</u>. Or <u>rex move</u>.

②  is also not an isomorphism! Projection to a common summand.

$\underline{w}' \rightleftarrows \underline{w}$  is an idempotent, but not the identity.

So two path morphisms $\underline{w} \rightsquigarrow \underline{w}'$ typically will NOT be equal.

NONETHELESS...

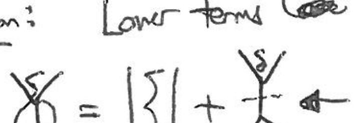
FACT: Two path morphisms $\underline{w} \rightsquigarrow \underline{w}'$ will be equal MODULO ~~LOWER~~ TERMS. ← How to think.

(a) <u>Localization</u>: Any $BS(\underline{w}) \overset{1}{\underset{\oplus}{\supset}} Q_w$ after localization. 

$$\begin{array}{ccc} BS(\underline{w}') & \supset & Q_w \\ \uparrow & & \uparrow 1 \\ BS(\underline{w}) & \supset & Q_w \end{array}$$

Def: Lower term $\Leftrightarrow$ zero on Q_w after localization.

(b) <u>Factorization</u>: Lower terms (are linear combos of things which) factor through shorter expressions.

Ex: 

Fact ⇒ Loc since shorter exp. have no Q_w summand.

(c) <u>1-tensor</u>: Lower terms ~~are~~ maps which kill $c_{bot} = 1 \otimes 1 \otimes \ldots \otimes 1 \in BS(\underline{w})$

Define LOWER TERMS, p3

Ex: Above. Exercise: All other terms in JW kill c_{bot}

Fact ⇏ 1-tensor. But 1-tensor obvious, proof good enough

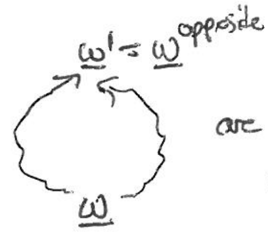
Special -ish (a)

★ will be MLT. (b)

Special (c) Do me first!

Why is Fact true? JW stuff $\Rightarrow$ equal mult. for Π loops. But what about Zams?
This requires a new relation!

So for each $I \underset{3}{\overset{f}{\subset}} S$, we will have a relation saying



are equal up to some precise lower terms

This will imply Fact. Which we want desperately!!

Examples: $A_1 \times A_2$ A_3

where are these lower terms??

Miracle (unexplained): In types $A_1 \times I_2(m)$, A_3, B_3 there is a choice of $\underline{w}$ such that the two paths agree on the nose! (Not all choices of $\underline{w}$ work!!!)

Possible explanation in type A: higher Bruhat order. But why?

In type H_3, lower terms are necessary (and also uncomputed...)

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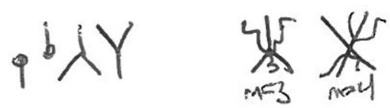
§2) Gens + Relations Redux:

Thm (E-W): Let $\mathcal{D}$ ~~...~~ be monoidal cat w/ presentation

Ob: $I \underset{f}{\subset} 1$ $I \underset{f}{\subset} 2$ $I \underset{f}{\subset} 3$
$s = B_s$

Mor:



Relations: 1-color 2-color Zam.

Let $\mathcal{F}: \mathcal{D} \to \mathbb{BS}Bim$ be defined as before. Then $\mathcal{F}$ is an equiv.

So pictoral computation for the win!

But many crazy pictures can be drawn, and it's not always easy to simplify. Want a BASIS for morphism spaces.

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§3) Libedinsky's Light Leaves

Recall $\underline{H}(\underline{w}) = \sum_{\underline{e} \subset \underline{w}} v^{def(\underline{e})} H_{\underline{w}^{\underline{e}}}$

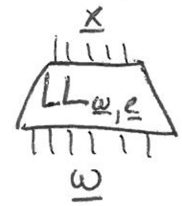
$$\Rightarrow (\underline{H}(\underline{w}), \underline{H}(\underline{y})) = \sum_{x} \sum_{\substack{\underline{e} \subset \underline{w} \\ \underline{f} \subset \underline{y}}} v^{def(\underline{e}) + def(\underline{f})} \quad \text{s.t. } \underline{w}^{\underline{e}} = \underline{y}^{\underline{f}} = x \quad = \text{grd rank } \mathrm{Hom}^{\bullet}(BS(\underline{w}), BS(\underline{y}))$$

So basis should be parametrized by triples $(\underline{e}, \underline{f}, x)$ where

$\underline{e} \subset \underline{w}$ $\quad \underline{f} \subset \underline{y}$ $\quad \underline{w}^{\underline{e}} = \underline{y}^{\underline{f}} = x \in W.$

Actually, the basis splits into two "halves", $(\underline{e}, x)$ and $(\underline{f}, x)$. Cellular theory, next lecture.

Construction: $(\underline{w}, \underline{e}, x)$ Fix arbitrary rex $\underline{x}$ for x. We build



inductively.

Follow the bruhat stroll.

Example

$\underline{w} =$	sts	sts	sts
$\underline{e} =$	001	100	101
	UUU	UUD	UUD
	+1 +1 0	0 +1 −1	0 1 0

ok so far

$m_{st} = 3$

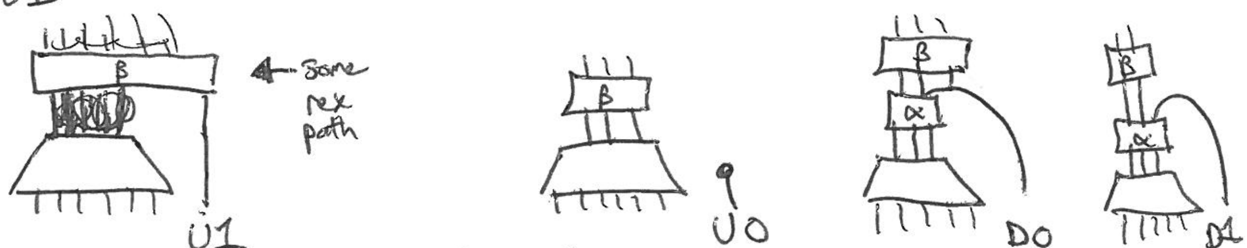
$\underline{w} =$ stst $\qquad$ $\underline{w} =$ ststs

$\underline{e} =$ 1111 $\qquad$ $\underline{e} =$ 11100

UUUD $\qquad$ UUUDD

can get nasty. Which rex path do you choose?? It matters!! No canonical choice.

Inductive Formula:



Procedure is well defined up to various choices of rex moves. (and choice of rex for $\underline{x}$)

Just choose arbitrarily. But in practice, we call anything constructible in this way by the name "light leaf".

Thm (E-W Lib) / Def: For $(\underline{w}, \underline{e}, x)$, $(\underline{y}, \underline{f}, x)$ let $\mathbb{LL}_{\underline{e},\underline{f}}$ denote

double leaf

Then $\{\mathbb{LL}_{\underline{e},\underline{f}}\}$ forms a basis for $\mathrm{Hom}^{\bullet}(BS(\underline{w}), BS(\underline{y}))$ as a right R-module

Consequence: Let I be an ideal in W for Bruhat order. I.e. $I = \leq w$ or $< w$.

$\mathcal{D}_I \equiv$ morphisms factoring through a rex for $v \in I$. $= \mathrm{span}\{\mathbb{LL}_{\underline{e},\underline{f}}\}_{\underline{w}^{\underline{e}} \in I}$ ① Thm.

This is an ideal in $\mathcal{D}$.

When w is understood, $\mathcal{D}_{<w}$ is lower terms. ② Choice of β, β' irrelevant mod $\mathcal{D}_{<x}$, so $\mathbb{LL}$ is canonical modulo $\mathcal{D}_{<x}$