The Classical Approach to Soergel Bimodules

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Setup: (W, S) Coxeter system

$\mathfrak{h}$ (geometric) realization of (W, S)

$R = S(\mathfrak{h}^*)$ = polynomial functions on $\mathfrak{h}$

$W \circlearrowright R$ & R^s = s-invt polys

Consider all of R-Bim for the moment.

graded R-bimodules

Standard bimodules:

Let $x \in W$.

Def: Define $R_x \in R\text{-Bim}$ by

1. $R_x = R$ as a left R-module
2. As a right R-module, $\underset{R_x}{\underset{\uparrow}{m}} \cdot \underset{R}{\underset{\uparrow}{r}} = x(r)m$.

Visualization:

Recall B_s can be viewed as

"border patrol"

B_s

semi-porous wall where s-invt polys can slide through

For R_x, we have

"puberty"

R_x

$f = x(f)$

completely-porous wall where anything can slide through but act by x

Facts:

① $R_x \underset{R}{\otimes} R_y \cong R_{xy}$

② $\mathrm{Hom}(R_x, R_y) = \begin{cases} R & \text{if } x=y \\ 0 & \text{else} \end{cases}$ since $\mathfrak{h}$ is a faithful rep

Define StdBim = full $(\underbrace{\oplus}_{\text{additive}}, \underbrace{(n)}_{\text{graded}}, \underbrace{\otimes}_{\text{tensor}})$ subcat of R-Bim gen by $\{R_x\}_{x\in W}$.

① & ② $\Rightarrow$ StdBim is a realization of the additive 2-groupoid of W over R.

- Tensor is multiplication, but
- Now we have more than just id maps, have mult by elts of R (polys in boxes)

with $\underset{s}{|}\boxed{f} = \boxed{sf}\underset{s}{|}$ $\forall s \in S$.

Filtrations:

Consider the SESs

$$(\Delta) \quad R_s(-1) \hookrightarrow B_s \twoheadrightarrow R_{id}(1)$$

$1 \longmapsto d_s := \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s)$

$f \otimes g \longmapsto fg$

$$(\nabla) \quad R_{id}(-1) \hookrightarrow B_s \twoheadrightarrow R_s(1)$$

$1 \longmapsto c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$

$f \otimes g \longmapsto f s(g)$

In alternate notation: $f \otimes g \longmapsto f \otimes g$ ← here $\otimes$ is the wall | from before

Defining props of c_s & d_s

$f c_s = c_s f$

$f d_s = d_s\, s(f)$ $\forall f \in R$

So, B_s is filtered by both R_{id} and R_s, but with no particular order (& grading shifts depend on this order).

Soon, we will fix this by specifying an order.

Def: A bimodule $M \in R\text{-Bim}$ is said to have property $(*)$ if it has a finite filtration with subquotients $\{\bigoplus R_x(?)\}$.

Facts:

① Having $(*)$ is closed under $\otimes$.

Suppose $0 \subseteq B^1 \subseteq \dots \subseteq B^m = B$ w/ subquots $\{\bigoplus R_x(?)\}$
& $0 \subseteq C^1 \subseteq \dots \subseteq C^n = C$

If more explanation is needed.

⌜Lemma: If M can be filtered so that subquots have $(*)$, then M has $(*)$.⌟

$\forall i,j$, $B^{i+1}/B^i \otimes C^{j+1}/C_j$ is a std bimod,

so $B \otimes C^{j+1}/C^j$ has $(*)$.

By Lemma, $B \otimes C$ has $(*)$.

② Having $(*)$ is closed under direct summands.

We are working in a Krull-Schmidt cat.

A filt of $B \oplus C$ by indec subquots (R_x is indec) will give a filt of B & C separately.

Observations:

i) Bott-Samelsons have (*) by SESs & ①

ii) Summands of Bott-Samelsons (& therefore Soergel bimods) have (*) by ②

iii) R_x are not Soergel bimods.
They can appear as submods & quots of Bott-Samelsons (as in SESs from before), but not as summands.
(Except $R = R_{id} = B_{id}$).

Time to pin down this order/grading shift issue!

<u>Δ- & ∇-filtrations:</u>

<u>Def</u>: A Δ-filtration (or standard) on $B \in \mathbb{S}Bim$ is a finite filtration by R-Bim $0 \subset B^m \subset \ldots \subset B^1 \subset B^0 = B$ such that
$$B^i / B^{i+1} \cong \bigoplus R_x(v) \quad \text{with } \ell(x) = i \ \& \ v \in \mathbb{Z}.$$

Analogously,

<u>Def</u>: A ∇-filtration (or costandard) on $B \in \mathbb{S}Bim$ is a finite filtration by R-Bim $0 \subset B^0 \subset \ldots \subset B^m = B$ such that
$$B^i / B^{i-1} \cong \bigoplus R_x(v) \text{ with } \ell(x) = i \ \& \ v \in \mathbb{Z}.$$

<u>NOTE</u>: The SES (Δ) gives a Δ-filtration on B_s & (∇) gives a ∇-filtration on B_s.

Thm (Soergel 2006)

Any $B \in \mathbb{S}Bim$ has a unique Δ/∇-filt.

Pf (Sketchy Sketch):

Having (Δ) is preserved under taking summands (same reason as for $(*)$).

Having (Δ) is closed under $\otimes$ with B_s.

↳ Gives (Δ) for all Bott-Samelsons. □

The subquots do not depend on the choice of refinement, but order & grading shifts are very different!

Remark: Support filtrations give an explicit construction of Δ/∇-filts!

One can define two notions of character.

We will use ch_Δ.

$$ch_\Delta : \mathbb{S}Bim \longrightarrow \mathbb{H}$$

$$ch_\Delta(B) = \sum_{x \in W} v^{\ell(x) + \text{shift on } R_x} H_x$$

E.g.: From SES, $ch_\Delta(B_s) = vH_{id} + H_s = \underline{H}_s$.

Remark: ① This gives an inverse to the isomorphism

$$\varepsilon : \mathbb{H} \xrightarrow{\sim} [\mathbb{S}Bim]$$

$$\underline{H}_s \longmapsto [B_s]$$

② In the course of the proof of the theorem above, Soergel showed

$$ch_\Delta(M \otimes B_s) = ch_\Delta(M) \cdot \underline{H}_s$$

Recall: Soergel conj: $\forall x \in W$, $ch_\Delta(B_x) = \underline{H}_x$.

Localization:

Soergel bimodules become much simpler after localization, (become "like" standards).

Let Q be the homogeneous fraction field of R.

Even though Q is graded, $Q \cong Q(2)$ in this case.

Def: The "localization" of $B \in R\text{-Bim}$ is $B \otimes_R Q$.

NOTATION: $B_s^Q := B_s \otimes_R Q$, $BS^Q(\underline{w}) := BS(\underline{w}) \otimes_R Q$,

$Q_s := R_s \otimes_R Q$.

NOTE THAT: $BS^Q(\underline{w}) \cong Q \otimes_{Q^{s_1}} \cdots \otimes_{Q^{s_n}} Q(n)$ if $\underline{w} = s_1 \cdots s_n$

Observation:

Suppose M is free as an R-module.

Then, it includes into its own localization (injection since M is free).

But $\mathrm{Hom}(B_x, B_y)$ is free as a right R-module,

so get injection:

$$\mathrm{Hom}(B_x, B_y) \hookrightarrow \mathrm{Hom}(B_x, B_y) \otimes_R Q.$$

Localization is a faithful functor!

Back to SESs:

After localization, these SESs split each other:

$$R_s(-1) \longrightarrow B_s \longrightarrow R_s(1)$$
$$1 \longmapsto d_s = \tfrac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s) \longmapsto \alpha_s$$

&

$$R_{id}(-1) \longrightarrow B_s \longrightarrow R_{id}(1)$$
$$1 \longmapsto c_s = \tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \longmapsto \alpha_s$$

But α_s is invertible now! So,

$$B_s^Q \cong Q(1) \oplus Q_s(1).$$

By expansion:

$$BS^{Q}(\underline{w}) \cong \bigoplus_{\underline{e} \subset \underline{w}} Q_{\underline{w}^{\underline{e}}}(\ell(w))$$

This is key! Bott-Samelsons are easy after localization.

Why is localization useful?

① Proposition: For a rex $\underline{w}$, $\exists!$ indec summand B_w in $BS(\underline{w})$ which contains Q_w after localization, and B_w is not a summand of any shorter sequence.

Pf: Look at the decomposition of $BS^{Q}(\underline{w})$ above. Since Q_w is indec and appears exactly once, there is certainly a unique summand which contains it. Again by looking above, it is clear that Q_w does not appear in the localization of any shorter sequence. □

NOTE: This does not prove that all the other summands in $BS(\underline{w})$ do appear in some shorter sequence.

② Localization is a faithful functor, so can check if two morphisms of Soergel bimods are equal by checking after localization.

- Geordie writes computer programs exploiting this fact.

(3) There are nice diagrammatics for playing with localization!