my talk

groups acting on categories;
the Coxeter 2-groupoid

(1) $G \curvearrowright$ set, v.s., top. spc.
bij., linear iso., homeo.

<u>weak action</u> $G \curvearrowright \mathcal{C}$: $g \longmapsto F_g$ autoequivalence
s.t. $F_g \circ F_h \cong F_{gh}$ $\forall g, h$

<u>strict action</u> : $g \longmapsto F_g$ autoequiv.
s.t. $F_g \circ F_h = F_{gh}$
unique functor for each group element

(2) $Aut(\mathcal{C})$
- autoequivs
- $F \otimes G = F \circ G$
- inv. nat. trans.

ΩG (2-groupoid of G)
- G
- $g \otimes h = gh$
- $Hom(g,h) = \begin{cases} \{id\} & g=h \\ \emptyset & g \neq h \end{cases}$

& variant $\Omega G'$

<u>defn.</u> A <u>strict group action</u> $G \curvearrowright \mathcal{C}$ is a strict monoidal functor $\Omega G \to Aut(\mathcal{C})$.
i.e., $F(g) \otimes F(h) = F(g \otimes h)$

In practice, we often give group actions via generators & relations: $G \curvearrowright X$
$s : X \to X$ $\quad G = \langle S \mid R \rangle$
& they satisfy r's

How do we do this for categories?

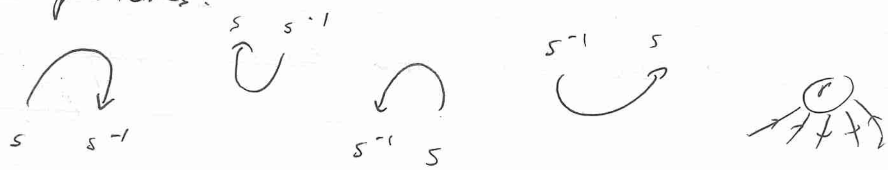
ΩP for a presentation $P = \langle \mathcal{S}, R \rangle$

- words in $\mathcal{S} \sqcup \mathcal{S}^{-1}$
- concatenation
- cups, caps, & $r \leftrightarrow \pi$'s

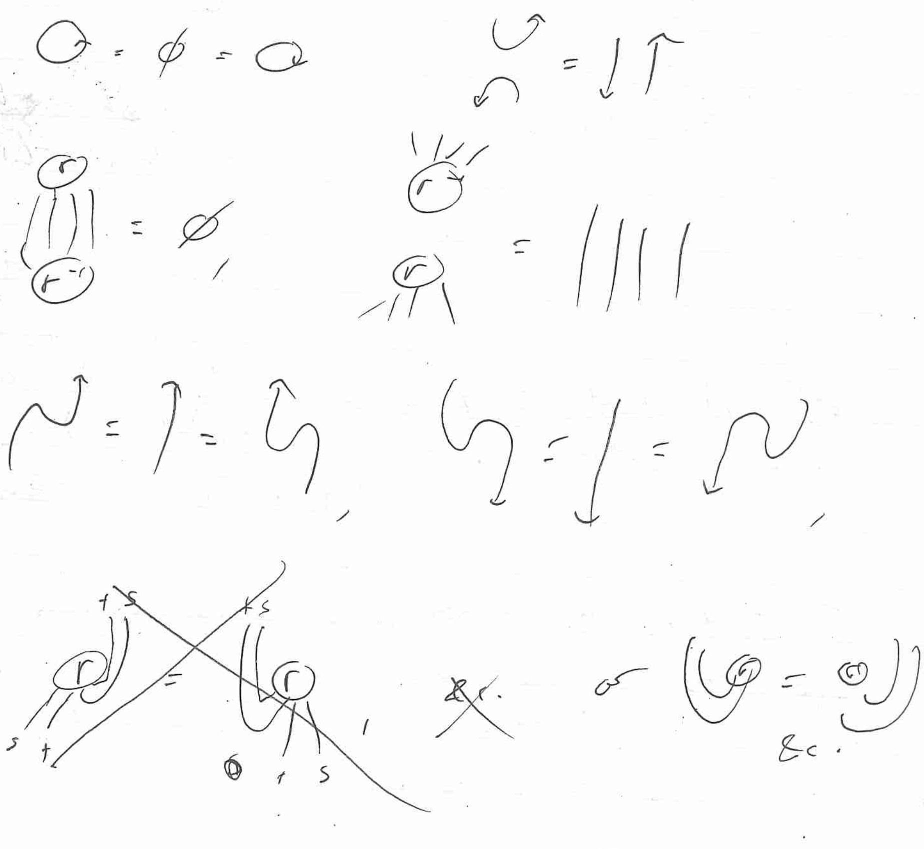
  modulo
  - they're iso
  - biadjoint s, s^{-1}
  - cyclic r

in pictures:

g:



r:



n.b. $\Omega P \to \mathrm{Aut}(\mathcal{C})$ is a cyclic weak action

we have a functor $\Omega P \to \Omega G$

For $Z \subseteq \mathrm{End}(\mathbb{1})$ (closed diagrams) let

$$\Omega(\mathcal{S}, R, Z) := \Omega(\mathcal{S}, R) \Big/ \left\langle \begin{matrix} z = \mathrm{id}_{\mathbb{1}} \text{ for} \\ \text{all } z \in Z \end{matrix} \right\rangle .$$

(3-presentation)

We say $P = (\mathcal{S}, R, Z)$ is acyclic is $\Omega P \xrightarrow{\sim} \Omega G$.
In this case, $\Omega P \to \mathrm{Aut}(\mathcal{C})$ is actually a strict action! Here a recipe for a strict action defined by g's & r's:

- F_s for each $s \in \mathcal{S}$, w/ biadjoint F_s^{-1}
- isos for each $r \in R$
- which satisfy each $z \in Z$

e.g. $\mathcal{S}_a = G$, $R = \{ g \cdot h = (gh) \}_{g,h}$
$Z = \{$all associativity rels$\}$

e.g. for any $(\mathcal{S}, R)$, an extension $(\mathcal{S}, R, \mathrm{End}(\mathbb{1}))$

both acyclic, both useless!

(3) $\mathrm{End}(\Pi) = \left(\begin{array}{c}\text{closed} \\ \text{diagrams} \\ \text{in } \Omega P\end{array}\right) = \left(\begin{array}{c}\Pi\text{-based} \\ \text{loops in } \Gamma_e\end{array}\right)$

expression graph Γ_e (same for any $g \in G$):

v. expressions in $S \sqcup S^{-1}$ for e

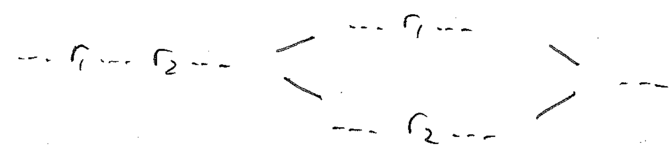
e. applications of $ss^{-1} \leftrightarrow \phi$

$s^{-1}s \leftrightarrow \emptyset$

$r \leftrightarrow \emptyset$

The ~~...~~ quotient by $Z \subseteq \mathrm{End}(\Pi)$ is the same as killing certain loops.

Of course some come for free:

~~...~~ 

finding an acyclic (S, R, Z) $\Leftrightarrow$ killing all loops (monoidally)

(4) Coxeter presentation S : $s, t, u, \dots$

R : $s^2 = \dots = 1$, $(st)^{m_{st}} = \dots = 1$

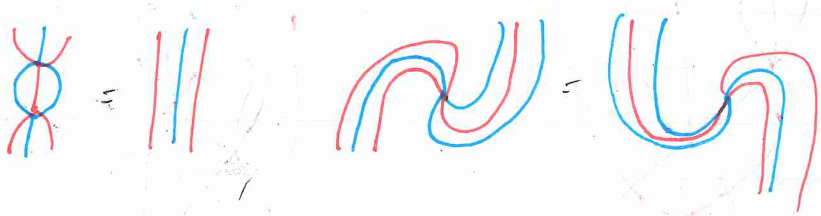
fact: $s^2 = 1 \ \forall s$ $\rightsquigarrow$ can use unoriented diagrams

$r \in R$'s have cyclic symmetry $\rightsquigarrow$ can use $sts\dots = tst\dots$ instead of $ststst\dots = 1$

~~call this~~ ~~let's do some~~ in pictures:

g: 

r: 



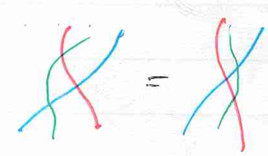
what do we need to kill? let's see some $\Gamma_{w_0}^{aux}$'s:

A_1^3

sut — ust

stu / \ uts

tsu — tus

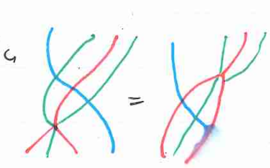


$A_2 \times A_1$

utst — tust — tsut

usts / \ tstu

suts — stus — stsu



Zam.

A3 stsuts = stusts

tstuts stutst

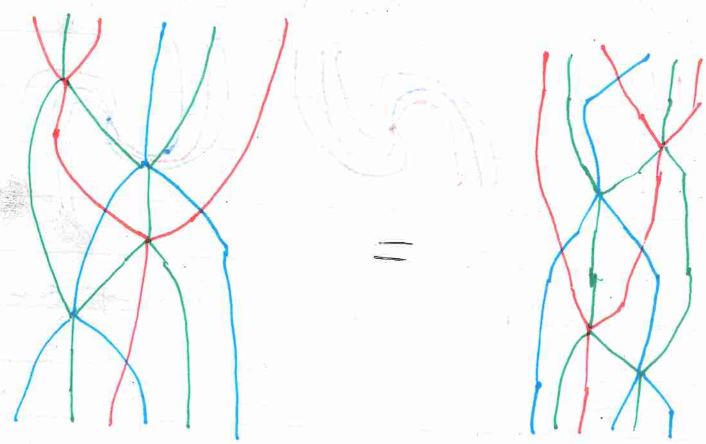
tsutus sutust

tustus = tsutsu ustust = sutsut for free

tustsu ustsut

tutstu utstut

utustu = utsutu

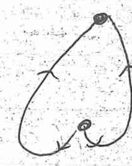


(5) <u>Thm. (old)</u>: For w_0 Coxeter rex's, the only cycles are Zams.

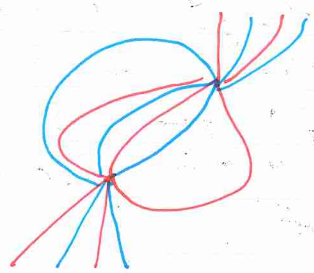
But for non-reduced, there are more:

<u>e.g.</u>

$$ss^{-1}$$
$$ss^{-1}s^{-1}s$$
$$r \quad ss \quad r^{\vee}$$



<u>e.g.</u> $m_{st}=5$



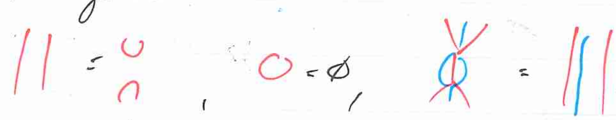
ststss

stst tststs

ttstst

(6) <u>Thm (Elias-Williamson)</u>: In general we need:

standard relations:



isotopy & cyclicity:

☞ can do planar graphs up to isotopy

generalised Zams: $A_1 \times I_2(m)$

A_3 (as above)

B_3

H_3

(7) vague idea of proof:

~~complex~~ presentation $\rightsquigarrow$ 2-complex X_P

$P=(S,R)$

one 0-cell
1-cell for each $s\in S$
2-cell attached along each $r\in R$

$\pi_1(X_P)\cong G$

if we add Z, this gives 3-cells which kill off parts of $\pi_2(X_P)$.

$P=(S,R,Z) \rightsquigarrow X_P$ 3-complex $\rightsquigarrow \Pi(X_P)_{\le 2}$ a 2-categorical approx. of $\pi(X_P)$.

$\cong$

ΩP.

P acyclic $\Leftrightarrow$ $\pi_2(X_P)$ trivial
in which case, $\Pi \simeq \Omega G$ & $X_P \cong \mathrm{Skel}^3(BG)$.

$\Pi(X_P)_{\le 2}$:
0. 0-cells
1. 1-cells, formal inverses
2. g. 2-cells, formal inverses
   r. from the 3-cells

& for the Coxeter case explicit models for BG save the day.

~~Add two cells~~

e.g. $(\emptyset, \{\text{empty word}\})$ $X_G \cong S^2$, $\pi_2 \cong \mathbb{Z}$
signed count of e-words

e.g. $G=\{e\}$, $P=(\{a\},\{a\})$  $X_P\cong D^2$

e.g. $P=(\{s\},\{s^2\},\{z\})$ (along $\oplus$  $\mathbb{R}P^2$, $\mathbb{R}P^3$