Our goal: Understand + compute w/ morphisms b/w SBim, in order to answer the Big Q.

Easiest way to work w/ an algebra — generators + relations. Start w/ something abstract, like $\mathbb{H}_W$.

Choose symbols for certain elements — i.e. H_s. Now can form $\boxed{\text{words}}$ $H_s H_t H_s$. (+ linear combos)

Relations are rules for replacing certain words w/ others, like $H_s H_s = v H_s + v^{-1} H_s$.

Key technology: words. What makes it tick? Associativity (+ unit) axiom.

But now look at a monoidal category ... words aren't the right thing!! $\varphi: A \to B$, $\psi: C \to D$

Then $(\varphi \otimes 1) \circ (1 \otimes \psi) : A \otimes C \to B \otimes D$ and your brain hurts. 2 kinds of composition ...

one of the axioms $\to$ $\overset{\shortparallel}{(1 \otimes \psi) \circ (\varphi \otimes 1)}$ need some kind of planar diagram.

Planar diagrams are the correct tool for the job! It's a notational convention, but an incredibly useful one.

Analogously: words for categories (algebra w/ multiple objects)
planar diagram for 2-categories.

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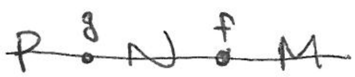
§1 Linear diagrams for 1-categories

Our convention is the dual of what you're used to.

Old way: $P \xleftarrow{g} N \xleftarrow{f} M$ Obj look like pts
Mor look like intervals

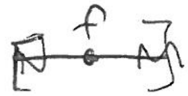
Same data, but apparent positioning.

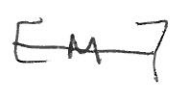
"dualize"



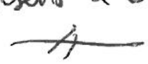
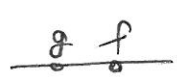
Obj intervals
Mor pts

$P \leftarrow N \longleftarrow M$

In picture: A (generic) point is actually an object.
A ( ) interval is a morphism  | from RHS to LHS

Composition    identity  (this is why the dual picture is better)

Axioms of category $\Longleftrightarrow$ Diagram/linear isotopy unambiguously represents a morphism.
(positioning is irrelevant)
parentheses

i.e. If f represents a known map
g    then  is also known.

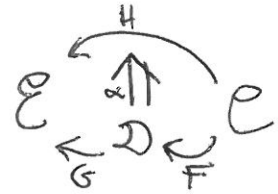
§2.1 Planar Diags for 2-cats

First, some examples of 2-cats:

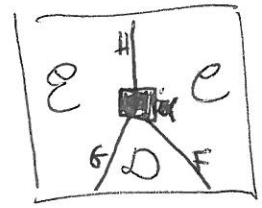
CAT
Ob: categories
1-mor: functors
2-mor: natural transformation

BIM
Ob: Rings
1-mor: bimodules
2-mor: bimodule maps

Old way of drawing 2-morphism

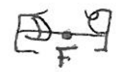


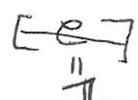
New way



Generic pt $\Longleftrightarrow$ object

( ) horiz line interval $\Longleftrightarrow$ 1-mor from RHS to LHS, as above





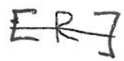
$= \mathbb{1}_{\mathcal{C}}$ composition

( ) rectangle $\Longleftrightarrow$ 2-mor from bottom to top

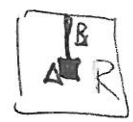
 $= \mathrm{id}_F$

 $= \mathrm{id}_{\mathbb{1}_{\mathcal{C}}}$

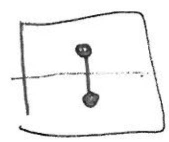
EXAMPLE IN BIM:

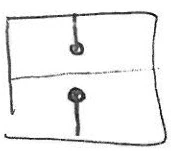
 is ${}_R R_R$

 is ${}_R {B_s}_R$

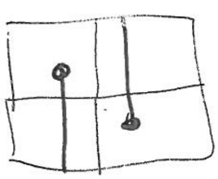
Encode $R \xrightarrow{\Delta} B_s$, $1 \mapsto c_s$ as 

Also have map $B_s \xrightarrow{\mu} R$, $f \otimes g \mapsto fg$ encode as 

Then  is $R \to R$, $1 \mapsto c_s \mapsto \mu(c_s) = \alpha_s$ is mult by α_s!!

meanwhile,  is some endomorphism of B_s. $B_s \to R \to B_s$

so is



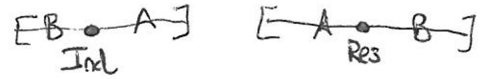
$B_s \to B_s B_s \to B_s$

$\mu \otimes 1 \circ 1 \otimes \Delta$

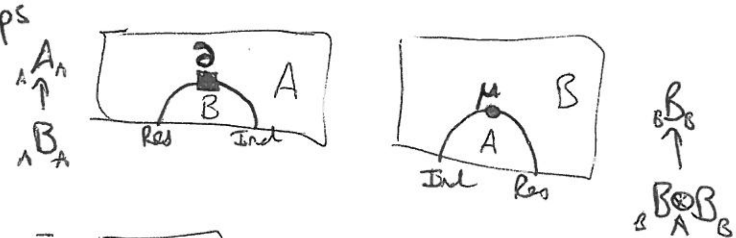
but these are equal. No accident!

Axioms of 2-cats $\Longleftrightarrow$ Diagram / Rectilinear isotopy unambiguously represents a 2-morphism.

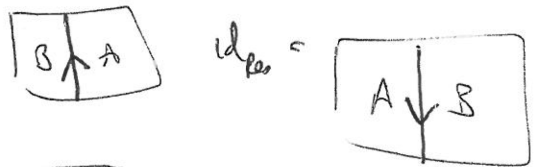
Ex: $A \subset B$ a Frob. ext

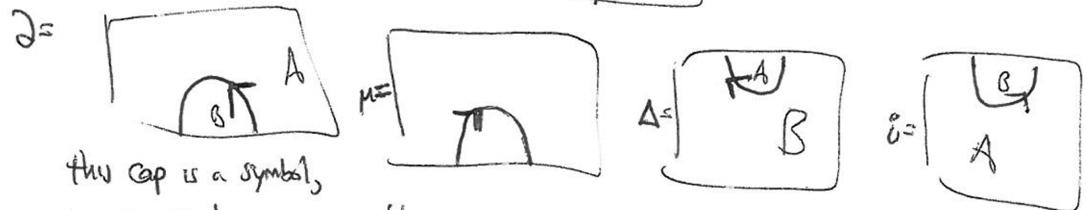


Equipped with 4 maps



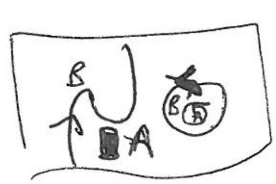
Try better notation: id_{Ind} =



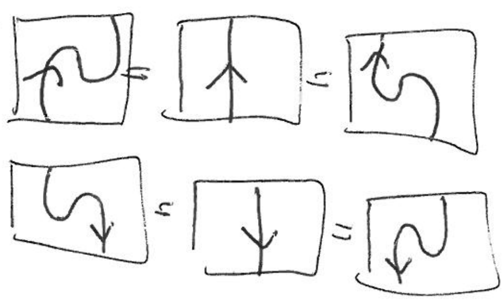


this cap is a symbol, no a priori meaning for a "sideways" identity!!

With this convention, we have a meaning for any oriented labeled 1-manifold!



Axioms of Frob Ext $\Longrightarrow$



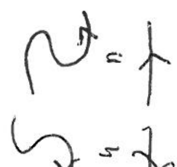
so topological isotopy doesn't affect a morphism!!

Advantage: diagrams!!

Formalize: Whenever $\mathcal{D} \underset{E^\vee}{\overset{E}{\leftrightarrows}} \mathcal{C}$ with $E \dashv E^\vee$ we have chosen $EE^\vee \to \mathbb{1}$, $\mathbb{1} \to E^\vee E$ draw as

have

with



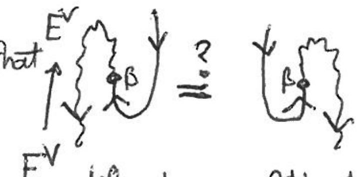
"rightward isotopy"

no meaning for 

but if also $E^\vee \dashv E$, get  as well.

However, suppose $E \dashv E^\vee \dashv E$ and $F \dashv F^\vee \dashv F$ and

Nothing guarantees that



left mate   right mate

If they are equal, say β is cyclic wrt the fixed choice of adjunctions. Draw both sides as . Without this, could never involve β in our isotopies.

Thm:

If all 1-mor have fixed biadjoints AND all 2-mor are cyclic THEN Axioms of biadjunction + cyclicity $\Longrightarrow$ Diagram / the isotopy unambiguously represents a 2-morphism.

Given such a thing, you SHOULD use diagrams, because isotopy makes your life easier!

Rmk: When "taking biadjoints" is functorial, all 2-morphs are automatically cyclic.
Common situation in geometry + rep theory.

$\boxed{\text{Ex}}$: fd Rep $\mathfrak{g}$ $\otimes V$, $\otimes V^*$ are biadjoint, * is a functor.

SBim is such a monoidal category!