Ben Elias — Oregon 8/2014 — Lecture 1

<u>This talk</u>: Rough intro to KL conjectures, our grand motivation. Once I explain what they're all about, what kind of questions are involved, etc., then I'll give an overview of the workshop.

KL Conj about Rep Theory, specifically $\mathcal{O}_0$ ← relatively difficult to work with concretely.

No need for advance knowledge: • this talk, I'll state some facts + you'll believe them (it won't be bad)

- Nick's talk later will explain the basics + justify these facts.
- Rest of workshop will ignore $\mathcal{O}_0$ entirely! Instead, <u>Soergel bimodules</u>, an <u>algebraic replacement</u> (← more on this concept later, but it's easier to work with.)

Ultimate goal: <u>be able to work w/ SBim + understand Soergel Conjecture</u> ($\Rightarrow$ KL conjecture)

↖ develop intuition, makes it easier to learn $\mathcal{O}_0$.

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§1 Projectives in $\mathcal{O}_0$

Fix $\mathfrak{g}$ a f.d. $\mathbb{C}$ s.s. Lie alg ⇝ Weyl group W (Ex: $\mathfrak{gl}_n$, $W = S_n$)

↳ BGG category $\mathcal{O}$ — some kind of nice (mostly) ∞-dim'l reps

$\cup$

$\mathcal{O}_0$ ← fixed central character, $\lambda = 0$ "most interesting".

<u>Facts</u>: (1) For each $w \in W$ have 3 objects in $\mathcal{O}_0$.

$$P_w \twoheadrightarrow \Delta_w \twoheadrightarrow L_w$$

indec proj. — standard/Verma — simple

(• $\mathcal{O}_0$ is abelian)

Axiomatization: $\mathcal{O}_0$ is a <u>highest weight category</u> [CPS]. Won't be so important for this workshop.

Consequence: $[\mathcal{O}_0] \cong \mathbb{Z}[W]$ with 3 bases: $\{[L_w]\}$ $\{[\Delta_w]\}$ $\{[P_w]\}$ (• $\mathcal{O}_0$ has finite homl dimension)

↖ $\mathbb{Z}$-modules

Big Q: What are the c.o.b. matrices? $[P_w] = \sum_{y \in W} p_{y,w} [\Delta_y]$ $\quad p_{y,w} \in \mathbb{Z}$.

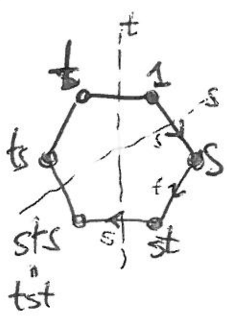
Rephrase: We know how big Δ_w is (dimension of weight spaces). So $p_{y,w}$ tells you how big P_w is. Finding $p_{y,w}$ called finding <u>character</u>.

<u>Ex</u>: <u>Weyl Character formula</u> L_1 is the only f.d. simple so you ran into it earlier in life.

WCF says $[L_1] = \sum (-1)^{\ell(y)} [\Delta_y]$. But other L_w are harder!

Some graphical notation to visualize $[\mathcal{O}_0]$: encode $\mathbb{Z}[W]$ by putting integers on the <u>dual Coxeter complex</u> of W. (← later.)

<u>Ex</u>: $W = S_3$, 6 elements, arrange in hexagon



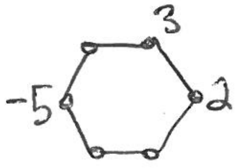
Let's name $s = (12) = X|$, $t = (23) = |X$ "simple reflection"

Left mult by simple refl = reflection

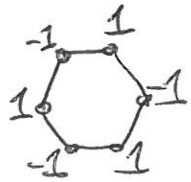
Right mult —— // —— = edge

so label + orient edges

This is dual Cox. complex.

So 

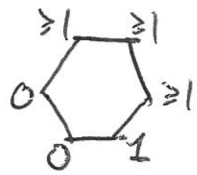
encodes $3+2s-5ts \in \mathbb{Z}[W]$, interpreted as $3[\Delta_1]+2[\Delta_s]-5[\Delta_{ts}] \in [\mathcal{O}_0]$

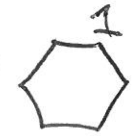
$[L_1] =$ 

Finally, ② • Any indec projective P_w has a Δ-filtration, i.e. a filt. whose subquotients are Δ_y.

$\Longrightarrow p_{y,w} \in \mathbb{Z}_{\geq 0}$.

- Δ_w appears exactly once
- Δ_y appears $\Longleftrightarrow$ $y \leq w$ in Bruhat order.

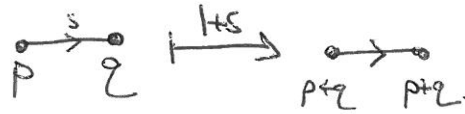
So $[P_{st}]$ has form 

Ex: $P_1 =$ 

$= \Delta_1$ our starting point

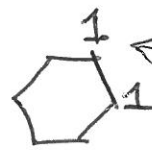
③ $\mathcal{O}_0$ has endofunctor Θ_s for each simple refl. wall-crossing functors.

- Θ_s preserves projectives (not nec. indecomposable though)
- Θ_s acts on $[\mathcal{O}_0]$ by right mult by $(1+s)$.



Key tool for exploring projectives!

Ex: $P_1 =$ 

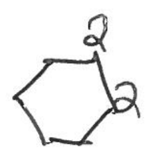
$\Theta_s P_1 =$ 

is it indecomposable? no summands P_w for $w \neq 1, s$

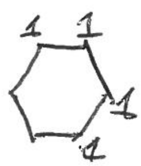
says P_s must appear once

if P_1 also a summand, would be at least 2 ※

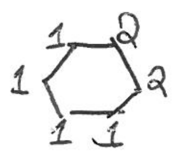
$\rightsquigarrow$ $\Theta_s P_1 = P_s$ and we found P_s!

$\Theta_s\Theta_s P_1 =$ 

$= P_s \oplus P_s$ let's stick to red exp for now.

$\Theta_t\Theta_s P_1 =$ 

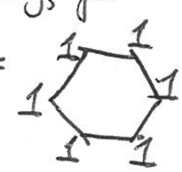
$\rightsquigarrow$ (similar arg) must be P_{st}!

$\Theta_s\Theta_t\Theta_s P_1 =$ 

... our "algorithm" fails.

Is it P_{sts}? $P_{sts} \oplus P_1$? $P_{sts} \oplus P_s$?

It turns out to be $P_{sts} \oplus P_s$.

Subtracting, get $P_{sts} =$ 

Obvious conclusion –

Verma's Thm: $p_{y,w} = 1$ iff $y \leq w$. OOPS! Wrong in sl_4.

Character considerations: $w = s_1 s_2 \cdots s_d$ a reduced expression

then $P_w \overset{\oplus}{\subseteq} \Theta_{s_d} \cdots \Theta_{s_1} P_1$ w/ mult. 1. All other summands are P_y, $y < w$.

Similarly, $\Theta_s P_w = P_{ws} \oplus \bigoplus P_y^{\oplus m(w,s,y)}$ $ws > w$

If you could find $m(w,s,y)$ then you could inductively find $[P_{ws}]$.

④ For P, Q projective, $\dim \operatorname{Hom}(P,Q) = ([P],[Q])$ dot product in $\mathbb{Z}[W]$

Ex: $\dim \operatorname{Hom}(P_s, \Theta_s P_{st}) = 4 = \dim \operatorname{Hom}(\Theta_s P_{st}, P_s)$ Symmetric.

But I told you $m(st,s,s) = 1$ so somewhere is a 1D subspace giving the inclusion/projection of the summand. How to find it? How to know it was 1D??

KL Conj designed to answer these questions, via some simple algebraic combinatorics!!!

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§2 KL basic

Now we work in more generality – Coxeter groups.

Def: A Coxeter group is a group w/ a presentation

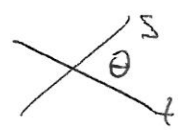
$$W = \langle s \in S \mid s^2 = 1, \underbrace{sts\ldots}_{m_{st}} = \underbrace{tst\ldots}_{m_{st}} \rangle$$

for some $m_{st} \in \{2, 3, \ldots, \infty\}$

($s \in S$: simple reflections; $s^2=1$: quadratic; $sts\ldots = tst\ldots$: braid)

braid, given quad, $\Leftrightarrow (st)^{m_{st}} = 1$.

Think: s is reflection across hyperplane, st = rotation by 2θ, m_{st} encodes angle.



$m_{st} = 2$, $\theta = 90°$, s,t commute.

Chart:

Coxeter Groups ⊃ finite Cox Gps ⊃ Weyl Gps (intersection)

⊃ Crystallographic Cox Gps ⊃ Affine Weyl Gps

$m_{st} \in \{2, 3, 4, 6, \infty\}$

These have associated KM Lie algebras, Geometric Constructions, etc.

The rest don't.

Now $\mathbb{Z}[W]$ has presentation $\mathbb{Z}\langle s \in S \rangle / \begin{matrix}(s+1)(s-1) = 0 \\ \text{braid}\end{matrix}$

Def: The Hecke alg of W is the $\mathbb{Z}[v, v^{-1}]$-alg with presentation

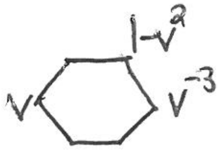
$$\mathbb{H}_W = \mathbb{Z}[v^{\pm}]\langle H_s, s \in S \rangle \Big/ \begin{matrix}(H_s + v)(H_s - v^{-1}) = 0 \\ \text{braid}\end{matrix}$$

$\mathbb{H}_W / {v=1} = \mathbb{Z}[W]$

Chose red. exp. for $w \in W$, $w = s_1 s_2 \ldots s_d$. Let $H_w = H_{s_1} \cdots H_{s_d}$.

Matsumoto Lemma: Two reduced exps for w are related by braid relns $\Rightarrow$ H_w is well-defined, indep of choice.

$\{H_w\}$ is a basis for $\mathbb{H}_W$, standard basis. H_1 = identity.

Ex1 

is $(1-v^2)H_1 + v^{-3}H_s + vH_{ts}$.

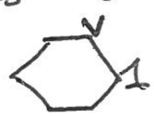
Now the weirdness: $\exists!$ involution $\mathbb{H}_W \to \mathbb{H}_W$ for which $\bar{v} = v^{-1}$
bar involution $b \mapsto \bar{b}$

$\overline{H_s} = H_s^{-1} \overset{\text{exercise}}{=} H_s + (v - v^{-1})$

$\overline{ab} = \bar{a}\,\bar{b}$

Hard to compute

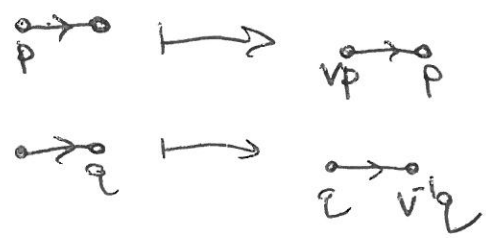
Exercise: $\underline{H}_s \equiv H_s + v$ is bar-invariant = self-dual, so mult by $\underline{H}_s$ preserves self-dual elements.



($v=1$, get $1+s$)

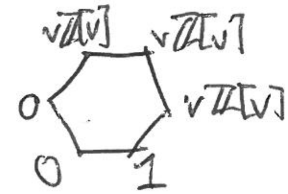
Analogous to before, finding self-duals directly is hard, so build inductively.

Right mult by $\underline{H}_s$:



Thm (K-L): $\exists!$ basis $\{\underline{H}_w\}$ of $\mathbb{H}_W$ over $\mathbb{Z}[v^{\pm}]$ st.

- $\underline{H}_w$ is self-dual
- $\underline{H}_w = H_w + \sum_{y<w} h_{y,w} H_y$ with $h_{y,w} \in v\mathbb{Z}[v]$

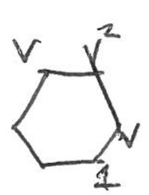
Ex $\underline{H}_{st}$ = 

Big Q: Find them. Exercise: Uniqueness.

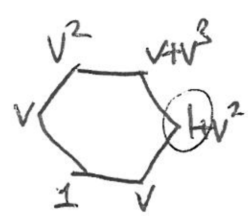
Inductive construction/algorithm: $\underline{H}_1 = H_1$

$1 \cdot \underline{H}_s = \underline{H}_s$ 

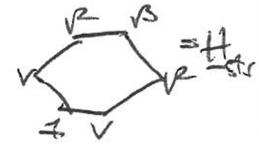
✓ notation okay.

$1 \cdot \underline{H}_s \underline{H}_t$ = 

$= \underline{H}_{st}$

$1 \cdot \underline{H}_s \underline{H}_t \underline{H}_s$ = 

must subtract self-dual, so subtract $\underline{H}_s$

get 

$= \underline{H}_{sts}$

This is the same as our $\mathcal{O}_0$ calc, except the extra data (poly vs. integer) tells you exactly how to complete the algorithm, what to subtract.

$2 \rightsquigarrow 1+v^2$ it's the constant term. A true algorithm.

Obs: When we subtracted $\underline{H}_s$, we subtracted $v\underline{H}_1$ ... but was already a vH_1 there, so no negative coeffs. Miraculous. A priori, no reason that $h_{y,w}$ needn't have negative coefficients.

Conj (K-L): The algorithm works for $\mathcal{O}_0$ too. I.e. $p_{yw} = h_{yw}(1)$ (= mult of Δ_y in P_w)

Where do the polynomials come from, in context of $\mathcal{O}_0$? Actually, $\mathcal{O}_0$ is secretly graded.

$\exists$ cat. $\mathcal{O}_0^{\mathbb{Z}}$, graded (has shift functor (n), $n \in \mathbb{Z}$) $\quad \mathrm{Hom}^\bullet(P,Q) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(P,Q(n))$ a gded v.s.

has "lifts" of all L_w, Δ_w, P_w

it satisfies $[\mathcal{O}_0^{\mathbb{Z}}] \cong \mathbb{H}_W$

$[P(1)] = v[P]$

$\Theta_s \Leftrightarrow$ right mult by $\underline{H}_s$

$\mathrm{gdim}\, \mathrm{Hom}^\bullet(P,Q) = ([P],[Q])$ ← dot product

Ex: $(\underline{H}_s\underline{H}_s, \underline{H}_s) = 1+2v^2+v^4$ so our 4d hom space was really graded! the 1D subspace was just the degree 0 part!!!

$[P_w] = \sum p_{y,w}[\Delta_w]$ $\quad p_{y,w} \in \mathbb{Z}_{\geq 0}[v^{\pm}]$ encodes graded multiplicity.

(graded) **Conj (K-L):** $[P_w] = \underline{H}_w$. $\Longrightarrow h_{y,w} = p_{y,w}$ and all coeffs are positive!!

Observations:

① Defining $\mathcal{O}_0^{\mathbb{Z}}$ is pretty nasty + unnatural. Grading is NOT obvious.

② KL Conj $\overset{\text{explains}}{\Longrightarrow}$ positivity miracle. But positivity observed + conjectured for ALL Coxeter groups, and no KL Conj to explain it!

The last step] Clearly $(\underline{H}_w, \underline{H}_y) = \delta_{wy} + v\mathbb{Z}[v]$

Inductive construction showed $\quad \underline{H}_w\underline{H}_s = \underline{H}_{ws} + \sum_{y<ws} \mu(w,s,y)\underline{H}_y \quad \mu(w,s,y) \in \mathbb{Z}$

Want to show $\mu(w,s,y) = m(w,s,y)$ $\quad \Theta_s P_w = P_{ws} \oplus \bigoplus P_y^{\oplus m(w,s,y)}$

Assuming KL Conj for all $y < ws$, get $\dim \mathrm{Hom}_0(P_y, \Theta_s P_w) = \mu(w,s,y) = \dim \mathrm{Hom}_0(\Theta_s P_w, P_y)$

Composition gives $\mathrm{Hom}_O(P_y, \Theta_s P_w) \times \mathrm{Hom}_O(\Theta_s P_w, P_y) \longrightarrow \mathrm{End}_O(P_y) = \mathbb{C}$

the (local) intersection form.

Key (easy) general fact: rank of form = multiplicity of P_y as summand in $\Theta_s P_w$.

So $\mu = m \iff$ intersection form is non-degenerate !!

KL Conj $\iff$ ALL int forms are nondegenerate.

So to conclude, finding the correct 1DC4D is "numerical" – once working in graded version, just look in degree 0. The "easy" part of KL.

But showing that the intersection form is full rank is hard, and is a question about computing morphisms.

Knowing graded dimensions isn't enough... not numerical. Must really be able to compute!

Computing in $\mathcal{O}_0$ is hard!

That's why we work instead with Soergel Bimodules, an alternate version of $\mathrm{Proj}\ \mathcal{O}_0^{\mathbb{Z}}$.
(actually, a lift)

Advantages: ① Obviously graded ② Easy to define ③ Works for all Coxeter groups.

④ More recently, have tools to compute w/ morphisms!

Rough outline:
- Learn S. Bim
- Learn how to compute w/ morphisms
- Learn how to prove intersection forms (in SBim context) are nondegenerate, via Hodge theory.

Can prove this abstractly, w/o needing to be able to compute... but you'd never have done it! Computation ⇝ intuition ⇝ understanding.

For this reason, I am a firm believer in exercises. Everything else about H/w you will learn via the exercises.