§1 Hecke alg redux (Didn't want to overload before)

Notation: $\underline{w} = (s_1, s_2, \ldots, s_d)$ $s_i \in S$ is an expression of length $\ell(\underline{w}) = d$.

Remove underline, $w = s_1 \ldots s_d \in W$. $\underline{w}$ expresses w. Reduced exp. or rex is minimal length.

Abuse notation and write $\underline{w} = s_1 \ldots s_d$, underline means expression, not element, is important.

Def: A subexpression is... what you think. $\underline{e} \subset \underline{w}$, $\underline{e} = (e_1, e_2, \ldots, e_d)$ $e_i \in \{0,1\}$ expresses $\underline{w}^{\underline{e}} = s_1^{e_1} s_2^{e_2} \ldots s_d^{e_d}$.

Bruhat order: $v \le w$ if $\exists$ $\underline{e} \subset \underline{w}$ (rex) s.t. $\underline{w}^{\underline{e}} = v$.

Subexpressions get an extra decoration U/D via the Bruhat Stroll.

Ex: $\underline{w} = stss$, $\underline{e} = 1001$, UUDD, def: 0

st ← s t ← 1 Start

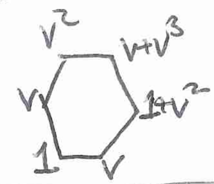
$\underline{w} = stss$, $\underline{e} = 0011$, UUUD, +2

Def: The defect of $\underline{e}$ is +1 per U0, −1 per D0. It measures regret. Alas!

Now, let $\underline{H}(\underline{w}) = \underline{H}_{s_1} \underline{H}_{s_2} \cdots \underline{H}_{s_d}$ (very much depends on the expression. Almost never equal to $\underline{H}_w$)

Deodhar Defect Formula:
$$\underline{H}(\underline{w}) = \sum_{\underline{e} \subset \underline{w}} v^{\operatorname{def}(\underline{e})} H_{\underline{w}^{\underline{e}}} \quad (2^d \text{ terms})$$

Ex: $\underline{H}_s = v H_1 + H_s$ ($\underline{e} = 0$, U; $\underline{e} = 1$, U)

Ex: $\underline{H}_s \underline{H}_t \underline{H}_s =$ 

Finally, there are 4 main structures on $\mathbb{H}_W$:

Bar Inv.
$\bar{v} = v^{-1}$
$\overline{\underline{H}_s} = \underline{H}_s \Leftrightarrow \overline{H_s} = H_s^{-1}$ $\overline{H_x} = H_{x^{-1}}^{-1}$
$\overline{ab} = \bar{b}\bar{a}$
$\overline{\underline{H}(\underline{w})} = \underline{H}(\underline{w})$

ω antiinvolution
$\omega(v) = v^{-1}$
$\omega(\underline{H}_s) = \underline{H}_s \Leftrightarrow \omega(H_s) = H_s^{-1}$ $\omega(H_x) = H_x^{-1}$
$\omega(ab) = \omega(b)\omega(a)$
$\omega(\underline{H}(\underline{w})) = \underline{H}(\underline{w}^{op})$

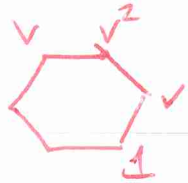
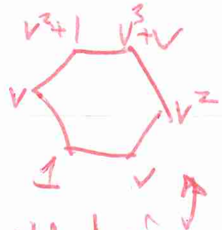
std trace
$\varepsilon: \mathbb{H}_W \to \mathbb{Z}[v, v^{-1}]$
$\varepsilon(v) = v$
$\varepsilon(H_x) = \delta_{x1}$
Claim: $\varepsilon(ab) = \varepsilon(ba)$
Claim: ~~$\varepsilon(H_x H_y^{-1}) = \delta_{xy}$~~ OOPS
$\varepsilon(H_x H_{y^{-1}}) = \delta_{xy}$

std pairing
$(a,b) \equiv \varepsilon(\omega(a)b) \in \mathbb{Z}[v, v^{-1}]$
So $(a, vb) = v(a,b) = (v^{-1}a, b)$
Claim: ~~$(H_x, H_y) = \delta_{xy}$~~ $(\overline{H_x}, H_y) = \delta_{xy}$
Claim: $(ab, c) = (a, c\,\omega(b))$ $= (b, \omega(a)c)$
$\Rightarrow \underline{H}_s$ is self-biadjoint
i.e. $(\underline{H}_s a, b) = (a, \underline{H}_s b)$
$(a, b\underline{H}_s) = (a\underline{H}_s, b)$
See BOOK

More exercises. All important, b/c they arise as decategorification of certain structures!

LECTURE 3 BONUS PAGE

EARLIER IN LECTURE 1 I GAVE THE PAIRING ON $\mathbb{H}_W$ TO BE THE DOT PRODUCT, THEN SAID I WAS LYING.

$a=$  $b=$  $(a,b) = ?$

↑ these are coeffs in std basis

Dot product would work if $\{H_x\}$ was an ONB, but it is not. The dual basis is $\{\overline{H_x}\}$.

HOWEVER, if $a=\overline{a}$ then the above are also the coeffs of $\overline{a}$ in $\{H_x\}$, i.e. the coeffs of a in $\{\overline{H_x}\}$, so dot product works. (Don't require $b=\overline{b}$)

Similarly, if $b=\overline{b}$ then dot product works.

We will be dealing almost entirely w/ self-dual elements like $\underline{H}_{\underline{w}}$ and $\underline{H}_w$. So we will "pretend" $(\,,\,)$ is the dot product.

Return to setup of previous lecture:

2) Bott-Samelson bimodules

$R = \mathbb{R}[\mathfrak{h}^*] \circlearrowleft W$

Ex: $R = \mathbb{R}[x_1, \dots, x_n] \circlearrowleft S_n$

R^I = invts under $s \in I$

$R^I \subset R$ a frob. Ext.

Def: For $s \in S$: $B_s = R \otimes_{R^s} R(1)$ an R-bimodule, graded $(\deg 1 \otimes 1 = -1)$

(B_s =) this is $\mathrm{Ind}_{R^s}^R \circ \mathrm{Res}_{R^s}^R (1)$.

Visualize elements of B_s: $\sum f_1 \,\big|_s\, f_2$ semiporous membrane border control — only R^s can osmose through.

Thus actions of R are evident.

Recall: $f = g + h\alpha_s$ for $g, h \in R^s$. So $f \,|\, = g \,|\, + \alpha_s h \,|\, = \,|\, g + \alpha_s \,|\, h$

$\Longrightarrow$ as Right R-mod, B_s is free w/ basis $\left\{ \,|\, \text{ and } \alpha_s \,|\, \right\}$ ($1 \otimes 1$ degree -1; $\alpha_s \otimes 1$ degree $+1$)

About Exercise: Why free? b/c $R^s \subset R$ is free.

Def: A Bott-Samelson bimodule $BS(\underline{w}) \equiv B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes_R B_{s_d}$

$= R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_d}} R(d)$ $\quad \deg(1 \otimes 1 \otimes \dots \otimes 1) = -d$.

visualize: $f_1 \,|\, f_2 \,\}\, f_3 \cdots f_d \,|\, f_{d+1}$ Europe before EU.

Exercise: As right R-mod, $BS(\underline{w})$ free w/ basis $\{\alpha_{s_1}^{e_1} \otimes \alpha_{s_2}^{e_2} \otimes \dots \otimes \alpha_{s_d}^{e_d} \otimes 1\}$ $\underline{e} \subset \underline{w}$

graded rank $= (v + v^{-1})^d$.

It has many R-actions, but we work in cat of R-bimodules. Morphisms must be linear on left+right, but can do funky stuff to the middle!

Examples + Computations

① $\exists$ bimodule map $B_s \longrightarrow B_s \otimes B_s$; $1 \otimes 1 \longmapsto 1 \otimes 1 \otimes 1$; $\Rightarrow f \otimes g \longmapsto f \otimes 1 \otimes g$. Check: well defined.

as a graded map, has degree -1. More correctly, $B_s \longrightarrow B_s \otimes B_s(-1)$

WARNING: Two different kinds of tensors. To avoid confusion, we'll no longer write $\otimes_R$. $B_s B_s$.

② $\exists$ bimod map $B_s \otimes B_s \longrightarrow B_s$

$1\otimes f\otimes 1 \longmapsto \partial_s(f)\otimes 1 = 1\otimes \partial_s(f)$

(Couldn't do ∂_t, needs to be R^s-linear for middle...)

again, has graded degree -1.

$1\otimes 1\otimes 1 \longmapsto 0$.

③ Actually, $B_s\otimes B_s \cong B_s(1)\oplus B_s(-1)$! as bimodules

$\{f|1|g\}$ ⊕ $\{f|\alpha_s|g\}$

(Think: $R\cong R^s(0)\oplus R^s(-2)$ as R^s-bimodules

$R\otimes_{R^s} R\otimes_{R^s} R$ and so forth.)

Map from ① is inclusion of first summand. Map from ② is projection to second. Exercise: Find missing inclusion, projection

⇝ in split Groth ring of graded R-bimodules, $[B_s][B_s]=(v+v^{-1})[B_s]$

recall: in H_W, $H_sH_s=(v+v^{-1})H_s$.

④ $B_s\otimes B_t$ Now $(\mathfrak{h}^*)^s+(\mathfrak{h}^*)^t=\mathfrak{h}^*$ when $a_{st}=-2$. (i.e. $m_{st}=\infty$)

$f_1|f_2|f_3$ ⟹ R^s, R^t generate R. ⟹ f_2 can be "slid out", i.e. $|f_2| = \sum g_1|\;|g_3$

Thus $1\otimes1\otimes1$ generates B_sB_t as a bimodule. (⟹ B_sB_t is indecomposable, unlike B_sB_s b/c degree -2 space is 1D)

Hence $R\otimes_{R^{st}} R(2) \twoheadrightarrow B_sB_t$

$f\otimes g \longmapsto f\otimes 1\otimes g$.

When $m_{s,t}=2$, this is an isomorphism! Exercise: Construct inverse map.

When $m_{s,t}\geq 3$, LHS is bigger, there is a kernel. Nor can one even define a bimodule map $B_sB_t\longrightarrow R\otimes_{R^{st}}R(2)$, $1\otimes1\otimes1\longmapsto 1\otimes 1$. Exercise: Why not?

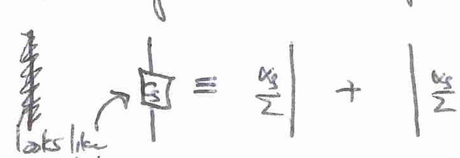
But... $m_{st}=2$ $[B_s][B_t]=[R\otimes_{R^{st}}R(2)]=[B_t][B_s]$ seeing the pattern?

$H_sH_t = H_{st} = H_tH_s$

⑤ $R\xrightarrow{\Delta} B_s$ satisfies $fc_s=c_sf$, as must to be image of bimodule map from R.

$1\longmapsto 1\otimes\frac{\alpha_s}{2}+\frac{\alpha_s}{2}\otimes 1 \equiv c_s$

Visual notation:



⑥ $m_{st}=3$ $\quad B_sB_tB_s \cong R\otimes_{R^{st}}R(3) \oplus B_s$

$\quad \| $

$\{f \mid 1 \otimes 1 \mid g\}$ ← (to $R\otimes_{R^{st}}R(3)$)

$\oplus$

$\{f \mid \boxed{c} \mid g\}$ ← (to B_s)

EXERCISE

Categorifies $\underline{H}_s\underline{H}_t\underline{H}_s = \underline{H}_{sts} + \underline{H}_s$

$\underline{H}_t\underline{H}_s\underline{H}_t = \underline{H}_{tst} + \underline{H}_t$

thus <u>Prop (Soergel)</u>: $\mathbb{H}_W \longrightarrow [R\text{-bim}]$ is a homomorphism of $\mathbb{Z}[v^{\pm}]$-algebras.

$\underline{H}_s \longmapsto [B_s]$ $\quad \underline{H}(\underline{w}) \mapsto [BS(\underline{w})]$

---

§3 Soergel bimodules

Def: A Soergel bimodule is a <u>summand</u> of a BS Bim. Closed under $\oplus, \otimes, (n)$.

SBim $\subset$ R-Bim is a full additive monoidal subcategory.

Ex: $R\otimes_{R^{st}}R(3)$ is a SBim ($m_{st}=3$). not all so easy to describe!!!

Ex: Kernel of ④, Cokernel of ⑤ are NOT SBim! NOT abelian!!

Thm: (Soergel Categorification Theorem)

① $\exists!$ indecomposable summand $B_{\underline{w}} \overset{\oplus}{\subset} BS(\underline{w})$ which does not appear in shorter expressions. ← (rex)

Ex: $R\otimes_{R^{st}}R(3) = B_{sts}$

② $\exists$ canonical isom $B_{\underline{w}} \cong B_{\underline{w}'}$ when $w = w'$. So we'll write it as B_w henceforth.

③ $\{B_w(n)\}_{n\in\mathbb{Z}}$ forms a complete list of non-isom. indecomposables in SBim.

$\Longrightarrow [\text{SBim}] = \mathbb{Z}[v^{\pm}]\langle [B_w] \rangle$ ← basis

④ $\mathbb{H}_W \xrightarrow{\sim} [\text{SBim}]$ an isom. $\quad$ Pf: ① gives uppertriangularity, so injective; ③ says same size.

⑤$^{\text{hard}}$ $\text{Hom}(B, B')$ is free as a right R-module (that's clear)

and $\text{grdrank Hom}(B,B') = ([B],[B'])$ $\quad$ <u>Soergel Hom Formula</u>.

Ex: $\text{Hom}(R,R) = R$, $(1,1) = 1$

$\text{Hom}(R,B_s) = R(-1)$ $\quad (1,\underline{H}_s) = v$

$\text{Hom}(B_s,B_s) = R \oplus R(-2)$ $\quad (\underline{H}_s,\underline{H}_s) = 1+v^2$

(ident; left mult by α_s)

etc...

> **BIG Q:**
> What basis is $[B_w]$?
> Is it $\underline{H}_w$???