Start to develop the background. Really, it's all a big study of the reflection repn.

Fix (W,S). Define the (symmetric) Cartan matrix of (W,S) to be $A = \begin{pmatrix} 2 & & a_{s,t} & \\ & 2 & & \\ & & 2 & \\ a_{t,s} & & & \ddots \\ & & & & 2 \end{pmatrix}$

with $a_{s,t} = a_{t,s} = \begin{cases} -2\cos\frac{\pi}{m_{s,t}} & \text{when } m_{s,t} < \infty \\ -2 \text{ (or anything suitably generic)} & \text{when } m_{s,t} = \infty \end{cases}$

Let $\mathfrak{h}^*$ be the $|S|$-dim $\mathbb{R}$-v.s. w/ basis $\{\alpha_s\}_{s\in S}$, called simple roots.

$W \circlearrowright \mathfrak{h}^*$ by $s(\alpha_t) = \alpha_t - a_{s,t}\alpha_s \rightsquigarrow$ $s(\alpha_s) = -\alpha_s$, $s(\alpha_t) = \alpha_t + 2\cos\frac{\pi}{m_{s,t}}\alpha_s$ Exercise: Well-defined.

Rmk: Many ways to generalize. Properties of $2\cos\frac{\pi}{m_{s,t}}$ explored in exercises.

Ex: $W = S_n \circlearrowright \mathrm{Span}(x_1,\dots,x_n) = V$ $\quad \mathfrak{h}^* = \left(V/\sum x_i = 0\right)$ $\quad \alpha_i = x_i - x_{i+1}$ For Weyl groups very familiar.

From $\mathfrak{h}^*$ we will construct some interesting commutative rings

Def: $R = \mathrm{Sym}(\mathfrak{h}^*) = \mathbb{R}[\alpha_s]$ a poly ring. Graded, w/ $\deg \alpha_s = 2$. (Why? $R = H^*_T(pt)$, $T = (\mathbb{C}^*)^{|S|}$)

Not excited yet? For $I \subset S$, let $R^I = R^{W_I} = \{f \in R \mid sf = f \ \forall s \in I\}$

Ex con't: $R = \mathbb{R}[x_1,\dots,x_n]/_{e_1 = 0}$. $\quad W_I = S_3 \times S_1 \times S_2 \times \dots$ $\quad R^I = \mathbb{R}[x_1+x_2+x_3,\ x_1x_2+x_1x_3+x_2x_3,\ x_1x_2x_3,\ x_4,\ x_5+x_6,\ x_5x_6, \dots]/e_1$

R^I is also "just" a poly ring! w/ same transcendence degree.

Thm (Chevalley): Suppose I is finitary, i.e. W_I is finite. Then R^I is a poly ring w/ transcendence degree $|S|$, generated by alg. ind. polys in degrees determined by W_I.

Ref: Humphreys, "Coxeter Groups ..." w/ many fun numerological facts about these degrees.

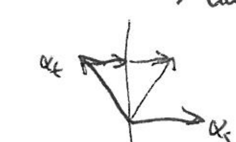
Examples: ① Nonexample - $G = \mathbb{Z}/2\mathbb{Z} \circlearrowright \mathbb{R}^2$ by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. $\mathbb{R}[x,y]^G$ is gen by $u = x^2$, $v = xy$, $w = y^2$

$\mathbb{R}[x,y]^G = \mathbb{R}[u,v,w]/_{uw = v^2}$.

Usually, get interesting singularities this way.

② $|S| = 2$. $\quad R = \mathbb{R}[\alpha_s,\alpha_t] = R^\emptyset \quad R^s = \mathbb{R}[\alpha_s^2,\ \alpha_t + \cos\frac{\pi}{m_{s,t}}\alpha_s]$

$R^t = \mathbb{R}[\alpha_t^2,\ \alpha_s + \cos\frac{\pi}{m_{s,t}}\alpha_t]$



$R^{s,t} = ?$ $\quad m_{s,t} < \infty$, Chevalley applies, $R^{s,t} = \mathbb{R}[z, Z]$ degree (2)m; quadratic (degree 4) $\quad$ Exercises

$m_{s,t} = \infty$, Chevalley fails. If $a_{s,t} = -2$, $R^{s,t} = \mathbb{R}[\alpha_s + \alpha_t]$ $\quad$ a poly ring, but wrong transcendence degree.

$a_{s,t}$ = generic, $R^{s,t} = \mathbb{R}[z]$

(Remember - don't care if W finite, only if W_I finite.)

Why emphasize transcendence degree? Because it means $R^I \subset R$ is a finite extension!!

Key Example: $R^s \subset R$. $R^s = \mathbb{R}[\alpha_s^2, (\mathfrak{h}^*)^s]$ codim 1

Usual $\mathbb{Z}/2\mathbb{Z}$ theory: $R = R^s \oplus R^{-s}$, but one can observe $R^{-s} = R^s \cdot \alpha_s$.

$\{f \mid sf=f\}$ $\{f \mid sf=-f\}$

So R is free over R^s w/ basis $\{1, \alpha_s\}$, any $f \in R$ can be written $f = g + h\alpha_s$ $g, h \in R^s$.

Easy way to find coeffs g, h is via Demazure operator / divided difference operator.

Def: $\partial_s : R \to R^s$ $\quad \partial_s(f) = \dfrac{f - sf}{\alpha_s}$

Facts: ① Well-defined (numerator in R^{-s}) ② degree -2, write $\partial : R \to R^s(-2)$

③ R^s-linear ④ $\partial_s(R^s) = 0$, so $\partial_s^2 = 0$, exact. ⑤ $\partial_s(\alpha_t) = a_{s,t}$

$\partial_s(\alpha_s) = 2$

⑥ $\partial_s(fg) = \partial_s(f) g + s(f) \partial_s(g)$ "twisted Leibniz rule"

⑦ $f = g + h\alpha_s$ ⑧ The pairing $(f, g) \mapsto \partial_s(fg)$ is perfect, i.e.

$R \times R \to R^s$

$g = \frac{1}{2}\partial_s(f\alpha_s)$

$h = \frac{1}{2}\partial_s(f)$ it has dual bases $\{1, \frac{\alpha_s}{2}\}$ and $\{\frac{\alpha_s}{2}, 1\}$ $\partial(a_i b_j) = \delta_{ij}$

What really makes this example so useful? Formalism: Frobenius Extension

Def: A commutative ring extension $A \subset B$ is a Frob Ext if it is equipped w/ $\partial : B \to A$, (part of the structure)

A-linear, and B is free over A w/ dual bases $\{b_i\}$ $\{b_j^*\}$ s.t. $\partial(b_i b_j^*) = \delta_{ij}$.

When A, B are graded, $\deg \partial = -2\ell$, and homogeneous dual bases exist, it's a graded Frob Ext of degree ℓ.

Thm (Chevalley Plus): I finitary $\Rightarrow$ $R^I \subset R$ is a graded Frob ext of degree $\ell(w_I)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$

$I \subset J$ finitary $\Rightarrow$ $R^J \subset R^I$ is graded Frob Ext of degree $\ell(J) - \ell(I)$. $\quad \ell(I)$.

(Not possible when I not finitary, $R^I \subset R$ is infinite rank)

Not done, Frob Ext is a structure, not a property... what is $\partial_I : R \to R^I$?

⑨ ∂_s, ∂_t satisfy braid relation $\quad m=3 \quad \partial_s\partial_t\partial_s = \partial_t\partial_s\partial_t$.

So for any $w \in W$, choose red exp $w = s_1 \ldots s_\ell$, and $\partial_w = \partial_{s_1} \ldots \partial_{s_\ell}$ indep of choice.

The map $\partial_I : R \to R^I$ is ∂_{w_I}. The map $\partial^I_J : R^I \to R^J$ is $\partial_{w_J w_I^{-1}}$.

More examples of Frobenius extensions: ① $H \subset G$ groups, finite index. $\mathbb{C}[H] \subset \mathbb{C}[G]$ $\partial(g) = \begin{cases} g & g \in H \\ 0 & \text{else} \end{cases}$

Rmks: $\partial^2 \neq 0$, no Leibniz rule. Those special.

② (Comm) Frob Alg is Frob Ext / k. X sm. proj vty, $A = H^*(X, \mathbb{C})$ is Frobenius,

$\partial(\beta) = \begin{cases} 1 & \beta = \text{fund class} \\ 0 & \text{else} \end{cases}$ (i.e. integration)

③ W Weyl gp. $R/R^W_+ \cong H^*(G/B) = C$ a Frobenius algebra. Dual bases for $\partial = \partial_{w_0}$ = integration are given by Schubert calculus — combinatorics involving roots!

Top class is $\prod_{\alpha \in \Phi^+} \alpha$, one basis is $\partial_w(\prod \alpha)$.

BUT we're doing relative version $R^W \subset R$ ($R^W = H^*_G(pt)$, $R = H^*_B(pt)$) and Schubert calc does NOT give dual bases!

I.e. $\partial_{w_0}(\sigma_i \sigma_j^*) \in R^W_+$ but not zero. No nice closed formulas known!!!

Open Problem: Find root-theoretic descriptions of dual bases for $R^I \subset R^J$ for all Cox. gps.

Why do we care about frob ext?? (Finite groups ~) Frobenius Reciprocity. Induction + Restriction are biadjoint!

$A \subset B$ arbitrary. ${}_B B_A$ bimodule gives functor ${}_B B \otimes_A (\cdot) : A\text{-mod} \to B\text{-mod}$ Induction

${}_A B_B$ ——— // ——— ${}_A B \otimes_B (\cdot) : B\text{-mod} \to A\text{-mod}$ Restriction

Then $\mathrm{Ind} \dashv \mathrm{Res}$ i.e. $\mathrm{Hom}_B(\mathrm{Ind}\, M, N) \cong \mathrm{Hom}_A(M, \mathrm{Res}\, N)$, naturally.

Adjunction is a structure, not a property. Fix this guy. Do it with unit + counit of adjunction

when $M = \mathrm{Res}\, N$, $\mathrm{Hom}(\mathrm{Ind}\,\mathrm{Res}\, N, N) \cong \mathrm{Hom}(\mathrm{Res}\, N, \mathrm{Res}\, N) \ni \mathbb{1}$

some guy $\longleftarrow$ $\mathbb{1}$

give nat'l trans $\mathrm{Ind} \circ \mathrm{Res} \longrightarrow \mathbb{1}_{B\text{-mod}}$

counit, a B-bimodule map ${}_B B \otimes_A B_B \longrightarrow {}_B B_B$. It's just multiplication.

Sim, $N = \mathrm{Ind}\, M$, get <u>unit</u> $\mathbb{1}_{A\text{mod}} \longrightarrow \mathrm{Res} \circ \mathrm{Ind}$

$$ {}_A A_A \longrightarrow {}_A B_A \quad \text{inclusion.} $$

units + counits satisfy some natural compatibilities, seen later.

For a frob ext, also have $\mathrm{Res} \dashv \mathrm{Ind}$! Maps in other direction.

$$ {}_A B_A \longrightarrow {}_A A_A \qquad \partial \qquad \text{"trace"} $$

$$ {}_B B_B \xrightarrow{\Delta} {}_B B \otimes_A B_B \qquad \text{"comult"} \quad 1 \longmapsto \Delta(1) = \sum b_i \otimes b_i^* $$

indep of choice of dual bases

<u>Exercise</u>: B-bimodule maps

For graded frob ext of degree l, shift to make things more symmetric. Work with Ind and $\mathrm{Res}(l)$

$$ B \otimes_A B(l) \longrightarrow B \qquad \text{degree } +l $$
$$ B \longrightarrow B \otimes_A B(l) \qquad \text{degree } +l $$
$$ B(l) \longrightarrow A \qquad \text{degree } -l $$
$$ A \longrightarrow B(l) \qquad \text{degree } -l $$

We will build some fantastic stuff, using only Induction + Restriction b/w R, R^I.