Supplementary Notes to Lecture 1.5

1) Usual defn of a strict action: (simplified slightly)

Data - $g \mapsto F_g$ $\forall g \in G$, $e \mapsto \mathbb{1}$.

$\alpha_{g,h} : F_g \circ F_h \xrightarrow{\sim} F_{gh}$; $\alpha_{g,h}^{-1}$ $\forall g,h \in G$ } s.t.

$$\begin{tikzcd}[column sep=small] & F_{gh}\circ F_k \arrow[dr] \arrow[dd, phantom, "\#"] & \\ F_g\circ F_h\circ F_k \arrow[ur] \arrow[dr] & & F_{ghk} \\ & F_g\circ F_{hk} \arrow[ur] & \end{tikzcd}$$

Associativity axiom

Claim: Ass axiom $\Rightarrow$ if $g_1 \cdots g_r = g$ then $\exists!$ natl isom $F_{g_1}\circ \cdots \circ F_{g_r} \xrightarrow{\sim} F_g$ using α maps.

2) How does this relate to the definition in lecture? Strictification. The defn given in lecture ($F_g \circ F_h = F_{gh}$ ← equal, not isom) is weird categorically.... just like strict monoidal categories where $(A\otimes B)\otimes C = A \otimes (B \otimes C)$ instead of $(A\otimes B)\otimes C \cong A\otimes(B\otimes C)$.

But strict monoidal cats are important b/c diagrammatics are only for strict mon. cats.

Not a big deal! Given $\mathcal{C}$ monoidal, its strictification $\mathcal{C}^{str}$ has

$Ob(\mathcal{C}^{str})$ = words in objects in $\mathcal{C}$. $\otimes$ is concatenation. ← strict!

$Hom_{\mathcal{C}^{str}} = Hom_{\mathcal{C}}(A_1 \otimes (A_2 \otimes (\ldots)), B_1 \otimes (B_2 \otimes (\ldots)))$.

$\boxed{\mathcal{C} \xrightarrow{\sim} \mathcal{C}^{str} \text{ equiv.}}$

When talking about monoidal cats, one is usually working with the strictification subconsciously — words, not objects

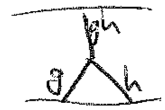
Rmk: Even when $\mathcal{C}$ is strict, $\mathcal{C}^{str} \neq \mathcal{C}$. But $\mathcal{C}^{str} \simeq \mathcal{C}$.

Given usual strict action, get $(\Omega G)^{str} \longrightarrow Aut(\mathcal{C})^{str}$

$g_1 g_2 \cdots g_r \longmapsto F_{g_1} \circ \cdots \circ F_{g_r}$

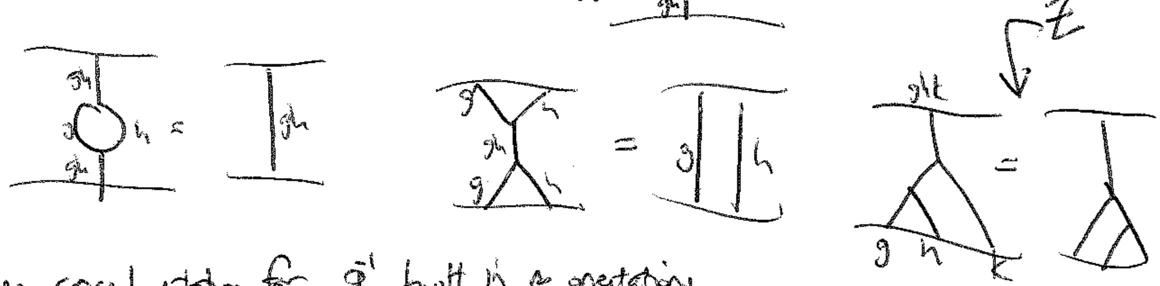
if $\shortparallel$ g, unique map $\longmapsto$ unique map from claim above

3) diagrammatic way to describe usual defn of strict action:

generating objects:  $g \in G$ generating 1-morphisms α 

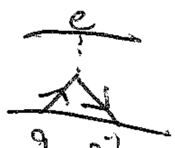
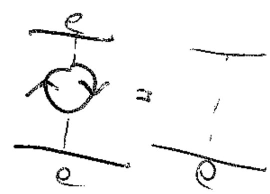
α^{-1} 

relations



<u>Note</u>: There is no special notation for g^{-1} built in, no orientations.

HOWEVER, if you draw g as  and g^{-1} as 

then $\alpha_{g,g^{-1}}$ looks like  $\alpha_{g^{-1},g}$  and  etc., get

the relations Alex mentioned for oriented diagrams, + biadjunction (exercise)

<u>Rmk</u>: Following Alex's rubric for this presentation (Z as above is associativity axiom)

you get something which looks bigger — have $g\uparrow$ and $g\downarrow$ w/o using any convention!

But w/ some extra axioms, reduces back to the ordinary definition.

4) Zams = cycles in rex graph for w_0 in <u>rank 3</u> $\overset{\text{Finite}}{\text{Coxeter gps}}$

e.g., all cycles in rex graph for D_4 come from boring cycles + Zams for $A_3 \subset D_4$ and $A_1 \times A_1 \times A_1 \subset D_4$.