Kazhdan-Lusztig polynomials of matroids

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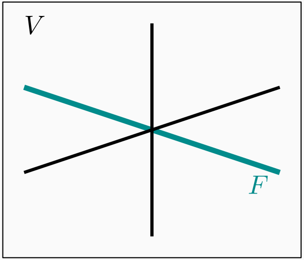
Arrangements and Flats

Let V be a finite dimensional vector space

$\mathcal{A}$ a finite set of hyperplanes in V with $\bigcap_{H \in \mathcal{A}} H = \{0\}$

$F \subset V$ a flat (intersection of some hyperplanes)

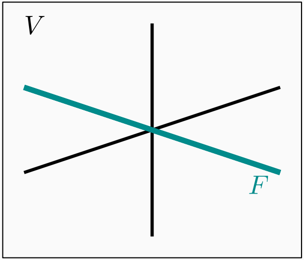


Definition

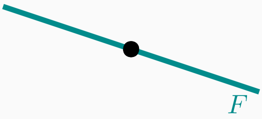
The **contraction of $\mathcal{A}$ at F** is the arrangement

$$\mathcal{A}^F := \{H \cap F \mid F \not\subset H \in \mathcal{A}\}$$

in the vector space F.



$\mathcal{A}$



$\mathcal{A}^F$

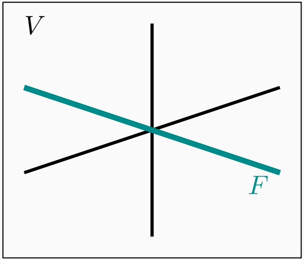
Arrangements and Flats

Definition

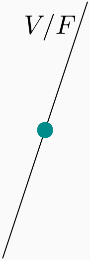
The **localization of $\mathcal{A}$ at F** is the arrangement

$$\mathcal{A}_F := \{H/F \mid F \subset H \in \mathcal{A}\}$$

in the vector space V/F.



$\mathcal{A}$



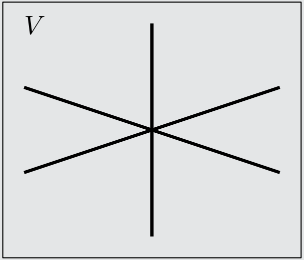
$\mathcal{A}_F$

Characteristic Polynomial

Let $\chi_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ be the **characteristic polynomial** of $\mathcal{A}$.

If V is a vector space over $\mathbb{F}_q$, $\chi_{\mathcal{A}}(q) = |V \setminus \bigcup_{H \in \mathcal{A}} H|$.

Example



$\chi_{\mathcal{A}}(t) = t^2 - 3t + 2$

Kazhdan-Lusztig Polynomial

Theorem

There exists a unique way to assign to each arrangement $\mathcal{A}$ a polynomial $P_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ subject to the following conditions:

- *If* $\dim V = 0$, $P_{\mathcal{A}}(t) = 1$
- *If* $\dim V > 0$, $\deg P_{\mathcal{A}}(t) < \frac{1}{2} \dim V$
- $t^{\dim V} P_{\mathcal{A}}(t^{-1}) = \sum_{F} \chi_{\mathcal{A}_F}(t) P_{\mathcal{A}^F}(t)$.

$P_{\mathcal{A}}(t)$ is called the **Kazhdan-Lusztig polynomial** of $\mathcal{A}$.

Remark

The theory of Kazhdan-Lusztig-Stanley polynomials provides a common generalization of these polynomials and classical Kazhdan-Lusztig polynomials.

Geometric Interpretation

$$V \hookrightarrow \bigoplus_{H \in \mathcal{A}} V/H \cong \bigoplus_{H \in \mathcal{A}} \mathbb{A}^1 \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1$$

Definition

We define the **Schubert variety of $\mathcal{A}$**

$$Y_{\mathcal{A}} := \bar{V} \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1.$$

Have $H^*(Y_{\mathcal{A}}) \curvearrowright IH^*(Y_{\mathcal{A}})$, both concentrated in even degree.

Geometric Interpretation

Theorem (Huh-Wang, P-Xu-Young, Elias-P-Wakefield)

- $\sum t^i \dim H^{2i}(Y_{\mathcal{A}}) \;=\; \sum_F t^{\operatorname{codim} F}$
- $\sum t^i \dim IH^{2i}(Y_{\mathcal{A}}) \;=\; \sum_F t^{\operatorname{codim} F} P_{\mathcal{A}^F}(t) \;=:\; Z_{\mathcal{A}}(t)$
- $\sum t^i \dim \left(IH^{2i}(Y_{\mathcal{A}}) \middle/ H^2(Y_{\mathcal{A}}) \cdot IH^{2i-2}(Y_{\mathcal{A}}) \right) \;=\; P_{\mathcal{A}}(t)$

Corollary

The polynomial $P_{\mathcal{A}}(t)$ has non-negative coefficients.

Remark

The definition of $P_{\mathcal{A}}(t)$ makes sense for matroids, but when the matroid is not realizable, non-negativity is still a conjecture. Work in progress by Braden-Huh-Matherne-P-Wang.

Example

Let $\mathcal{A}_n$ be an arrangement of n generic hyperplanes in $\mathbb{C}^{n-1}$.

What are the flats?

- For each $k < n-1$, there are $\binom{n}{k}$ flats of codimension k.
  For such a flat F,
  $$\mathcal{A}_n^F \cong \mathcal{A}_{n-k}$$
  and $(\mathcal{A}_n)_F$ is Boolean of rank k.
- There is a unique flat of codimension $n-1$.

$$t^{n-1}P_{\mathcal{A}_n}(t^{-1}) = \sum_{k=0}^{n-2}\binom{n}{k}(t-1)^k P_{\mathcal{A}_{n-k}}(t) \ + \ \chi_{\mathcal{A}_n(t)}$$
$$= \sum_{k=0}^{n-2}\binom{n}{k}(t-1)^k P_{\mathcal{A}_{n-k}}(t) \ + \ \frac{(t-1)^n + (-1)^n(t-1)}{t}$$

Example

Put it in a generating function:

$$\Phi(t,u) \;:=\; \sum_{n=2}^{\infty} P_{\mathcal{A}_n}(t)u^{n-1}$$

Then our recursion becomes

$$\begin{aligned}
\Phi(t^{-1},tu) &= \sum_{n=2}^{\infty} t^{n-1} P_{\mathcal{A}_n}(t^{-1})u^{n-1} \\
&= \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n-2} \binom{n}{k} (t-1)^k P_{\mathcal{A}_{n-k}}(t) \;+\; \chi_{\mathcal{A}_n}(t) \right) u^{n-1} \\
&= \cdots \\
&= \frac{(1+u)\,\Phi\!\left(t, \frac{u}{1-u(t-1)}\right) + u(t-1)\,(1-u(t-1))}{(1+u)\,(1-u(t-1))^2}
\end{aligned}$$

Example

Theorem (P-Wakefield-Young)

- $\Phi(t,u) = \frac{2}{u} \cdot \frac{(2t+1)u - 1 + \sqrt{1 - 2(2tu+1)u + u^2}}{1 - (2tu+1)^2}$
- $P_{\mathcal{A}_n}(t) = \sum_{i \geq 0} \frac{1}{i+1} \binom{n}{i} \binom{n-i-2}{i} t^i$

Corollary (Gedeon-P-Young)

The polynomial $P_{\mathcal{A}_n}(t)$ is real-rooted for all n.

Remark

The polynomial $P_{\mathcal{A}}(t)$ is conjecturally real-rooted for any $\mathcal{A}$, but this is the only non-trivial family of examples for which we can prove it!

Thanks!