## POLYTOPES AND COXETER MATROIDS: EXERCISES

## ALEX FINK

- (1) Draw all the polytopes of rank 2 matroids on four elements, up to symmetries of the ground set.
- (2) Does the matroid polytope of the Fano matroid have an octahedral face?
- (3) Prove that the standard permutohedron in R<sup>n</sup> is the Minkowski sum of
  (a) all of the hypersimplices in R<sup>n</sup>.
  (b) a sellection of comments
  - (b) a collection of segments.
- (4) Loday's realisation of the associahedron is the generalised permutohedron which is the Minkowski sum of all simplices of the form  $conv\{e_k, e_{k+1}, \ldots, e_\ell\}$  in  $\mathbb{R}^n$ . What is its rank function? What are its vertices?
- (5) (a) A polytope subdivision  $\Delta$  of a polytope P is a geometric polyhedral complex whose total space is P. Given such a polytope subdivision, prove that

$$\sum_{F} (-1)^{\dim P - \dim F} \mathbf{1}_F = \mathbf{1}_P,$$

where the sum is over cells F of  $\Delta$  not contained in  $\partial P$ , and  $1_F$  is the indicator function of F.

- (b) Let  $C_n$  be the Catalan matroid on  $\{1, \ldots, 2n\}$ , and let  $\tilde{C}_n$  be  $C_n \setminus \{1, 2n\}$ . Let  $\sigma$  be the permutation of  $\{2, \ldots, 2n-1\}$  given by  $\sigma(i) = i+2 \mod 2n-2$ . Prove that  $P(\tilde{C}_n), P(\sigma \tilde{C}_n), \ldots, P(\sigma^{n-2}\tilde{C}_n)$  are the maximal cells in a polytope subdivision of a hypersimplex.
- (6) Construct the stratification of  $Gr(2, \mathbb{K}^4)$  into matroid realisation spaces, and the poset formed by the strata under containment of closures. Does the result depend on  $\mathbb{K}$ ?
- (7) Let W be a dihedral group. Characterise the Coxeter matroids for W whose vertices lie in a single W-orbit.
- (8) A matroid M on ground set E is a *projection* of a matroid N on E if there is some matroid M' on a ground set  $E \amalg F$  such that M = M'/F and  $N = M' \setminus F$ .

Fix a finite set E and naturals  $0 < r_1 < \cdots < r_k < |E|$ . A flag matroid on E of ranks  $(r_1, \ldots, r_k)$  is a collection of matroids on E of respective ranks  $r_1, \ldots, r_k$ , each of which is a projection of the next. Describe how to associate a type A Coxeter matroid, i.e. a polymatroid, to a flag matroid. Prove that distinct flag matroids have distinct polymatroids.

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(9) A ribbon graph is a two-dimensional manifold with boundary obtained as the neighbourhood of a graph embedded in a smooth surface, not necessarily orientable. More precisely, a ribbon graph is obtained by taking a collection of discs, some of them *vertices* and the others *edges*, and repeatedly identifying a closed segment on the boundary of an edge with a closed segment on the boundary of a vertex, such that all these segments are disjoint, and each edge is involved in exactly two identifications.

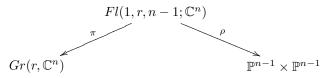
A *basis* of a ribbon graph G is a subset S of its edges such that the union of S and all the vertices of G has exactly one boundary component. Prove that the set of bases of a ribbon graph is a Coxeter matroid for the type C maximal parabolic subgroup excluding the long root.

(Note that, if the ribbon graph comes from a graph embedded in the plane, then its bases are the usual spanning trees of the graph.)

(10) Take a concrete  $x \in Gr(r, \mathbb{C}^n)$ : perhaps (r, n) = (2, 4) and

$$x = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Define the morphisms



given by retaining only the r-dimensional, respectively the 1 and (n-1)dimensional, spaces in the flags. Let  $\mathcal{O}(1)$  be the line bundle on  $Gr(r, \mathbb{C}^n)$ determining the Plücker embedding.

Compute  $\rho_*\pi^*([\mathcal{O}(1)] \cdot [\mathcal{O}_{\overline{(C^{\times})^n x}}])$  in algebraic K-theory, and express it in the basis of powers of structure sheaves of hyperplanes in the two  $\mathbb{P}^{n-1}$ factors. You should get the Tutte polynomial of the matroid of x.

Equivariant localisation is the best tool for this, I reckon.

(11) (Open, but perhaps not difficult?) Let  $\Sigma_{A_{n-1}}$  be the permutahedral fan. Let  $MW^c$  be the group of codimension c Minkowski weights on  $\Sigma_{A_{n-1}}$ , i.e. associations of weights to the codimension c faces of  $\Sigma_{A_{n-1}}$  satisfying the balancing condition.

Let  $ch: MW^c \to MW^1$  be the linear map given by Minkowski sum with the (c-1)-dimensional skeleton F of the normal fan of the simplex  $conv\{-e_1, \ldots, -e_n\}$ , with the weight on a cone  $\sigma$  of ch(m) being the sum of the weights on cones  $\tau$  of  $m \in MW^c$  such that the Minkowski sum  $\tau + F$ contains  $\sigma$ . (In general, to make this operation yield a balanced fan, one needs a correction factor coming from an index of a sublattice, but in the present situation this is trivial.)

Characterise the kernel of ch.

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