BEN ELIAS

ORGEON 8/2014 LECTURE 6 ①

Recall: $\mathbb{BSB}im \equiv$ full monoidal subcat of (graded) R-bimodules w/ objects $BS(\underline{w})$)

$\mathrm{Hom}^\circ =$ all graded maps.

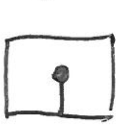
This is a monoidal cat (w/ biadjunction) so should draw morphisms as planar diagrams (modulo isotopy)

Warning: Before we visualized elements $f \mid g \} h$ now we draw morphisms, connection soon but don't get confused.

1] Type A_1 $S = \{s\}$ Ob: B_s ~~R s R s R ○ ○ ○~~ all labeled s, all regions labeled R, omit

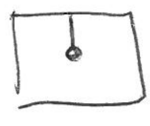
B_s is a frob alg in $\mathbb{BSB}im$, so have 4 structure maps.

$\mu: B_s \to R$, $f \otimes g \mapsto fg$



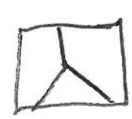
degree $+1$

$\Delta: R \to B_s$, $1 \mapsto c_s = \frac{\alpha_s}{2} \otimes 1 + 1 \otimes \frac{\alpha_s}{2}$



$+1$

$B_s B_s \to B_s$, $1 \otimes f \otimes 1 \mapsto \partial_s(f) \otimes 1$



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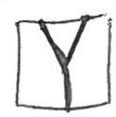
$B_s \to B_s B_s$, $1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1$



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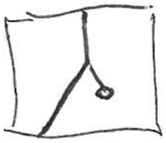
EXAMPLES: ⓐ



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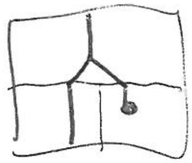
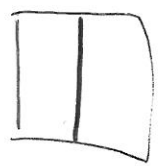


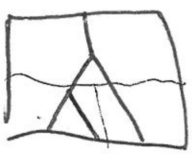
$1 \otimes 1 \mapsto 1 \otimes \frac{\alpha_s}{2} \otimes 1 + 1 \otimes 1 \otimes \frac{\alpha_s}{2} \mapsto 1 \otimes 1$ and 0



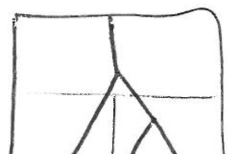
$1 \otimes 1 \mapsto 1 \otimes 1$

so

ⓑ



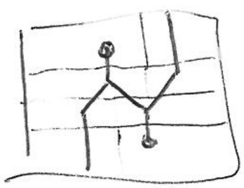
$1 \otimes f \otimes g \otimes 1 \mapsto \partial_s(f) \otimes g \otimes 1 \mapsto \partial_s(f)\partial_s(g) \otimes 1$



$1 \otimes f \otimes g \otimes 1 \mapsto 1 \otimes \partial_s(g) f \otimes 1 \mapsto \partial_s(\partial_s(g) f) \otimes 1$

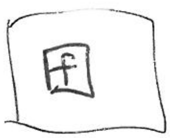
so

ⓒ

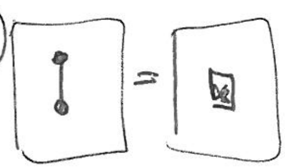


exercise

Another generator: $R \to R$, $1 \mapsto f$



Ex: ⓓ



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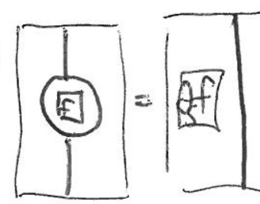
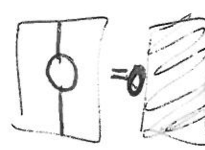


$\Longrightarrow$



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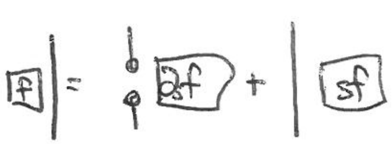
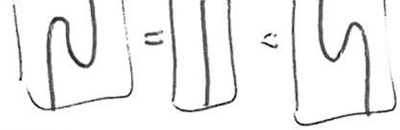


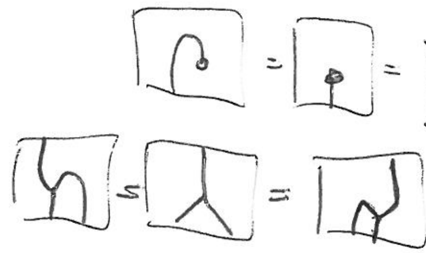
linear combination of diagrams.

Thm: (Elias–Khovanov) Ⓐ  generate all R-bimodule morphisms in $\mathbb{BS}$Bim.

Write 

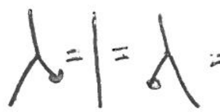
$dg = 0$

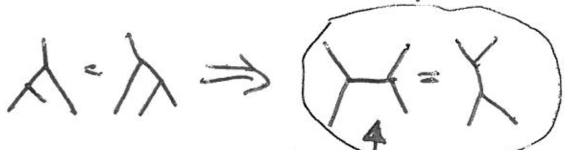
Ⓑ The following relations hold in $\mathbb{BS}$Bim: ⓘ Isotopy 



cyclicity of $\boxed{f}$ plus state next that this means is $\boxed{b}$

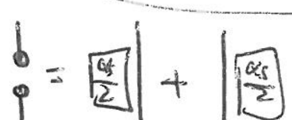
ditto ... we used a symbol w/ 120° symmetry b/c the symmetry exists!

② Unit:  ⇒ Counit 

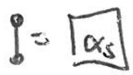
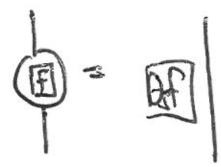
Assoc: 

what is this ??

 or anything else isotopic.

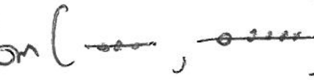
③ Decomp: 

(for more generally,  for dual bases of R over R^J)

④ Eval  

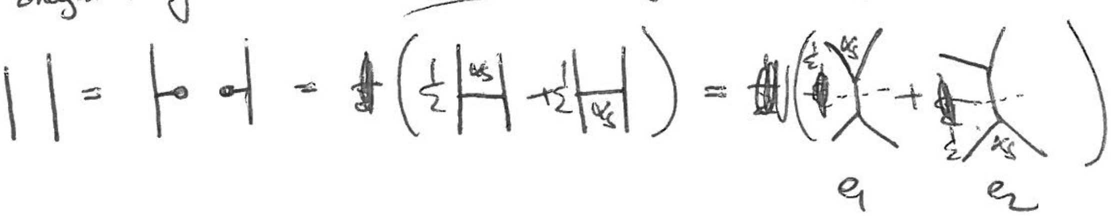
⑤ Polynomial forcing relation

Moreover, these are all the relations. I.e. let $\mathcal{D}$ denote the monoidal cat w/ objects 

③ and $\mathrm{Hom}($  $)$ = linear combos of  / relations

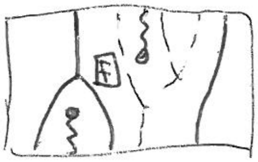
$F : \mathcal{D} \longrightarrow \mathbb{BS}\mathrm{Bim}$ as above. Then F is an equivalence of categories.

Most important feature of $\mathbb{BS}$Bm for groth gp is $B_sB_s \cong B_s(1) \oplus B_s(-1)$. How to show this "diagrammatically" i.e. morphism-theoretically? $\mathrm{id}_{B_sB_s} = e_1 + e_2$, e_i factors thru B_i

 Exercise.

Now practice!

Goal: General case (W,S) Can already construct many morphisms



and the relations Ⓑ still hold. Call these Universal Soergel diagrams/morphisms

Exercise: Any Universal morphism with empty body $(\in \mathrm{End}(R))$ reduces to a polynomial.
$\Longrightarrow$ nothing in negative degree

$\mathrm{Hom}(B_s, R)$ reduces to  etc. etc.

Can Ⓐ be true... are these all morphisms? No.

Exercise: $s \neq t$. The minimal degree of any universal morphism $B_s B_t B_s B_t B_s \to B_t B_s B_t B_s B_t$ is 2.

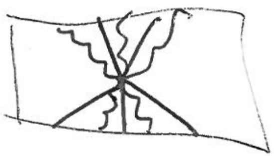
But if $m_{st} \leq 5$, there should be a degree zero map

$$BS(tstst) \leftarrow B_{ststs} \leftarrow BS(ststs), \quad BS(ststs) \to BS(tstst)$$

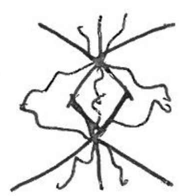
SHF $\Rightarrow$ degree 0 maps are 1D.

Can pin down the precise morphism by asking
$|\otimes|\otimes\_\otimes| \longrightarrow |\otimes|\otimes\_\otimes|$. (works since $B_{ststs} \Rightarrow |\!\!\_\!\_\otimes|$.)

Qn: What is a formula for ~~this~~ map in terms of polys? Ans: Gosh, its awful.

Regardless, we draw this map  $2m_{st}$-valent vertex. Diagrammatic relations will be easy, even tho algebraic formula is not!

Ⓐ I could just write down some relations + ask you to believe me, but actually there is some beautiful deep math at play here + I want you to understand. (Even if you only care about type A.)

Key Relation:  $=$ proj to $B_{ststs} \overset{\oplus}{\subset} BS(ststs) \in$ Universal, what is it?
↑ idempotent
It's a Jones-Wenzl projector.

(Incomplete) Thm (E-W): Ⓐ  ... generate all morphisms. Really, it's dihedral!

Ⓑ What are additional relations?? In order for i

Ⓒ $\mathcal{F}: \mathcal{D} \xrightarrow{\sim} \mathbb{BS}\mathrm{Bim}$.

Next goal: $S = \{s,t\}$. Can already construct many morphisms

Are these all ??

If $m_{st} < \infty$, NO. $\exists$ degree 0 map (1D space of)

$ts\,ts$?? $\underbrace{stst}_{m_{st}}$

If $m_{st} = \infty$, actually YES...

but none are possible using just the stuff above (exercise)

How to approach...

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§2 $\mathfrak{sl}_2$-reps

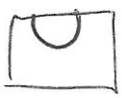
Bat? Switch? You'll find out.

$V = \mathbb{C}^2$ be std rep of $\mathfrak{sl}_2$, basis $\{e_1, e_2\}$

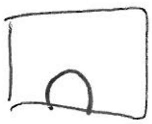
Denote $V^{\otimes n}$ as ——•—•—•—•—• ← Objects of $\mathrm{Fund}\,\mathfrak{sl}_2 \subset \mathrm{Rep}\,\mathfrak{sl}_2$

Now, $V^{\otimes 2} \cong S^2 V \oplus \Lambda^2 V$, $\Lambda^2 V = \mathbb{C} = V^{\otimes 0}$, so $\exists\ V^{\otimes 0} \leftrightarrow V^{\otimes 2}$

Let

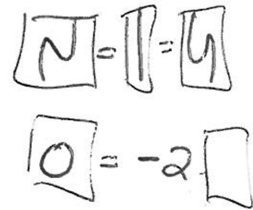


$e_1 \otimes e_2 - e_2 \otimes e_1 \mapsfrom 1$

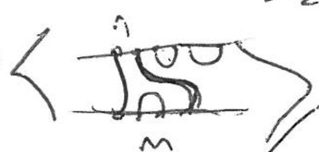


$-1 \mapsfrom e_1 \otimes e_2$, $1 \mapsfrom e_2 \otimes e_1$, $0 \mapsfrom e_1 \otimes e_1,\ e_2 \otimes e_2$

Check:



Def: $\mathcal{TL}$ has objects $n \in \mathbb{N}$; $\mathrm{Hom}(m,n) = $



crossingless matchings (remove closed components), $\bigcirc = 2$

Thm: $\mathcal{TL} \xrightarrow{\sim} \mathrm{Fund}\,\mathfrak{sl}_2$

Rmk: $\exists$ a q-deformation where $\bigcirc = -q - q^{-1} = -[2]$

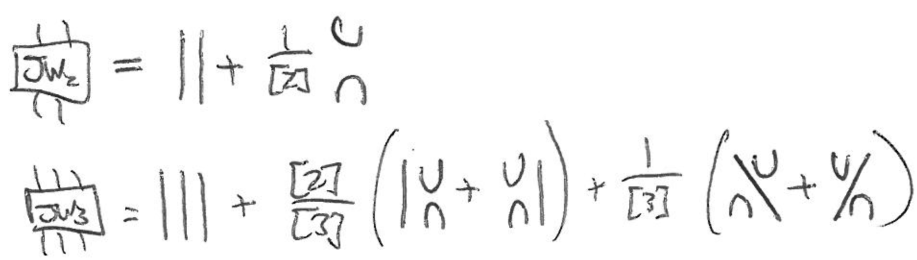
$\mathcal{TL}_q \xrightarrow{\sim} \mathrm{Fund}\,U_q(\mathfrak{sl}_2)$

Main Implication: $V_n \overset{\oplus}{\subset} V^{\otimes n}$ so $\exists$ idemp $V^{\otimes n} \to V_n \to V^{\otimes n}$

$\in TL_n \equiv \mathrm{Hom}_{\mathcal{TL}}(n,n)$

Jones-Wenzl Projector

Ex:



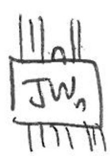
$[3] = [2][2] - 1$

Properties: (Exercises)

① JW_n can be defined when quantum binomials $\begin{bmatrix} n \\ k \end{bmatrix}$ are invertible for $k \le n$ (rep theory of $U_q(\mathfrak{sl}_2)$ at roots of unity)

② Nice recursion formulas.

③ JW_n is ! morphism s.t.



$= 0$ for all caps and coeff of $||||||$ is 1.

Can use recursion formulas to give diag/morph. theoretic proof that

$$V\otimes V_n \cong V_{n+1}\oplus V_{n-1} \text{ for } n\geq 1 \qquad V\otimes V_0 = V_1$$

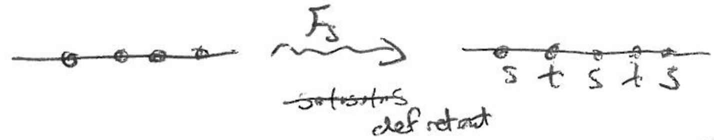
$$\big([2][n+1] = [n+2]+[n] \qquad [2][1]=[2]\big)$$

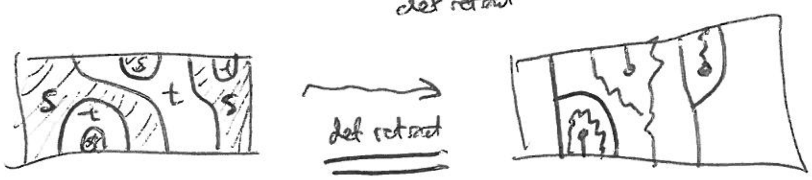
$$\big(\underline{H}_s\,\underline{H}_{tsts} = \underline{H}_{ststs} + \underline{H}_{sts} \qquad \underline{H}_s\,\underline{H}_t = \underline{H}_{st}\big)$$

BREAK

§3 Triumphant Return to dihedral case

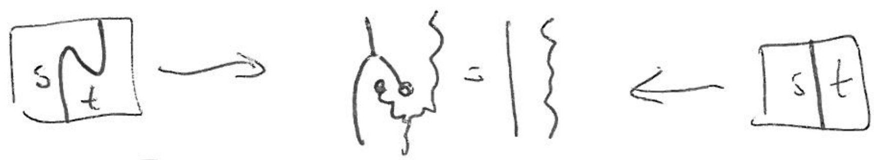
$S=\{s,t\}$ $\mathcal{TL} \underset{F_t}{\overset{F_s}{\rightrightarrows}} \mathbb{BS}Bim$ (not monoidal)

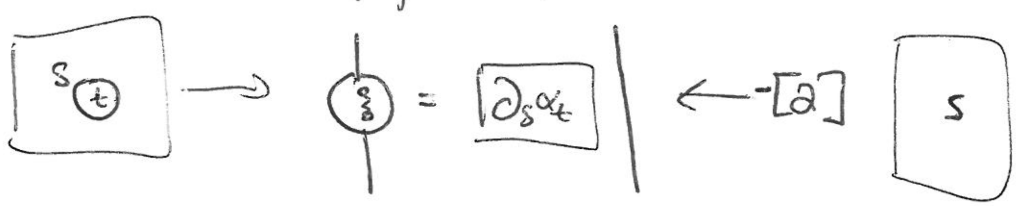




this is degree 0.

Well defined?





well defined when $a_{s,t} = -(q+q^{-1})$ $\partial_s\alpha_t = a_{s,t}$

but $a_{s,t} = -2\cos\frac{\pi}{m_{st}} = -(\zeta_{2m}+\zeta_{2m}^{-1})$ so works when q is the correct root of unity!!

$\Rightarrow$ $[m_{st}]=0$, $[m_{st}-1]=1$, $\begin{bmatrix} m_{st}-1 \\ k\end{bmatrix}$ invertible

Pause. Rehash. Fix dihedral gp, fix m. Set $q=\zeta_{2m}$. Then $\mathcal{TL}_q \underset{F_t}{\overset{F_s}{\rightrightarrows}} \mathbb{BS}Bim^\circ$

when $m=\infty$, $q=1$. $\mathcal{TL} \longrightarrow \mathbb{BS}Bim^\circ$.
$\mathcal{TL} \cong$ Fund $\mathfrak{sl}_2$

Rmk! Weird + unnatural? Actually, comes from a 2-functor which is very natural!

This is (quantum) geometric Satake (at a root of unity!)   exercises.

Patience until later in the week.

Consequence: $JW_n \rightsquigarrow$ some idempotent, projects $BS(\underline{w}) \rightleftarrows B_w$.

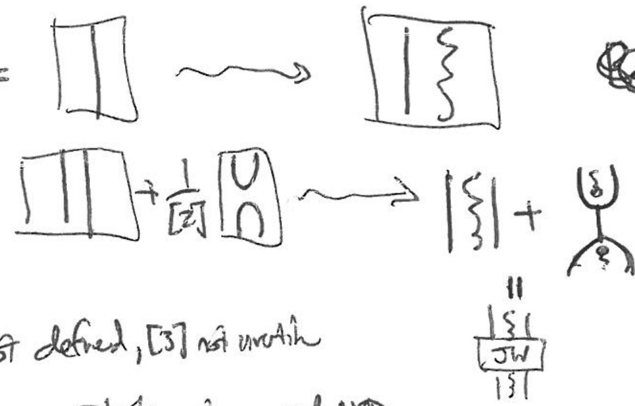
Ex: Type A_2

$m=3$

$[2] = \zeta_6 + \zeta_6^{-1} = 1$

$a_{s,t} = -1$

$[3] = 0$

$JW_1 =$  $B_{st} = \mathrm{Im}\ id_{B_sB_t} = B_sB_t$

$JW_2 =$ ... $+ \frac{1}{[2]}$ ... $B_{sts} = \mathrm{Im}\ JW$

JW_3 is not defined, $[3]$ not invertible

but $stst$ not a red exp.

Spose we define $B_{st(n+1)} = \mathrm{Im}\ JW_n \overset{\oplus}{\subset} BS(\underline{st\ldots})$

$V\otimes V_n \cong V_{n+1} \oplus V_{n-1} \Rightarrow B_t B_{st(n+1)} = B_{ts(n+2)} \oplus B_{ts(n)}$

categorifies $\underline{H}_t\, \underline{H}_{st(n+1)} = \underline{H}_{ts(n+2)} + \underline{H}_{ts(n)}$

$\Rightarrow [B_w] = \underline{H}_w \underset{SHF}{\Longrightarrow} \mathrm{End}(B_w) = id +$ higher degree

$\Longrightarrow B_w$ is indecomposable $\Rightarrow B_w$ is the indec. Soergel bim as desired.

Diagrams a huge help ... try to come up with a formula for JW_n in terms of what it does to $f_1 \otimes f_2 \otimes \ldots \otimes f_{n+2}$ ... good luck! and fast change...

One thing is missing. $F_s(JW_{m-1})$ gives $B_{sts\ldots} \overset{\oplus}{\subset} BS(sts\ldots)$ but $B_{sts\ldots} \cong B_{tst\ldots}$

$F_t(JW_{m-1})$ —"— $B_{tst\ldots} \overset{\oplus}{\subset} BS(tst\ldots)$

as noted, this isomorphism can NOT come from $\mathcal{TL}$ (neither functor, of course, but also) need new morphism

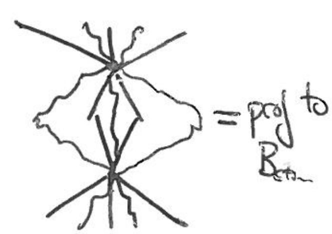
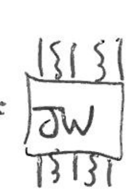
$$B_tB_sB_t\ldots \uparrow B_{sts\ldots} \uparrow B_sB_tB_s\ldots \qquad ???? \uparrow \text{ on polynomials}$$

for simplest formulas see my thesis.

$m=5$

2m-valent vertex

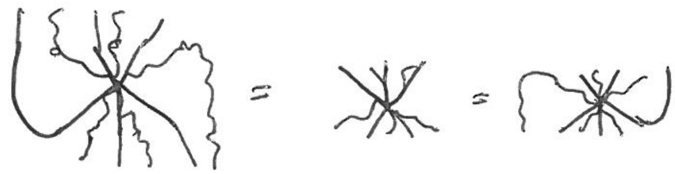
degree 0.

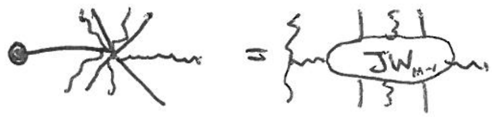
so  = proj to $B_{sts\ldots}$ = 

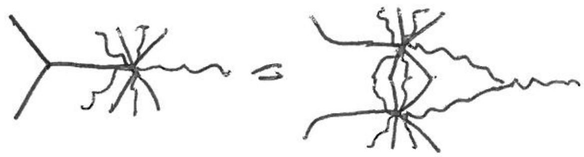
Type $I_2(m)$

Thm (Elias): (A)  generate all morphisms.

(B) Relations:

Isotopy 

Dot:  ← degree 2 version, another way to def retract

Tri: 

(C) These are all the relations, i.e. $F: \mathcal{D} \xrightarrow{\sim} \mathrm{BSBim}$.